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ON A PAPER OF MAWHIN ON SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract. The first part of the paper deals with classification of solutions to the equations

$$u'' + \sigma g(t, u^{(i)}) = 0, \qquad i = 0, 1;\ \sigma^2 = 1,\ t \geqslant 0.$$

The second part is devoted to systems of the form

$$u''(t) = A(t)u^{(i)}(t) - g(t, u(h(t)), u'(h(t))), \qquad t \in [0, 1]$$

$$u^{(i)}(0) = u^{(i)}(1) = 0, \qquad i = 0, 1.$$

A wide literature has been devoted to the study of the differential equations

$$(1) \qquad u''(t) + g(t, u(t)) = f(t), \qquad t \in [0, 1],$$

or

$$(2) \qquad u''(t) + g(t, u'(t)) = f(t), \qquad t \in [0, 1],$$

with the Neumann: $u'(0) = u'(1) = 0$ or the periodic boudary value problem: $u(0) - u(1) = u'(0) - u'(1) = 0$.

In recent years interesting results on these problems have been reported in [1], [3], [7].

Using elementary methods we demonstrate here for (1) and (2) certain nonoscillatory and oscillatory results, connected with these problems.

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In the second part we demonstrate sufficient conditions for the existence of solutions to generalizations of (1) and (2). The proofs are based on topological degree theory. Certain results are related to those in paper [7].

I. Let us consider the second order differential equation

$$u''(t) + \sigma g(t, u'(t)) = 0, \qquad \sigma^2 = 1, \tag{3}$$

in which the function $g(t, u)$ is continuous for $(t, u) \in [0, \infty) \times \mathbb{R}^1$ with values in $\mathbb{R}^1$.

We consider solutions $u(t)$ of (3) which exists in $[0, \infty)$ and $u(t) \not\equiv$ const. in every interval $(\alpha, \beta) \subset [0, \infty)$.

We assume that

$$g(t, u)u > 0 \qquad \text{for} \quad u \neq 0. \tag{4}$$

Instead of (3) let us consider

$$v'(t) + \sigma g(t, v(t)) = 0, \qquad t \geqslant 0,\ v(t) = u'(t),\ \sigma^2 = 1. \tag{5}$$

Then the following theorem is true.

THEOREM 1. *Let $v(t)$ denote the solution of* (5). *Then this solution has the following properties:*

When $v(0) > 0$ (< 0), then $v(t) > 0$ (< 0) for $t > 0$. When $\sigma = -1$ and $v(0) > 0$ (< 0), then additionally $v(t)$ is decreasing (increasing) and $v(t) \to c \geqslant 0$ $(c \leqslant 0)$.

When $\sigma = 1$ and $v(0) > 0$ (< 0), then additionally $v(t)$ is increasing (decreasing) for $t \geqslant 0$ and tends to elimit $g \leqslant \infty$ $(\geqslant -\infty)$.

Of course $v = 0$ is a solution of (5) *satisfying $v(0) = v(1) = 0$.*

PROOF. For $t > 0$, $\sigma = -1$ and $v(0) > 0$ (< 0) we have $g(t, v(t)) > 0$ (< 0) and $v(t)$ is monotonic for $t \geqslant 0$.

Let us suppose that there exists $t_2 > 0$ and a neighbourhood U_{t_2} of t_2 such that for every $t_1,\ t_3 \in U_{t_2},\ t_1 < t_2 < t_3$ we have

$$v(t_1)v(t_3) < 0.$$

Then the function $v^2(t)$ is monotonic because of $v(t)$ and

$$v^2(t_3) - v^2(t_1) = (v(t_3) - v(t_1))(v(t_3) + v(t_1)).$$

The left hand side and the first factor of the right hand side of the last equality are of constant sign. The second factor changes its sign and we get a contradiction. Hence $v(t)$ is monotonic for $t \geqslant 0$.

