Annales Mathematicae Silesianae 11. Katowice 1997, 39 – 53

Prace Naukowe Uniwersytetu Śląskiego nr 1665

UNIFIED APPROACH TO BOUNDED, PERIODIC AND ALMOST PERIODIC SOLUTIONS OF DIFFERENTIAL SYSTEMS

JAN ANDRES* AND BOHUMIL KRAJC

Abstract. The criteria for an entirely bounded solution of a quasi-linear differential system are developed via asymptotic boundary value problems. The same principle allows us to deduce at the same time the existence of periodic orbits, when assuming additionally periodicity in time variables of the related right-hand sides. For almost periodicity, the situation is unfortunately not so straightforward. Nevertheless, for the Lipschitzean uniformly almost periodic (in time variables) systems, we are able to show that every bounded solution becomes almost periodic as well.

1. Introduction

The main purpose of this paper consists in discussion whether or not we can consider at the same time the existence problems for bounded, periodic and almost periodic solutions of differential systems. The idea is, roughly speaking, to deduce the results for periodicity or almost periodicity from those for boundedness, provided the same for the generating vector fields. For the sake of simplicity, we are concerned only with the quasi-linear systems which have been already systematically studied (see e.g. [1], [2], [8], [9], [11], [6], [7] and the references therein). As we will see, our problem can be affirmatively answered only partly, because almost periodicity brings some serious troubles. On the other hand, the theorems given here still generalize or improve some of their analogies obtained separately (see e.g. those by P. Bohl, G.I. Birjuk and M.A. Krasnosel'skii in [8, pp. 360–361, 426], [14, p. 100]).

1991 *Mathematics Subject Classification.* 34B15, 34C11, 34C25, 34C27.

Key words and phrases: Asymptotic boundary value problems, boundedness, periodicity, almost periodicity, unified approach.

* Supported by grant no 0053 1996 of FRVŠ.

2. Unified approach

Consider the problem

$$\begin{cases} X' = AX + F(t,X), \\ X \in S, \end{cases} \tag{1}$$

where $X = (x_1, \ldots, x_n)$, $A = [a_{jk}]$ is a constant $(n \times n)$-matrix which is hyperbolic (i.e. the eigenvalues have nonzero real parts), $F : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is a continuous function and S is a (nonempty) bounded, closed subset of the Fréchet space $C(\mathbb{R}, \mathbb{R}^n)$ endowed with the topology of the uniform convergence on compact subintervals of $\mathbb{R}$.

We start with a very special case of the main result in [4].

PROPOSITION 1. *Problem* (1) *is solvable if there exists a (nonempty), closed, convex, bounded subset Q of $C(\mathbb{R}, \mathbb{R}^n)$ such that, for any $q(t) \in Q$, the problem*

$$\begin{cases} X' = AX + F(t,q(t)) \ , \\ X \in S \cap Q \end{cases} \tag{2}$$

has a unique solution.

Defining (the evidently nonempty, closed, convex, bounded subset of $C(\mathbb{R}, \mathbb{R}^n)$)

$$S_1 := \{s(t) \in C(\mathbb{R}, \mathbb{R}^n) : \sup_{t \in \mathbb{R}} \|s(t)\| \leqslant C\}, \tag{3}$$

where C is a suitable constant which will be specified below and putting $Q = S = S_1$, we can get immediately

PROPOSITION 2. *System*

$$X' = AX + F(t,X) \tag{4}$$

admits, under the above assumptions, an entirely bounded solution if the problem

$$\begin{cases} X' = AX + F(t,q(t)) \ , \\ X \in S_1 \ , \end{cases} \tag{5}$$

has a solutions for each $q(t) \in S_1$.

For the existence of periodic solutions, we employ still the following well-known Massera theorem (see e.g. [17, p.164]).

LEMMA 1. *For the linear system*

$$X' = AX + P(t), \tag{6}$$

where A is as above and $P(t)$ is a continuous, T-periodic n-vector defined on $\mathbb{R}$, the existence of a solution which is bounded in the future implies the existence of a T-periodic one.

