Annales Mathematicae Silesianae 11. Katowice 1997, 67 – 87

Prace Naukowe Uniwersytetu Śląskiego nr 1665

GENERALIZED PERIODIC SOLUTIONS OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS IN THE COLOMBEAU ALGEBRA

JAN LIGĘZA

Abstract. It is shown that from the fact that the unique periodic solution of homogeneous system of equations is the trivial one it follows the existence of periodic solutions of nonhomogeneous systems of equations in the Colombeau algebra.

1. Introduction

We consider the following problem

$$x'_k(t) = \sum_{j=1}^{n} A_{kj}(t)x_j(t) + f_k(t), \tag{1.0}$$

$$x_k(0) = x_k(\omega), \quad \omega > 0, \quad k = 1, \dots, n, \tag{1.1}$$

where A_{kj}, f_k and x_k are elements of the Colombeau algebra $\mathcal{G}(\mathbb{R})$; $x_k(0)$ and $x_k(\omega)$ are understood as the value of generalized functions x_k at the points 0 and ω respectively and $k = 1, \dots, n$ (see [2]). Generalized functions A_{kj} and f_k are given, elements x_k are unknown (for $k, j = 1, \dots, n$). The multiplication, the derivative, the sum and the equality is meant in the Colombeau algebra sense. We prove theorems on the existence and uniqueness of solutions of problem (1.0)–(1.1). Our theorems generalize some results given in [8], [9], [11], [12].

1991 *Mathematics Subject Classification.* AMS classification: 34A10, 46F99.

Key words and phrases: generalized ordinary differential equations, periodic solutions, Colombeau algebra

2. Notation

Let $\mathcal{D}(\mathbb{R})$ be the set of all C^∞ functions $\mathbb{R} \to \mathbb{R}$ with compact support. For $q = 1, 2, \ldots$ we denote by $\mathcal{A}_q$ the set of all functions $\varphi \in \mathcal{D}(\mathbb{R})$ such that the relations

$$(2.1) \qquad \int_{-\infty}^{\infty} \varphi(t)dt = 1, \quad \int_{-\infty}^{\infty} t^k \varphi(t)dt = 0, \qquad 1 \leqslant k \leqslant q$$

hold.

Next, $\mathcal{E}[\mathbb{R}]$ is the set of all functions $R : \mathcal{A}_1 \times \mathbb{R} \to \mathbb{R}$ such that $R(\varphi, t) \in C^\infty(\mathbb{R})$ for every fixed $\varphi \in \mathcal{A}_1$.

If $R \in \mathcal{E}[\mathbb{R}]$, then $D_k R(\varphi, t)$ for any fixed φ denotes a differential operator in t (i.e. $D_k R(\varphi, t) = \frac{d^k}{dt^k}(R(\varphi, t))$ for $k \geqslant 1$ and $D_0 R(\varphi, t) = R(\varphi, t)$).

For given $\varphi \in \mathcal{D}(\mathbb{R})$ and $\varepsilon > 0$, we define φ_ε, by

$$(2.2) \qquad \varphi_\varepsilon(t) = \frac{1}{\varepsilon}\varphi\left(\frac{t}{\varepsilon}\right).$$

An element R of $\mathcal{E}[\mathbb{R}]$ is moderate if for every compact set K of $\mathbb{R}$ and every differential operator D_k there is $N \in \mathbf{N}$ such that the following condition holds; for every $\varphi \in \mathcal{A}_N$ there are $c > 0$, $\varepsilon_0 > 0$ such that

$$(2.3) \qquad \sup_{t \in K} |D_k R(\varphi_\varepsilon, t)| \leqslant c\varepsilon^{-N} \qquad \text{if} \quad 0 < \varepsilon < \varepsilon_0.$$

We denote by $\mathcal{E}_M[\mathbb{R}]$ the set of all moderate elements of $\mathcal{E}[\mathbb{R}]$.

By Γ we denote the set of all the increasing functions α from $\mathbf{N}$ into $\mathbb{R}^+$ such that $\alpha(q)$ tends to ∞ if $q \to \infty$.