When $\sigma = 1$ and $v(0) > 0$, then $v(t) > 0$ for $t \geqslant 0$, $v'(t) < 0$ and there exists a finite limit $c = \lim v(t) \geqslant 0$ when $t \to \infty$. When $\sigma = -1$ and $v(0) > 0$, then $v(t) > 0$ for $t > 0$, $v'(t) = g(t, v(t)) > 0$ and a limit $\lim v(t) = c \leqslant \infty, t \to \infty$ exists. The other situations can be treated similarly.

We shall now consider the equation

$$(3') \qquad u''(t) + \sigma g(t, u(t)) = 0, \qquad \sigma^2 = 1,\ t \geqslant 0, \quad \text{(see also [2])}.$$

Under the same assumptions as before: $g(t, u)$ is continuous for $(t, u) \in [0, \infty) \times \mathbf{R}^1$ with values in $\mathbf{R}^1$ and $g(t, u)u > 0$ for $u \neq 0$. Again we consider only solutions $u(t) \not\equiv \text{const}$ at every interval $(\alpha, \beta) \subset \mathbf{R}^1$.

THEOREM 2. *When* (4) *is satisfied and* $\sigma = -1$, *then for every solution* $u(t)$ *of* (3') *which exists in* $[0, \infty)$ *the set*

$$\{t:\ t \in [0, \infty),\ u(t)u'(t) = 0\}$$

has at most one point.

PROOF. Let us consider the function $\phi(t) = u(t)u'(t)$ where $u(t)$ satisfies (3'). Then $\phi'(t) = (u'(t))^2 + u(t)g(t, u(t)) \geqslant 0$.

THEOREM 3. *Let us suppose that for* $u(t) > 0$ (< 0) *for* $t \geqslant 0$ *and* $u(t)$ *nondecreasing (nonincreasing) in* $[0, \infty)$ *we have*

$$\int^{\infty} g(t, u(t))dt = +\infty\ (-\infty).$$

Then all solutions of (3') *with* $\sigma = 1$ *are oscillating in* $[0, \infty)$.

PROOF. Let us suppose that there exists a solution $u(t)$ of (3) with $\sigma = 1$ such that $u(t) > 0$ for $t > t_0 \geqslant 0$. Then

$$u'(t) = u'(t_1) - \int_{t_1}^{t} g(\tau, u(\tau))d\tau$$

and from (4) $u'(t)$ is decreasing for $t \geqslant t_1$. It is impossible that $u'(t) \geqslant 0$ for $t \geqslant t_1$, otherwise $u'(t) \to -\infty$ when $t \to \infty$. Hence there exists $t_2 > t_1$ such that $u'(t_2) < 0$ and

$$u'(t) = u'(t_2) - \int_{t_2}^{t} g(\tau, u(\tau))d\tau$$

is negative and decreasing, in contradiction with $u(t) > 0$ for $t \geqslant t_0$. Hence we get that $u(t)$ change its sign. This is true for every fixed $t_2 \geqslant t_1$ and $u(t)$ is oscillating.

Using elementary methods it is possible to derive theorems on the monotonicity of extrema of oscillating solutions to the equation

$$u''(t) + a(t)\phi(u(t))\phi'_u(u(t)) = 0, \qquad t \geqslant 0. \tag{6}$$

We now assume that $\phi \in C^1([(-\infty, +\infty), \mathbb{R}^1])$, $\phi(u)\phi'_u(u)u > 0$ for $u \neq 0$. $a(t) \in C^1([0,\infty), \mathbb{R}^1)$ and $a'(t)$ is of constant sign for $t \geqslant 0$. The following theorem is then true.

