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A CANTOR SET IN THE INTERSECTION OF SETS OF LARGE MEASURE

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Abstract. We present a proof of the following theorem. Let $E_n \subset [0,1]$ be a sequence of measurable sets with measures $\mu(E_n) \geqslant \delta > 0$. There is a subsequence whose intersection contains a Cantor set.

The problem, how large can be the intersection of infinitely many sets taken from a given sequence of sets was treated by P. Erdös, H. Kestelman, C. A. Rogers (1963), and by M. Laczkovich (1977). The problem was started anew by M. J. Pelling in Amer. Math. Monthly 101 (1994), p. 275, Problem 10373. The solution given in this paper[1)] is located in the classical theory of the Lebesgue integral, and seems to be more elementary than these ones mentioned above. So, the aim of this note is to present a proof of the following

THEOREM 1. *Let $E_n \subset [0,1]$ be a sequence of measurable sets with measures $\mu(E_n) \geqslant \delta > 0$. There is a subsequence whose intersection contains a Cantor set.*

PROOF. We may assume that for every n the sets E_n are closed. Consider the functions

$$f_n(x) = \mu([0,x] \cap E_n), \tag{1}$$

defined for $n = 1, 2, \ldots$, and $x \in [0,1]$. Functions f_n are continuous monotone and fulfil Lipschitz's condition with a common constant equal to 1.

By Ascoli–Arzela theorem, there exists a subsequence f_n, convergent uniformly. For simplicity, let us suppose that f_n is such a sequence. Let f be the limit of f_n.

The function f is absolutely continuous, satisfying the Lipschitz condition, thus $\int_0^1 f' = f(1) - f(0)$ ([3], p. 223), but $f(1) - f(0) \geqslant \delta$, so we have $\int_0^1 f' \geqslant \delta$.

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[1)]It was accepted as a solution in Amer. Math. Monthly **103** (1996), p. 934–935.

Thus there exist $\alpha > 0$ and a closed set $B \subset [0,1]$ with positive measure, such that

$$(2) \qquad f'(x) > \alpha,$$

for $x \in B$.

We shall define by induction closed intervals $I_{j_1 \dots j_k}$, $j_r \in \{0,1\}$, defining in a standard way a Cantor set, and a sequence $n_1 < n_2 < \dots$ of integers such that

$$(3) \qquad \mu(B \cap E_{n_1} \cap \dots \cap E_{n_k} \cap I_{j_1 \dots j_k}) > 0,$$

for every k and every interval $I_{j_1 \dots j_k}$.

To do this, take on the interval $[0,1]$, using the Lebesgue density theorem, two disjoint closed intervals I_{j_1}, $j_1 \in \{0,1\}$ such that

$$(4) \qquad \mu(B \cap I_{j_1}) > (1-\beta)\mu(I_{j_1}),$$

where $\beta > 0$ is such that

$$(5) \qquad (1-\beta) + \alpha(1-\beta) > 1.$$

From a known property of the derivative of monotone function, having in view the continuity of f, we get $\mu(f(I_{j_1})) \geqslant \int_{I_{j_1}} f'$ ([3], p. 187). Hence, by (2), it follows that $\mu(f(I_{j_1})) > \alpha\mu(B \cap I_{j_1})$. Since f_n converges to f, there exists an integer n_1 such that for both values 0 and 1 of j_1 the inequality $\mu(f_{n_1}(I_{j_1})) > \alpha\mu(B \cap I_{j_1})$ holds. By (1), we have $\mu(E_{n_1} \cap I_{j_1}) > \alpha\mu(B \cap I_{j_1})$. From the last inequality and from (4) we get

$$(6) \qquad \mu(E_{n_1} \cap I_{j_1}) > \alpha(1-\beta)\mu(I_{j_1}),$$

for both values of j_1. Now, from (4), (6) and (5), we get $\mu(B \cap E_{n_1} \cap I_{j_1}) > 0$ for both $j_1 \in \{0,1\}$. Thus the inductive construction for $k = 1$ is finished. Suppose that for $k \leqslant m$ there are defined intervals $I_{j_1 \dots j_k}$ and integers $n_1 < n_2 < \dots < n_k$ such that for any given $k \leqslant m$ the inequalities $\mu(B \cap E_{n_1} \cap \dots \cap E_{n_k} \cap I_{j_1 \dots j_k}) > 0$ hold for all sequences $j_1 \dots j_k$, where $j_r \in \{0,1\}$. Repeating the reasoning from the first step of induction with $B \cap E_{n_1} \cap \dots \cap E_{n_m}$ instead of B, and with $I_{j_1 \dots j_m}$ insteread of $[0,1]$, we get an integer n_{m+1} greater than n_m, common for all intervals $I_{j_1} \dots j_{m+1}$ and such that $\mu(B \cap E_{n_1} \cap \dots \cap E_{n_{m+1}} \cap I_{j_1 \dots j_{m+1}}) > 0$ for each $I_{j_1} \dots j_{m+1}$. Thus, the inductive construction is finished.

By (3), we have $E_{n_1} \cap \dots \cap E_{n_k} \cap I_{j_1 \dots j_k} \geqslant \emptyset$ for all the sets defined above. It follows that each set E_{n_i} has points in each interval $I_{j_1 \dots j_k}$. Therefore, the Cantor set defined by the intervals $I_{j_1 \dots j_k}$ lies in the set of accumulation points of each E_{n_i}, thus, in view of compactness of the sets E_n, it is contained in each E_n, thus in the intersection $E_{n_1} \cap E_{n_2} \cap \dots$, which ends the proof. $\square$

References

[1] P. Erdös, H. Kestelman, C. A. Rogers, *An intersection property of sets with positive measure*, Coll. Math. **11** (1963), 75–80.

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[3] I. P. Natanson, *Theory of functions of real variable* (Russian), Moscow–Leningrad 1950.

[4] M. J. Pelling, *Problem 10373*, Amer. Math. Monthly **101** (1994), 274.

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