Hence, replacing S_1 by (the nonempty, closed, convex, bounded subset of $C(\mathbb{R},\mathbb{R}^n)$):

$$S_2 := S_1 \cap \{s(t) \in C(\mathbb{R},\mathbb{R}^n) : s(t+T) \equiv s(t)\},$$

and putting $P(t) = F(t, q(t))$ in (5), we obtain as a direct consequence of Proposition 2 and Lemma 1 the following

PROPOSITION 3. *System* (4), *where* $F(t+T, X) \equiv F(t, X)$, *admits under the assumptions of Proposition 2 a T-periodic solution.*

REMARK 1. Obviously, the same can be also done when working only in the Banach space of continuous T-periodic functions.

For the existence of (uniform) almost periodic solutions, the situation however becomes different, which will be now discussed in detail. By the analogy to the periodic case, it is natural to use the concept of uniform almost periodicity for F, because then the composition $F(t, q(t))$, where $q(t)$ is almost periodic, is known (see e.g. [12, p. 14]) to be almost periodic as well.

DEFINITION (see e.g. [12], [17]) $F(t, X) \in C(\mathbb{R}^{n+1}, \mathbb{R}^n)$ is called *uniformly almost periodic in t*, if for any $\varepsilon > 0$ and any $D > 0$ there exists a positive number $l(\varepsilon, D)$ such that any interval of the length $l(\varepsilon, D)$ contains a τ for which

$$\|F(t+\tau, X) - F(t, X)\| < \varepsilon$$

for all $t \in \mathbb{R}$ and $\|X\| \leqslant D$.

Thus, one may expect to employ in a similar manner as in the periodic case, the generalization of the well-known theorem due to H. Bohr and O. Neugebauer (see [8, pp. 423–425]). This says that for the linear system

(6), where A is as above and $P(t)$ is this time almost periodic, every entirely bounded solutions is almost periodic.

However, such an approach fails. Replacing namely S_1 by the nonempty, convex, bounded subset of $C(\mathbb{R}, \mathbb{R}^n)$:

$$S_3 := S_1 \cap \{s(t) \in C(\mathbb{R}, \mathbb{R}^n) : s(t) \text{ is almost periodic}\},$$

one can check that S_3 is not closed in the given topology of the uniform convergence on compact subintervals of $\mathbb{R}$. This fact seems, unfortunately, essential in the negative sense. On the other hand, S_3 is a closed subset of the Banach space of all almost periodic functions endowed with the topology of the uniform convergence on the whole $\mathbb{R}$. But then the operators $X = T(Q)$ associated with the solutions of problem (2), where $Q = S = S_3$, are not necessarily compact with the new topology because of the failness of the well-known Ascoli-Arzela theorem, related there strictly to the compact intervals.

To conclude, we simply cannot go in this way. Fortunately, later we will be able to deduce almost periodicity directly from the integral form of the representation of bounded solutions, provided F is Lipschitzean in X.

3. Estimates for bounded solutions of linear systems

As we could see in the foregoing section, the estimates of entirely bounded solutions of (6) will be crucial. Hence, consider (6), where A is as above and $P(t)$ is a continuous bounded vector function such that

$$(\infty >)\ P := \sup_{t\in\mathbb{R}} \|P(t)\| \,. \tag{7}$$

The following well-known statements (for details see e.g. [8, pp. 358–360]) will be very useful in the sequel.

LEMMA 2. *Let A be hyperbolic and $P(t)$ satisfy* (7). *Then system* (6) *has exactly one entirely bounded solution*

$$X(t) = \int_{-\infty}^{\infty} G(t-s)\, P(s) ds \,, \tag{8}$$

where $G(t) \in C^\infty(\mathbb{R} \setminus \{0\})$ is a suitable $(n \times n)$-matrix function.

The matrix A can be written in the form $S(\mathcal{P}, \mathcal{N})S^{-1}$, where S is a regular matrix and $(\mathcal{P}, \mathcal{N})$ is a block diagonal matrix, having the Jordan canonical form, such that the eigenvalues of A with positive real parts make the diagonal of $\mathcal{P}$ and those with negative real parts make the diagonal of $\mathcal{N}$.