THEOREM 4. *Under the previous assumptions, for every oscillatory solution $u(t)$ of equation* (6) *the following properties are true:*

1. *When $a'(t) \geqslant 0$, then the sequence of consecutive maxima is nonincreasing and the sequence of consecutive minima nondecreasing.*
2. *When $a'(t) \leqslant 0$, then the sequence of consecutive maxima is nondecreasing and the sequence of consecutive minima is nonincreasing.*
3. *If additionally $\phi(-u)\phi'_u(-u) = \phi(u)\phi'_u(u)$, then the sequence of moduli of consecutive extrema is monotonic.*
4. *If $a(t) = const. > 0$, then the sequence of consecutive moduli of extrema is constant.*

PROOF. Let us consider the function

$$\phi(t) = (\phi(u(t)))^2 + (u'(t))^2(a(t))^{-1}, \tag{7}$$

where $u(t)$ denotes a solution of (6).

We have

$$\phi'(t) = -(u'(t))^2 a'(t)(a(t))^{-2}.$$

When $a'(t) \geqslant 0$, then $\phi'(t) \leqslant 0$ and $\phi(t)$ is nonincreasing. Denoting by $t_1, t_2, \ldots$ the abscisae of the consecutive moduli of extrema, we obtain

$$\phi(u(t_i)) > 0, \quad u'(t_i) = 0 \quad \text{and} \quad \phi(t_i) = (\phi(u(t_i)))^2.$$

But we have $\phi'(t) \geqslant 0$ and the assumption on ϕ show that the sequence $u(t_i)$ is nonincreasing. The proof may be finished in the same manner.

II. The next part of the paper will be devoted to certain boundary value problems for systems of equations with deviated argument

$$u''(t) = A(t)u^{(i)}(t) + g(t, u(h(t)), u'(k(t))), \qquad i = 0, 1 \tag{8$_1$}$$

$$(8_2) \qquad u^{(i)}(0) = u^{(i)}(1) = 0, \qquad i = 0, 1, \quad t \in [0,1]$$

We assume that A is a $n \times n$ matrix, $(\cdot,\cdot)$ the inner product in $\mathbb{R}^n$

$$(9) \qquad (Au, u) > -m(u, u), \qquad m < \pi^2 \quad \text{for} \quad u \neq 0,$$

$g(t, u, v)$ is locally Caratheodory in $[0,1] \times \mathbb{R}^{2n}$ with values in $\mathbb{R}^n$,

$$(10) \qquad |g(t, u, v)|_{\mathbb{R}^n} \leqslant M(|u|^{\alpha}_{\mathbb{R}^n} + |v|^{\beta}_{\mathbb{R}^n}) \qquad \text{for some} \quad 0 \leqslant \alpha, \beta < 1$$

and constant M. The deviations $h(t), k(t)$ of the argument t are continuous, $h, k : \; [0,1] \to [0,1], \;\; h(0) = 0.$

Under the assumptions given above we have:

THEOREM 5. *The problem* (8_1), (8_2) *with* $i = 0$ *has at least one solution* $u \in W^{2,1}([0,1], \mathbb{R}^n)$.

PROOF. It is well known that for $i = 0$ there exists a Green function $G(t, s)$ such that the solutions of

$$(11) \qquad u''(t) = Au(t), \quad u(0) = u(1) = 0, \quad t \in [0,1]$$

and

$$(12) \quad u''(t) = Au(t) + g(t, u(h(t)), u'(k(t))), \quad u(0) = u(1) = 0, \quad t \in [0,1]$$

can be written in the form

$$(13) \qquad u(t) = \int_0^1 G(t, s) Au(s) ds$$

or

$$(14) \qquad u(t) = \int_0^1 G(t, s)[Au(s) + g(s, u(h(s)), u'(k(s)))] ds,$$

where $G(t, s)$ satisfies the well known regularity and boundedness conditions.