We define an ideal $\mathcal{N}[\mathbb{R}]$ in $\mathcal{E}_M[\mathbb{R}]$ as follows; $R \in \mathcal{N}[\mathbb{R}]$ if for every compact set K of $\mathbb{R}$ and every differential operator D_k there are $N \in \mathbf{N}$ and $\alpha \in \Gamma$ such that the following condition holds: for every $q \geqslant N$ and $\varphi \in \mathcal{A}_q$ there are $c > 0$ and $\varepsilon_0 > 0$ such that

$$(2.4) \qquad \sup_{t \in K} |D_k R(\varphi_\varepsilon, t)| \leqslant c\varepsilon^{\alpha(q) - N} \qquad \text{if} \quad 0 < \varepsilon < \varepsilon_0.$$

The algebra $\mathcal{G}(\mathbb{R})$ (the Colombeau algebra) is defined as a quotient algebra of $\mathcal{E}_M[\mathbb{R}]$ with respect to $\mathcal{N}[\mathbb{R}]$ (see [2]).

We denote by $\mathcal{E}_0$ the set of all functions from $\mathcal{A}_1$ into $\mathbb{R}$. Next, we denote by $\mathcal{E}_M$ the set of all the so – called moderate elements of $\mathcal{E}_0$ defined by

$$(2.5) \quad \mathcal{E}_M = \Big\{ R \in \mathcal{E}_0 \text{: there is } N \in \mathbf{N} \text{ such that for every } \varphi \in \mathcal{A}_N \text{ there are } c > 0,\ \eta_0 > 0 \text{ such that } |R(\varphi_\varepsilon)| \leqslant c\varepsilon^{-N} \text{ if } 0 < \varepsilon < \eta_0 \Big\}.$$

Further, we define an ideal $\mathcal{N}$ of $\mathcal{E}_M$ by

(2.6) $\mathcal{N} = \Big\{ R \in \mathcal{E}_0$: there are $N \in \mathbf{N}$ and $\alpha \in \Gamma$ such that for every $q \geqslant N$ and $\varphi \in \mathcal{A}_q$ there are $c > 0$, $\eta_0 > 0$ such that $|R(\varphi_\varepsilon)| \leqslant c\varepsilon^{\alpha(q)-N}$ if $0 < \varepsilon < \eta_0 \Big\}$.

We define an algebra $\overline{\mathbb{R}}$ by setting

$$\overline{\mathbb{R}} = \frac{\mathcal{E}_M}{\mathcal{N}} \qquad \text{(see [2])}.$$

It is known that $\overline{\mathbb{R}}$ is not a field.

If $R \in \mathcal{E}_M[\mathbb{R}]$ is a representative of $G \in \mathcal{G}(\mathbb{R})$, then for a fixed t the map $Y : \varphi \to R(\varphi, t) \in \mathbb{R}$ is defined on $\mathcal{A}_1$ and $Y \in \mathcal{E}_M$. The class of Y in $\overline{\mathbb{R}}$ depends only on G and t. This class is denoted by $G(t)$ and is called the value of the generalized function G at the point t (see [2]).

We say that $G \in \mathcal{G}(\mathbb{R})$is a constant generalized function on $\mathbb{R}$ if it admits a representative $R(\varphi, t)$ which is independent on $t \in \mathbb{R}$. With any $Z \in \overline{\mathbb{R}}$ we associate a constant generalized function which admits $R(\varphi, t) = Z(\varphi)$ as it representative, provided we denote by Z a representative of Z (see [2]).

Troughout the paper K denotes a compact set in $\mathbb{R}$. We denote by $R_{A_{kj}}(\varphi, t)$, $R_{f_k}(\varphi, t)$, $R_{x_{0j}}(\varphi)$, $R_{x_{j(t_0)}}(\varphi)$, $R_{x_j}(\varphi, t)$ and $R_{x'_j}(\varphi, t)$ representatives of elements $A_{kj}, f_k, x_{0j}, x_j(t_0)$, x_j and x'_j for $k, j = 1, \dots, n$. Let $A(t) = (A_{kj}(t))$, $f(t) = (f_1(t), \dots, f_n(t))^T$, $x(t) = (x_1(t), \dots, x_n(t))^T$, $x'(t) = (x'_1(t), \dots, x'_n(t))^T$, $x_0 = (x_{10}, \dots, x_{n0})^T$, where T denotes the transpose. We put

$$R_A(\varphi, t) = (R_{A_{kj}}(\varphi, t)), \quad R_f(\varphi, t) = (R_{f_1}(\varphi, t), \dots, R_{f_n}(\varphi, t))^T,$$

$$R_x(\varphi, t) = (R_{x_1}(\varphi, t), \dots, R_{x_n}(\varphi, t))^T, \quad R_{x'}(\varphi, t) = (R_{x'_1}(\varphi, t), \dots, R_{x'_n}(\varphi, t))^T,$$