REMARK 2. The matrix function $G(t)$ from (8) takes the form (see [8])

$$G(t) = \begin{cases} -S(e^{\mathcal{P}t}, 0)S^{-1} & \text{for } t < 0, \\ S(0, e^{\mathcal{N}t})S^{-1} & \text{for } t > 0\,. \end{cases} \tag{9}$$

Hence, for a stable matrix A we have

$$X(t) = \int_{-\infty}^{t} e^{A(t-s)}P(s)ds\,. \tag{10}$$

If each eigenvalue of A has a positive real part, then

$$X(t) = -\int_{t}^{\infty} e^{A(t-s)}P(s)ds\,. \tag{11}$$

LEMMA 3. *The folowing estimate holds for $X(t)$ in* (8) *with $G(t)$ in* (9)*:*

$$\sup_{t\in\mathbb{R}} \|X(t)\| \leqslant P\int_{-\infty}^{\infty} \|G(t)\|dt \leqslant PC_1(A)\,, \tag{12}$$

where $C_1(A)$ is a real constant depending only on A.

For $X(t)$ in (10) *or* (11)*, inequality* (12) *can be expressed explicitly as follows:*

$$\sup_{t\in\mathbb{R}} \|X(t)\| \leqslant P\sum_{k=0}^{n-1} \frac{2^k\|A\|^k}{|Re\,\lambda|^{k+1}}, \tag{13}$$

where $|Re\,\lambda|$ denotes the minimum of the absolute values of the real parts of the eigenvalues of A. (The matrix norms above are compatible with the vectors ones).

PROOF. The formula (12) follows from (9) (for details see e.g. 8, pp. 358–360]). For verifying (13), assume at first that all eigenvalues of A have negative real parts. According to (10), we have

$$\|X(t)\| \leqslant P\int_{-\infty}^{t} \|e^{A(t-s)}\|ds$$

i.e.

$$\|X(t)\| \leqslant P \int_0^\infty \|e^{Au}\| du,$$

after the substitution $u = t - s$. Applying the refined version (see [3, p. 131]) of the Gel'fand-Shilov inequality, namely

$$\|e^{Au}\| \leqslant e^{-|\mathrm{Re}\,\lambda| u} \sum_{k=0}^{n-1} \frac{(2u\|A\|)^k}{k!}, \qquad u \geqslant 0,$$

we obtain

$$\|X(t)\| \leqslant P \sum_{k=0}^{n-1} \frac{(2^k\|A\|)^k}{k!} \int_0^\infty e^{-|\mathrm{Re}\,\lambda| u} u^k du.$$

To get (13), it is sufficient to integrate by parts the last integral. After the substitution $Y(t) = -X(t)$ into (6), one can readily check that (13) holds also in the case when all the eigenvalues have positive real parts. This completes the proof. $\square$

LEMMA 4. *If A is a additionally symmetrical matrix and the Euclidean vector norms are used then, under the assumptions of Lemma 3, the following estimate holds for $X(t)$ in (8):*

$$\|X(t)\| \leqslant P \frac{2}{|\lambda|},$$

where $|\lambda|$ is the minimum of the absolute values of the eigenvalues of A.

PROOF. In (9), an orthogonal matrix S can be employed; $(\mathcal{P}, \mathcal{N})$ is diagonal. Hence, by means of the spectral matrix norm we get

$$\begin{aligned}\|X(t)\| &\leqslant P \int_{-\infty}^{\infty} \|G(t-s)\| ds \\ &= P \left[\int_{-\infty}^{t} \|e^{(\mathcal{P},0)(t-s)}\| ds + \int_t^\infty \|e^{(0,\mathcal{N})(t-s)}\| ds \right] \leqslant \frac{2P}{|\lambda|}. \quad \square\end{aligned}$$

For the planar systems, the conclusions of Lemma 2 and 3 can be improved in the following way.