Instead of solutions to (11) and (12) we shall consider zero vectors of completely continuous vector fields ϕ and ψ in the Banach space $W^{2,1}([0,1], \mathbb{R}^n)$

$$(15) \qquad (\phi u)(t) =: u(t) - \int_0^1 G(t, s) Au(s) ds,$$

and

$$(16)\qquad (\psi u)(t) =: u(t) - \int_0^1 G(t,s)[Au(s) + g(s, u(h(s)), u'(k(s))]ds.$$

Using the same method as in [4] it is clear that the vector fields ϕ, ψ are completely continuous in $W^{2,1}([0,1], \mathbb{R}^n)$. Applying (9) we have

$$(17)\qquad (\phi u)(t) \neq 0 \quad \text{for} \quad u \in W^{2,1}([0,1], \mathbb{R}^n) \quad \text{and} \quad u \neq 0.$$

This is sufficient to conclude that problem (13) has only the $u = 0$ solution. In fact we have

$$(18)\qquad \begin{aligned} \int_0^1 (Au(t), u(t))dt &= \int_0^1 (u''(t), u(t))dt = -\int_0^1 (u'(t), u'(t))dt \\ &\geqslant -m\int_0^1 (u(t), u(t))dt > -\pi^2 \int_0^1 (u(t), u(t))dt. \end{aligned}$$

But Wirtinger's inequality gives for $u \in W^{2,1}([0,1], \mathbb{R}^n)$, $u(0) = u(1) = 0$.

$$(19)\qquad \pi^2 \int_0^1 (u(t), u(t))dt \leqslant \int_0^1 (u'(t), u'(t))dt.$$

Conditions (18) and (19) lead to

$$(20)\qquad \pi^2 \int_0^1 (u(t), u(t))dt \leqslant \int_0^1 (u'(t), u'(t))dt < \pi^2 \int_0^1 (u(t), u(t)dt,$$

which is impossible.

Hence the problem, (11) in $W^{2,1}([0,1], \mathbb{R}^n)$ has only the $u = 0$ solution and we can now consider the rotation $\gamma(\phi, S_R)$ (see [6]) of the completely continuous vector field ϕ on the sphere

$$(21)\qquad S_R = \{u : \ u \in W^{2,1}([0,1], \mathbb{R}^n), \ \ |u|_{W^{2,1}} = R > 0\}.$$

It is

$$(22)\qquad \phi(-u) = -\phi(u)$$

and $\gamma(\phi, S_R) \neq 0$.

We now discuss the difference

$$(23) \qquad (\phi u - \psi u)(t) = \int_0^1 G(t,s) g(s, u(h(s)), u'(k(s))) ds.$$

Condition (10) together with

$$(24) \qquad \inf_{u \in S_R} |\phi u|_{W^{2,1}} = R \inf_{n \in S_1} |\phi u|_{W^{2,1}}$$

shows that

$$(25) \qquad |\phi u - \psi u|_{W^{2,1}} < |\phi u|_{W^{2,1}} \quad \text{on spheres} \quad S_R,$$

with R sufficiently large. Thus problem (12) has at least one solution in $W^{2,1}([0,1], \mathbb{R}^n)$.

REMARK. In the case of ordinary differential equations better results are known (see e.g. J. Mawhin [8]).

Unfortunately the above method cannot be applied to the problem

$$(26) \qquad \begin{aligned} u''(t) =& Au'(t) + g(t, u(h(t)), u'(k(t))), \quad t \in [0,1] \\ & u'(0) = u'(1) = 0, \quad u(0) = u_0 \in \mathbb{R}^n, \end{aligned}$$

The obstacle is that the problem

$$(27) \qquad u'' = 0, \quad u'(0) = u'(1) = 0, \quad u \in W^{2,1}([0,1], \mathbb{R}^n), \quad t \in [0,1]$$

is not invertible and hence it is impossible to transform (8_1), (8_2) with $i = 1$ into an integral equation.

J. Mawhin in his paper [7] proposed a method which enables demonstration of the existence of solution to equation

$$(28) \qquad \begin{aligned} u''(t) =& g(t, u'(t)) + f(t), \quad t \in [0,1] \\ u'(0) =& u'(1) = 0 \end{aligned}$$

but only for special forms of the function f. Such a situation is typical for system (8_1), (8_2) with $i = 1$ as can be shown from the following example.