$$R_{x_0}(\varphi) = (R_{x_{10}}(\varphi), \dots, R_{x_{n0}}(\varphi))^T,$$

$$R_{x(t_0)}(\varphi) = (R_{x_{1(t_0)}}(\varphi), \dots, R_{x_{n(t_0)}}(\varphi))^T,$$

$$\int_{t_0}^{t} R_A(\varphi, s)ds = \left(\int_{t_0}^{t} R_{A_{kj}}(\varphi, s)ds \right),$$

$$\int_{t_0}^{t} R_f(\varphi, s)ds = \left(\int_{t_0}^{t} R_{f_1}(\varphi, s)ds, \dots, \int_{t_0}^{t} R_{f_n}(\varphi, s)ds \right)^T,$$

$$\|R_A(\varphi,t)\| = \sqrt{\sum_{k,j=1}^{n} |R_{A_{kj}}(\varphi,t)|^2}, \quad \|R_f(\varphi,t)\| = \sqrt{\sum_{j=1}^{n} |R_{f_j}(\varphi,t)|^2},$$

$$\|R_A(\varphi,t)\|_K = \sup_{t\in K} \|R_A(\varphi,t)\|, \quad \|R_f(\varphi,t)\|_K = \sup_{t\in K} \|R_f(\varphi,t)\|.$$

If $A_{kj}, f_j \in \mathcal{G}(\mathbb{R})$, $u = (u_1,\dots,u_n) \in \mathbb{R}^n$, $v = (v_1,\dots,v_n) \in \mathbb{R}^n$, $b_j \in \mathcal{N}[\mathbb{R}]$, $m_j \in \mathcal{N}$, $p_j \in \overline{\mathbb{R}}$ for $k,j = 1,\dots,n$, then we write $R_A(\varphi,t) \in \mathcal{E}_M^{n\times n}[\mathbb{R}]$, $R_f(\varphi,t) \in \mathcal{E}_M^n[\mathbb{R}]$, $A = (A_{kj}) \in \mathcal{G}^{n\times n}(\mathbb{R})$, $f = (f_1,\dots,f_n)^T \in \mathcal{G}^n(\mathbb{R})$, $b = (b_1,\dots,b_n)^T \in \mathcal{N}^n[\mathbb{R}]$, $m = (m_1,\dots,m_n) \in \mathcal{N}^n$, $p = (p_1,\dots,p_n) \in \overline{\mathbb{R}}^n$ and $(u,v) = \sum_{i=1}^{n} u_i v_i$.

We say that $x = (x_1,\dots,x_n)^T \in \mathcal{G}^n(\mathbb{R})$ is a solution of system (1.0) if x satisfies system (1.0); i.e. if $R_x(\varphi,t)$ is a representative of x, then there is $\eta \in \mathcal{N}^n[\mathbb{R}]$ such that (for every $\varphi \in \mathcal{A}_1$ and $t \in \mathbb{R}$)

$$R_{x'}(\varphi,t) = R_A(\varphi,t)R_x(\varphi,t) + \eta(\varphi,t).$$

If $x \in \mathcal{G}^n(\mathbb{R})$ admits an ω–periodic representative $R_x(\varphi,t)$ ($\omega > 0$) and x is a solution of system (1.0), then we say that x is the ω–periodic solution of system (1.0).

The class of all ω–periodic generalized functions will be denoted by $\mathcal{G}_\omega(\mathbb{R})$.

An ω–periodic function $p \in C^\infty(\mathbb{R})$ is called hereditarity ω–periodic if there is a ω–periodic function q such that $q' = p$ (see [19]). One can show (see [19]) that for every hereditarily ω–periodic function p there exists a unique hereditarily ω–periodic function q such that $q' = p$ (see [19]). Now we shall give the definition of a smooth integral of a function $p \in C^\infty(\mathbb{R})$ (see [1], [19]). Let β be a positive number and let the support of a nonnegative function $\varphi \in \mathcal{D}(\mathbb{R})$ equals $[\beta, 2\beta]$. Moreover, let $\int_{-\infty}^{\infty} \varphi(t)dt = \frac{1}{\omega}$ ($\omega > 0$) and let $\prod$ be the characteristic function of the interval $[0,\omega]$. We define