LEMMA 5. *Consider* (6), *where* A *is a* (2×2)*-matrix with the eigenvalues* λ_1, λ_2. *Then the following is true for* $C_1(A)$ *in* (12)*:*

(i)
$$\lambda_1 = \lambda_2 \neq 0 \implies C_1(A) = \frac{\|E\|}{|\lambda_1|} + \frac{\|A-\lambda_1 E\|}{\lambda_1^2},$$

(ii)
$$\operatorname{sgn}(\operatorname{Re}\lambda_1) = \operatorname{sgn}(\operatorname{Re}\lambda_2) \neq 0,\ \lambda_1 \neq \lambda_2 \implies$$
$$C_1(A) = \frac{1}{|\lambda_1-\lambda_2|}\left(\frac{\|A-\lambda_2 E\|}{|\operatorname{Re}\lambda_1|} + \frac{\|A-\lambda_1 E\|}{|\operatorname{Re}\lambda_2|}\right),$$

(iii)
$$\lambda_1 > 0 > \lambda_2,\ a_{21} \neq 0 \implies$$
$$C_1(A) = \frac{1}{|a_{21}|(\lambda_1-\lambda_2)}\left(\frac{\left\|\begin{matrix} a_{21}(a_{11}-\lambda_2) & (\lambda_2-a_{11})(a_{11}-\lambda_1) \\ a_{21}^2 & a_{21}(\lambda_1-a_{11})\end{matrix}\right\|}{\lambda_1} - \frac{\left\|\begin{matrix} a_{21}(\lambda_1-a_{11}) & (a_{11}-\lambda_2)(a_{11}-\lambda_1) \\ -a_{21}^2 & a_{21}(a_{11}-\lambda_2)\end{matrix}\right\|}{\lambda_2}\right),$$

$$\lambda_1 > 0 > \lambda_2,\ a_{12} \neq 0 \implies$$
$$C_1(A) = \frac{1}{|a_{12}|(\lambda_1-\lambda_2)}\left(\frac{\left\|\begin{matrix} a_{12}(a_{22}-\lambda_1) & -a_{12}^2 \\ (a_{22}-\lambda_2)(a_{22}-\lambda_1) & a_{12}(\lambda_2-a_{22})\end{matrix}\right\|}{\lambda_1} - \frac{\left\|\begin{matrix} a_{12}(\lambda_2-a_{22}) & a_{12}^2 \\ (a_{22}-\lambda_2)(\lambda_1-a_{22}) & a_{12}(a_{22}-\lambda_1)\end{matrix}\right\|}{\lambda_2}\right),$$

$$\lambda_1 > 0 > \lambda_2,\ a_{12} = a_{21} = 0 \implies C_1(A) = \frac{1}{\lambda_1} - \frac{1}{\lambda_2}.$$

(One can use row, column, spectral or Schmidt norms above).

PROOF. The standard fundamental matrix e^{At} associated with A reads (see e.g. [13, p. 15])

$$e^{At} = e^{\lambda_1 t}[E + (A-\lambda_1 E)t], \tag{14}$$

for the multiplicity case of eigenvalues, and

$$e^{At} = e^{\lambda_1 t}\frac{(A-\lambda_2 E)}{\lambda_1-\lambda_2} + e^{\lambda_2 t}\frac{(A-\lambda_1 E)}{\lambda_2-\lambda_1}, \tag{15}$$

for $\lambda_1 \neq \lambda_2$.

In the cases (i) and (ii), the entirely bounded solution $X(t)$ of (6) takes the form (10) or (11). Therefore, $G(t) = e^{At}$ for $t > 0$ or $G(t) = -e^{At}$ for $t < 0$, and we can make by the standard manner an estimate for the norm of (14) or (15), respectively. This already gives $C_1(A)$, after an integration.