The equation

$$u''(t) = \phi(t)u'(t) + k\phi(t), \quad u'(0) = u'(1) = 0, \quad t \in [0,1],$$
$$k > 0, \ \phi \text{ continuous are given}$$

has a suitable form. Substituting $u' = v$ we obtain

$$|k + v(t)| = k \exp\left(\int_0^t \phi(s)ds\right).$$

For $t = 0$ and $t = 1$, $k = k \exp\left(\int_0^1 \phi(s)ds\right)$.

When $k > 0$, $\int_0^1 \phi(s)ds \neq 0$, this equation has no solution. When k arbitrary and $\int_0^1 \phi(s)ds = 0$, then it has solutions.

Our aim now is a modification to (8_1), (8_2) with $i = 1$ of Mawhin's method.

Let us denote by X_0 the subspace of $W^{1,1}([0,1],\mathbb{R}^n)$ of the form

$$X_0 = \left\{v: \ v \in W^{1,1}([0,1],\mathbb{R}^n), \ v(0) = v(1) = 0\right\}.$$

We assume that A is a $n \times n$ matrix with integrable elements, $g(t,u,v)$ is a locally Caratheodory function in $[0,1] \times \mathbb{R}^{2n}$ and

$$|g(t,u,v)|_{\mathbb{R}^n} \leqslant M\left(|u|^{\alpha}_{\mathbb{R}^n} + |v|^{\beta}_{\mathbb{R}^n}\right), \quad 0 \leqslant \alpha, \beta < 1, \ M = \text{const}.$$

Suppose also that the boudary value problem

$$v'(t) = A(t)v(t) - \int_0^1 A(s)v(s)ds, \quad v(0) = v(1) = 0 \tag{29}$$

has in X_0 only the $v = 0$ solution.

The deviations $h, k : [0,1] \to [0,1]$ of the argument t are continuous and $h(0) = 0$.

REMARK. A sufficient condition for (29) is that

$$\sup_{0 \leqslant t \leqslant 1} |A(t)|_{C([0,1],\mathbb{R}^n)} < \frac{1}{2}.$$

In fact let $v \in C^{(1)}([0,1],\mathbb{R}^n)$, $v \neq 0$, $v(0) = v(1) = 0$ denote a solution of

$$v(t) = \int_0^t A(s)v(s)ds - \int_0^t \left(\int_0^1 A(t)v(t)dt\right)d\tau, \tag{30}$$

and $p =: \max\limits_{t\in[0,1]} |v(t)| > 0$.

Then

$$p \leqslant \max_{0\leqslant t\leqslant 1} \left| \int_0^t A(s)v(s)ds \right| + \max_{0\leqslant t\leqslant 1} t \left| \int_0^1 A(t)v(t)dt \right|$$

and

$$p < \frac{1}{2}p + \frac{1}{2}p$$

The last inequality is posible only if $p = 0$

Last time prof. Jean Mawhin showed me the following example. The integro–differential linear system with constant coefficients

$$\begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}' = \begin{pmatrix} 0, & -2\pi \\ 2\pi, & 0 \end{pmatrix} \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} - \int_0^1 \begin{pmatrix} 0, & -2\pi \\ 2\pi. & 0 \end{pmatrix} \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} dt,$$
$$v_1(0) = v_1(1) = 0$$
$$v_2(0) = v_2(1) = 0$$

has a family of nontrivial solutions

$$v_1(t) = C\sin 2\pi t, \quad v_2(t) = C(1 - \cos 2\pi t), \quad C \in \mathbb{R}^1$$

and has the form (30).

Hence (30) is a linear system of integro–differential equations

$$v'(t) = \mathcal{A}v(t), \quad v(0) = 0, \quad (\mathcal{A}v)(t) =: A(t)v(t) - \int_0^1 A(s)v(s)ds$$

with a non zero solution of the initial value problem $v(0) = 0$.