$$\int_0^{\omega+3\beta} \lambda(r-c)dr \int_r^t p(s)ds = \int_{C_\lambda}^t p(s)ds, \tag{2.7}$$

where

$$\lambda(t) = \int_{-\infty}^{\infty} \prod(t-s)\varphi(s)ds.$$

If c and λ are fixed, then integral (2.7) is a primitive function of p. A primitive function which is of the form (2.7) will be called a smooth integral of p. The smooth integral of order n we define by induction, letting

$$(2.8)\qquad \int_{C_\lambda}^{t} f(s)ds^0 = f, \quad \int_{C_\lambda}^{t} f(s)ds^n = \int_{C_\lambda}^{t}\left(\int_{C_\lambda}^{s} f(r)dr^{n-1}\right)ds.$$

One may prove (see [19]) that for every hereditarily ω–periodic function p, $\int_{C_\lambda}^{t} p(s)ds$ is also a hereditarily ω–periodic function which does not depend on the choice of c and λ.

3. The main results

First we shall introduce some hypotheses.

Hypothesis H_1

$$(3.0)\qquad A \in \mathcal{G}^{n\times n}(\mathbb{R}), \quad f \in \mathcal{G}^n(\mathbb{R});$$

the matrix $A \in \mathcal{G}^{n\times n}(\mathbb{R})$ admits an ω–periodic representative $R_A(\varphi, t)$ such that

$$(3.1)\qquad R_A(\varphi,t) = (R_A(\varphi,t))^T \quad \text{for every} \quad \varphi \in \mathcal{A}_1,$$

(3.2) the matrix $A \in \mathcal{G}^{n\times n}(\mathbb{R})$ admits ω–periodic representative $R_A(\varphi, t)$,

(3.3) $f \in \mathcal{G}^n(\mathbb{R})$ admits an ω–periodic representative $R_f(\varphi, t)$,
(3.4) the matrix $A \in \mathcal{G}^{n\times n}(\mathbb{R})$ admits a representative $R_A(\varphi, t)$ with the following properties:

a) $R_A(\varphi, t)$ is ω–periodic for every $\varphi \in \mathcal{A}_1$,

b) for every K there is $N \in \mathbf{N}$ such that for every $\varphi \in \mathcal{A}_N$ there are constants $c > 0$ and $\varepsilon_0 > 0$ such that

$$\left\|\int_0^t \|R_A(\varphi_\varepsilon, s)\|ds\right\|_K \leqslant c \qquad \text{for} \quad 0 < \varepsilon < \varepsilon_0,$$

the matrix $A \in \mathcal{G}^{n\times n}(\mathbb{R})$ $(n \geqslant 2)$ admits an ω–periodic representative $R_A(\varphi, t) = (R_{A_{kj}}(\varphi, t))$ with the following property: there is $N \in \mathbf{N}$

such that for every $\varphi \in \mathcal{A}_N$ there are constants $\varepsilon_0 > 0$ and $\gamma_0 > 0$ such that

(3.5) $$R_{A_{jj}}(\varphi_\varepsilon, t) \geqslant \gamma_0 \quad \text{if} \quad 0 < \varepsilon < \varepsilon_0,\ j = 1, \dots, n;\ t \in [0, \omega]$$

and

(3.6) $$R_{A_{kj}}(\varphi_\varepsilon, t) = -R_{A_{jk}}(\varphi_\varepsilon, t) \quad \text{for} \quad j \neq k;\ j, k = 2, \dots, n;$$

the matrix $A \in \mathcal{G}^{n\times n}(\mathbb{R})$ admits an ω–periodic representative $R_A(\varphi, t)$ with the following property: there is $N \in \mathbf{N}$ such that for every $\varphi \in \mathcal{A}_N$ there are constants $\varepsilon_0 > 0$ and $\gamma_0 > 0$ such that

(3.7) $$(u^T,\ R_A(\varphi_\varepsilon, t)u) \geqslant \gamma_0(u, u) \quad \text{if} \quad 0 < \varepsilon < \varepsilon_0,$$

$t \in [0, \omega]$ and $u \in \mathbb{R}^n$.