For (iii), we need to know the matrix S. One can see from (15) that, in the case $a_{21} \neq 0$, the first columns of $(A - \lambda_2 E)$ and $(A - \lambda_1 A)$ are the eigenvectors of A related to λ_1 and λ_2, respectively. Therefore, this columns generate S (see e.g. [13, pp. 56–59]). Hence, $G(t)$ in (9) takes an explicit form, and so we arrive at

$$\|G(t)\| \leqslant e^{\lambda_1 t}\|S \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} S^{-1}\| \qquad \text{for } t < 0$$

and

$$\|G(t)\| \leqslant e^{\lambda_2 t}\|S \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} S^{-1}\| \qquad \text{for } t > 0\,.$$

Now by means of

$$\int_t^{\infty} e^{\lambda_1(t-s)}\,ds = \frac{1}{\lambda_1} \qquad \text{and} \qquad \int_{-\infty}^{t} e^{\lambda_2(t-s)}\,ds = \frac{1}{-\lambda_2}\,, \tag{16}$$

we obtain $C_1(A)$ immediately.

In the case $a_{12} \neq 0$, the same procedure can be used to obtain $C_1(A)$.

To complete the proof, it is sufficient to put $S = E$ in the diagonal case $a_{12} = 0$, $a_{21} = 0$, and apply (16). □

4. Main results

Now, we can give the main results of our paper. It follows from the investigation in Part 2 that if we prove the existence of an entirely bounded solution of (4) by means of Proposition 2, then the existence of periodic solutions can be immediately deduced as well (see Proposition 3), when assuming additionaly the periodicity of F in t.

THEOREM 1. *Let A be a hyperbolic matrix and for $F(t,X)$ bounded in t assume*

$$(17)\qquad \limsup_{\|X\|\to\infty} \frac{\|F(t,X)\|}{\|X\|} \leqslant \mu, \quad \textit{uniformly w.r.t. } t\in\mathbb{R},$$

where μ is a sufficiently small constant. Then system (4) admits an entirely bounded solution.

If, additionally, F is periodic in t, then system (4) has at least one T-periodic solution. At last, if F is additionaly uniformly almost periodic in t and Lipschitzean in X with a sufficiently small Lipschitz constant L, then system (4) has at least one almost periodic solution.

PROOF. Since condition (17) implies the existence of a positive constant C (see (3)) such that

$$(18)\qquad C_1 \sup_{\substack{\|X\|\leqslant C\\ t\in\mathbb{R}}} \|F(t,X)\| \leqslant C,$$

where $C_1 > 0$ is an arbitrary fixed constant (see (12)), the first and second assertions follow directly from Propositions 2,3 and Lemmas 2,3. So it remains to prove the almost periodicity case, when following the idea e.g. in [8, pp. 426–427].

According to Lemmas 2 and 3, the bounded solution $X(t)$ satisfies

$$X(t) = \int_{-\infty}^{\infty} G(t-s)\,F(s,X(s))ds$$

with $\sup_{t\in\mathbb{R}} \|X(t)\| \leqslant C$. Assuming the uniform almost periodicity of F in t (see Definition), let $\tau = \tau(\varepsilon, C)$ be an almost period of F for ε, C. Thus (cf. [8])

$$X(t+\tau) = \int_{-\infty}^{\infty} G(t+\tau-s)\,F(s,X(s))ds = \int_{-\infty}^{\infty} G(t-s)\,F(s+\tau, X(s+\tau))ds\,,$$

and consequently, by means of Lipschitzeanity and (12), (18):

$$
\begin{aligned}
&\|X(t+\tau)-X(t)\| \\
&\quad\leqslant \int_{-\infty}^{\infty}\|G(t-s)\|\,\Big[\|F(s+\tau,X(s+\tau))-F(s,X(s+\tau))\| \\
&\qquad + \|F(s,X(s+\tau))-F(s,X(s))\|\Big]\,ds \\
&\quad\leqslant \Big[\sup_{\substack{\|X\|\leqslant C\\ t\in\mathbb{R}}}\|F(t+\tau,X)-F(t,X)\| + L\sup_{t\in\mathbb{R}}\|X(t+\tau)-X(t)\|\Big]\int_{-\infty}^{\infty}\|G(t)\|dt \\
&\quad\leqslant \Big[\varepsilon + L\sup_{t\in\mathbb{R}}\|X(t+\tau)-X(t)\|\Big]\,C_1(A).
\end{aligned}
$$

Since L is, by the hypothesis, sufficiently small, we arrive at

$$
\sup_{t\in\mathbb{R}}\|X(t+\tau)-X(t)\| \leqslant \frac{C_1(A)}{1-C_1(A)\,L}\varepsilon\,,
$$

i.e. $X(t)$ is almost periodic, which completes the proof. □

REMARK 3. Because of the absence of Lipschitzeanity, Theorem 1 generalizes, for example, its analogy for boundedness due to P. Bohl (see e.g. [8, pp. 360–361]) and coincides with its analogy for almost periodicity due to G.I. Birjuk (see e.g. [8, p. 426]).