Instead of (8_1), (8_2), $i = 1$ let us consider

$$(31) \qquad \begin{cases} v'(t) = Av(t) + g(t, u_0 + \int\limits_0^{h(t)} v(s)ds, v(k(t))), & t \in [0,1] \\ v(0) = v(1) = 0. \end{cases}$$

Together with the system (31) we introduce two completely continuous vector fields defined in the space X_0

$$(32) \qquad (\phi v)(t) = v(t) - \int_0^t Av(s)ds + t\left(\int_0^1 Av(t)dt\right), \quad t \in [0,1]$$

and

$$(33)\qquad \begin{aligned}(\psi v)(t) = v(t) - \int_0^t \left[Av(s) + g\left(s, u_0 + \int_0^{h(s)} v(\tau)d\tau, v(k(s))\right)\right] ds + \\ + t\left[\int_0^1 \left(Av(z) + g\left(z, u_0 + \int_0^{h(z)} v(s)ds, v(k(z))\right)\right) dz\right].\end{aligned}$$

It is important that $\phi, \psi : X_0 \to X_0$ and the vector fields (32), (33) are completely continuous in X_0.

THEOREM 6. *Under the assumptions formulated above the vector fields* (32), (33) *in X_0 have at least one zero vector, therefore the problems*

$$(34)\qquad v'(t) = Av(t) - \int_0^1 Av(t)dt, \quad v(0) = v(1) = 0, \quad t \in [0,1]$$

and

(35)

$$\begin{aligned}v'(t) = Av(t) + g\left(t, u_0 + \int_0^{h(t)} v(s)ds, v(k(t))\right) - \int_0^1 Av(s)ds \\ - \int_0^1 g\left(s, u_0 + \int_0^{k(s)} v(z)dz, v'(k(s))\right) ds, \quad v(0) = v(1) = 0, \ t \in [0,1],\end{aligned}$$

have at least one solution in X_0.

PROOF. Firstly we know from (29) that:

$$(36)\qquad (\phi v = 0) \Leftrightarrow (v = 0) \quad \text{for} \quad v \in X_0.$$

The solution $v(t) \equiv 0$ for $t \in [0,1]$ is the unique solution to the problem

$$(37)\qquad v(t) = \int_0^t A(s)v(s)ds - t\int_0^1 A(s)v(s)ds, \quad v(0) = v(1) = 0.$$

This last property together with

$$\phi(-v) = -\phi(v)$$

gives

$$\gamma(\phi, S_R) \neq 0, \quad R > 0, \tag{38}$$

and also

$$\inf_{v \in S_R \subset X_0} |\phi(v)|_{X_0} = R \inf_{v \in S_1 \subset X_0} |\phi(v)|_{X_0}, \quad R > 0. \tag{39}$$

Similarly as in theorem 5, for the difference $(\phi v - \psi v)(t)$ we have

$$\begin{aligned} (\phi v - \psi v)(t) = & \int_0^t g\left(s, u_0 + \int_0^{h(s)} v(z)dz, v(k(s))\right) ds \\ & - t \int_0^1 g\left(t, u_0 + \int_0^{h(t)} v(s)ds, v'(k(t))\right) dt. \end{aligned} \tag{40}$$

From the sublinearity of the function g and (39) there exists such an $R_0 > 0$ that for $v \in S_{R_0} \subset X_0$

$$|\phi v - \psi v|_{X_0} < |\phi v|_{X_0}. \tag{41}$$

This last inequality shows the homotopy of the vector fields ϕ and ψ on $S_{R_0} \subset X_0$. Therefore

$$\gamma(\psi, S_{R_0}) \neq 0. \tag{42}$$

Thus the vector field ψ has at least one zero vector and this is equvalent to the existence of at least one solution to the problem (33).

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