THEOREM 2. *Let all the eigenvalues of A have negative (positive) real parts and let there exist a constant $C \geqslant 0$ such that*

$$
\sup_{\substack{\|X\|\leqslant C\\ t\in\mathbb{R}}}\|F(t,X)\| \leqslant C \Big/ \sum_{k=0}^{n-1}\frac{2^k\|A\|^k}{|\mathrm{Re}\,\lambda|^{k+1}},
$$

where $|Re\,\lambda|$ denotes the minimum of the absolute values of the real parts of the eigenvalues of A (its lower estimate in terms of the coefficients of A can be found e.g. in [10]); the matrix norm is compatible with the vector one. Then the same assertion as in Theorem 1 holds, where for L (in the third part) it is sufficient to assume locally (i.e. on $\|X\| \leqslant C$) that

$$
L < 1 \Big/ \sum_{k=0}^{n-1}\frac{2^k\|A\|^k}{|\mathrm{Re}\,\lambda|^{k+1}}\,. \tag{19}
$$

PROOF. — follows immediately from Propositions 2, 3; Theorem 1 and Lemmas 2, 3. For the magnitude of the Lipschitzean constant L see (12), (13) and the proof of Theorem 1, concerning the sufficient condition $L < 1/C_1(A)$. $\square$

REMARK 4. Obviously, in this particular situation for A, Theorem 2 improves not only Theorem 1 (and so e.g. the mentioned one by G.I. Birjuk), but also, for example, its analogy for periodicity due to M.A. Krasnosel'skii (see e.g. [14, p. 100]).

Using Lemma 4, one can repeat the idea of the proof of Theorem 2 to obtain

COROLLARY 1. *Let A be a symmetrical matrix which is hyperbolic, $|\lambda|$ be the minimum of the absolute values of the eigenvalues of A and perturbation $F(t,X)$ be such that the estimate*

$$\sup_{\substack{\|X\|\leqslant C \\ t\in\mathbb{R}}} \|F(t,X)\| \leqslant C\frac{|\lambda|}{2}$$

holds for a constant $C \geqslant 0$ (with the Euclidean norms in the left-hand side of the inequality).

Then the same assertion as in Theorem 1 holds with the Lipschitz constant satisfying

$$L < \frac{|\lambda|}{2}\,. \tag{20}$$

Finally, we can formulate results for planar systems.

THEOREM 3. *Consider the system (4), where A is a hyperbolic (2×2)-matrix. Let $C_1(A)$ be same as in Lemma 5 and the nonlinearity $F(t,X)$ satisfy the estimate*

$$\sup_{\substack{\|X\|\leqslant C \\ t\in\mathbb{R}}} \|F(t,X)\| \leqslant C/C_1(A)\,,$$

for a constant $C \geqslant 0$.

Then the same assertion as in Theorem 1 holds for (4), where for L we require

$$L < \frac{1}{C_1(A)}\,. \tag{21}$$

PROOF – is trivial, when following the idea of the proof of Theorem 2. □

COROLLARY 2. *Let A be hyperbolic, $F(t,0)$ be bounded and $F(t,X)$ be Lipschitzean in X with a sufficiently small Lipschitz constant L. Then system* (4) *has exactly one entirely bounded solution. If, additionally, F is T-periodic or uniformly almost periodic in t, then system* (4) *has exactly one T-periodic or almost periodic solution, respectively.*

In particular, if all the eigenvalues of A have still negative (positive) real parts, then it is sufficient to take L (globally) as in (19). *For a symmetrical A or for a hyperbolic (2×2)-matrix A, conditions* (20) *or* (21) *are sufficient, respectively.*

PROOF. One can easily check that the growth restrictions imposed on F by the Lipschitzeanity imply those above. Thus, the existence part follows immediately from Theorems 1, 2 and 3.

For the uniqueness part, it is obviously enough to show that two bounded solutions must be identical. Hence, let $X(t)$ and $Y(t)$ be bounded solutions of (4). Because of Lemmas 2 and 3, they can be expressed as

$$X(t) = \int_{-\infty}^{\infty} G(t-s)\,F(s,X(s))ds$$

and

$$Y(t) = \int_{-\infty}^{\infty} G(t-s)\,F(s,Y(s))ds\ ,$$

where (see (12)) $\int_{-\infty}^{\infty} \|G(t)\|dt \leqslant C_1(A)$.

Therefore, we have

$$\|X(t)-Y(t)\| \leqslant \int_{-\infty}^{\infty} \|G(t-s)\|\;\|F(s,X(s))-F(s,Y(s))\|ds \leqslant$$

$$\leqslant L\int_{-\infty}^{\infty} \|G(t-s)\|\;\|X(s)-Y(s)\|ds \leqslant L\,C_1(A)\sup_{t\in\mathbb{R}}\|X(t)-Y(t)\|\ ,$$

and consequently

$$[1-L\,C_1(A)]\sup_{t\in\mathbb{R}}\|X(t)-Y(t)\| \leqslant 0 \qquad \text{as far as} \qquad L<1/C_1(A)\ ,$$

where $C_1(A)$ can be expressed in particular cases for A as above (see (19), (20), (21)). Since L is by the hypothesis so sufficiently small, we arrive at the desired identity $X(t) \equiv Y(t)$. This completes the proof. □

5. Bounded solutions of quasi-linear systems with time-variable coefficients

Now, consider the system

$$X' = A(t)X + F(t, X) , \tag{22}$$

where F is as above, but $A(t)$ is this time continuous (i.e. not necessarily constant). The exponential dichotomy criteria for $A(t)$ can be found in [15], [16]. Let us note that the appropriate analogies for (22) to Propositions 1, 2 and 3 can be developed without any problems in the same way. On the other hand, the situation for almost periodicity becomes again more complicated, even in the linear case. The mentioned generalized Bohr-Neugebauer theorem (see [8, pp. 423–425]) has not namely the simple analogy (cf. e.g. with the well-known Favard theorem in [8], [17]). That is why we restrict ourselves, in this section, only to the study of bounded (periodic) solutions to (22).

We start with the following lemma (see [8, p. 286]) for the linear system

$$X' = A(t) + P(t) , \tag{23}$$

where $P(t)$ is continuous and satisfies (7).

Let $\Lambda(t)$ be the maximum of the (real) eigenvalues of the symmetrical matrix $\frac{1}{2}[A(t) + A^T(t)]$ and denote $\Lambda^* = \sup_{t \in \mathbb{R}} \Lambda(t)$.

LEMMA 6. *If $\Lambda^* < 0$ and (7) takes place, then there exists a unique entirely bounded solution $X(t)$ of (23) such that*

$$\sup_{t \in \mathbb{R}} \|X(t)\| \leqslant \frac{P}{-\Lambda^*} .$$

THEOREM 4. *Let $\Lambda^* < 0$ and assume the existence of a positive constant C such that*

$$\sup_{\substack{\|X\| \leqslant C \\ t \in \mathbb{R}}} \|F(t, X)\| \leqslant -\Lambda^* C .$$

Then system (22) *admits an entirely bounded solution. If, additionaly,* $F(t+T,X) \equiv F(t)$ *and* $A(t+T) \equiv A(t)$, *then* (22) *admits a T-periodic solution.*

PROOF — is trivial in view of Lemma 6. □

6. Concluding remarks

If $P(t)$ is a measurable T-periodic function, then for a weak bounded solution $X(t)$ of (6) we have (see (8))

$$X(t+T) \equiv \int_{-\infty}^{\infty} G(t+T-s)\, P(s)ds = \int_{-\infty}^{\infty} G(t-s)\, P(s+T)ds \equiv X(t)\ ,$$

where the integrals are considered in the sense of Lebesgue.

So, one can easily develop, on the basis of the appropriately generalized Proposition 1 for (4) with the Carathéodory functions $F(t,X)$ (see [5]) the analogical statements for weak bounded and periodic solutions.

In [2], we have significantly improved and extended the results from [4], [5] (related to Proposition 1) for Carathéodory systems of differential equations as well as differential inclusions.

At the present time, we have been also systematicaly studying weak almost periodic solutions.

REFERENCES

[1] J. Andres, *Asymptotic properties of solutions to quasi-linear differential systems*, J. Comput. Appl. Math. **41** (1992), 57–64.

[2] J. Andres, G. Gabor, and L. Górniewicz, *Boundary value problems on infinite intervals*, To appear in Trans. Amer. Math. Soc.

[3] B.F. Bylov, R.E. Vinograd, D. M. Grobman, and V.V. Nemytskii, *Theory of Liapunov Exponents*, Nauka, Moscow, 1966 Russian.

[4] M. Cecchi, M. Furi, and M. Marini, *On continuity and compactness of some nonlinear operators associated with differential equations in non-compact intervals*, Nonlin. Anal., T.M.A. **9,2** (1985), 171–180.

[5] M. Cecchi, M. Furi, and M. Marini, *About the solvability of ordinary differential equations with asymptotic boundary conditions*, Boll. U.M.I.(6) **4-C,1** (1985), 329–345.

[6] M. Cecchi, M. Furi, and P.L. Zezza, *Linear boundary value problems for systems of ordinary differential equations on non-compact intervals* I, Ann. Mat. Pura Appl. **4,123** (1980), 267–285.

[7] M. Cecchi, M. Furi, and P.L. Zezza, *Linear boundary value problems for systems of ordinary differential equations on non-compact intervals* II, Ann. Mat. Pura Appl. **4,124** (1980), 367–379.

[8] B.P. Demidowitch, *Lectures on the Mathematical Stability Theory*, Nauka, Moscow, 1967. (Russian.)

[9] A.G. Kartsatos, *Locally invertible operators and existence problems in differential systems*, Tôhoku Math. J. **28** (1976), 167–176.

[10] T.V. Kostova, *Estimates of the real parts of eigenvalues of complex matrices*, Dokl. Bolg. Akad. Nauk (C.R. Acad. Bulgare Sci.) **38,1** (1985), 15–18.

[11] M.A. Krasnosel'skii, *The Operator of Translation along the Trajectories of Differential Equations*, Nauka, Moscow, 1966 Russian.

[12] M.A. Krasnosel'skii, B.Sh. Burd and You.S. Kolesov, *Nonlinear Almost Periodic Oscillations*, Nauka, Moscow, 1970 Russian.

[13] M. Ráb, *Methods for Solutions of Differential Equations* II, Lecture Notes of the Masaryk (J.E. Purkynje), Univ. in Brno, SPN, Prague, 1989 Czech.

[14] R. Reissig, G. Sansone, and R. Conti, *Nichtlineare Differentialgleichungen Höherer Ordnung*, Cremonese, Roma, 1969.

[15] R.J. Sacker, and G.R. Sell, *Existence of dichotomies and invariant splittings for linear differential systems* I, J. Diff. Eqns **15,3** (1974), 429–458.

[16] R.J. Sacker, and G.R. Sell, *Existence of dichotomies and invariant splittings for linear differential systems* II, J. Diff. Eqns **22,2** (1976), 478–496.

[17] T. Yoshizawa, *Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions*, Springer, Berlin, 1975.

DEPT. OF MATH. ANAL., FAC. OF SCIENCE
PALACKÝ UNIVERSITY
TOMKOVA 40
779 00 OLOMOUC-HEJČÍN
CZECH REPUBLIC

DEPT. OF APPL. MATH., FAC. OF ELECTRICAL ENG. AND INFORMATICS
TECHNICAL UNIV. OF OSTRAVA
17. LISTOPADU
708 33 OSTRAVA-PORUBA
CZECH REPUBLIC