Hypothesis H_2

(3.8) $$p_i, r \in \mathcal{G}_\omega(\mathbb{R}) \quad \text{for} \quad i = 1, \dots, n;$$

$p_i \in \mathcal{G}(\mathbb{R})$ and p_i admits a representative $R_{p_i}(\varphi, t)$ with the following properties: for every K there is $N \in \mathbf{N}$ such that for every $\varphi \in \mathcal{A}_N$ there are constants $c > 0$ and $\varepsilon_0 > 0$ such that

(3.9) $$\|\int_0^t |R_{p_i}(\varphi_\varepsilon, s)| ds\|_K \leqslant c \quad \text{if} \quad 0 < \varepsilon < \varepsilon_0 \quad \text{and} \quad i = 1, \dots, n;$$

the element $p_2 \in \mathcal{G}_\omega(\mathbb{R})$ admits an ω–periodic representative $R_{p_2}(\varphi, t)$ with the following property: there is $N \in \mathbf{N}$ such that for every $\varphi \in \mathcal{A}_N$ there are constants $\varepsilon_0 > 0$ and $\gamma_0 < 0$ such that

(3.10) $$R_{p_2}(\varphi_\varepsilon, t) \leqslant \gamma_0 \quad \text{if} \quad 0 < \varepsilon < \varepsilon_0 \quad \text{and} \quad t \in [0, \omega],$$

the element $p_2 \in \mathcal{G}_\omega(\mathbb{R})$ admits an ω–periodic representative $R_{p_2}(\varphi, t)$ with the following property: there is $N \in \mathbf{N}$ such that for every $\varphi \in \mathcal{A}_N$ there are constants $\varepsilon_0 > 0$, $\gamma_0 > 0$ and $\gamma_1 > 0$ such that:

(3.11) $$|R_{p_2}(\varphi_\varepsilon, t)| \geqslant \gamma_0 \quad \text{if} \quad 0 < \varepsilon < \varepsilon_0 \quad \text{and} \quad t \in [0, \omega]$$

and

$$(3.12)\qquad \int_0^{\omega} |R_{p_2}(\varphi_\varepsilon, t)dt \leqslant \frac{16}{\omega} - \gamma_1 \qquad \text{if} \quad 0 < \varepsilon < \varepsilon_0;$$

$p_2 \in L^1_{loc}(\mathbf{R})$ and p_2 is an ω–periodic function such that:

$$(3.13)\qquad \int_0^{\omega} p_2(t)dt \geqslant 0, \quad \omega \int_0^{\omega} |p_2(t)|dt \leqslant 16, \qquad p(t) \not\equiv 0;$$

elements $p_i \in \mathcal{G}_\omega(\mathbf{R})$ $(i = 1, \dots, n;\ n \geqslant 2)$ admit ω–periodic representatives $R_{p_i}(\varphi, t)$ with the following property: there are $N \in \mathbf{N}$ and $\beta > 0$ such that for every $\varphi \in \mathcal{A}_N$ there are constants $\varepsilon_0 > 0,\ \gamma_0 > 0$ and $\gamma_1 > 0$ such that

$$(3.14)\quad \sum_{i=0}^{n-1} R_{p_i^{(n-1)}}(\varphi_\varepsilon, t) \geqslant \gamma_0, \quad \max_{1 \leqslant i \leqslant n} \int_0^{\omega+3\beta} |R_{p_i}(\varphi_\varepsilon, t)|dt \leqslant a(\omega, \beta, \varepsilon) - \gamma_1,$$

if $0 < \varepsilon < \varepsilon_0,\ \ t \in [0, \omega]$ and

$$a(\omega, \beta, \varepsilon) = \left(\sum_{i=0}^{n-1} (\omega + 3\beta)^{n-i-1} \right)^{-1};$$

the element $p_n \in \mathcal{G}(\mathbf{R})$ admits an ω–periodic representative $R_{p_n}(\varphi, t)$ with the following property; there are $N \in \mathbf{N}$ and $\beta > 0$ such that for every $\varphi \in \mathcal{A}_N$ there are constants $\varepsilon_0 > 0,\ \gamma_0 > 0$ and $\gamma_1 > 0$ such that

$$(3.15)\qquad R_{p_n}(\varphi_\varepsilon, t) \geqslant \gamma_0, \quad \int_0^{\omega} R_{p_n}(\varphi_\varepsilon, t)dt \leqslant (\omega + 3\beta)^{-n+1} - \gamma_1,$$
$$\text{if} \quad 0 < \varepsilon < \varepsilon_0 \quad \text{and} \quad t \in [0, \omega].$$

Now we shall give theorems on the existence and uniqueness of the solution of problem (1.0)–(1.1). Apart from problem (1.0)–(1.1) we shall examine the homogeneous problem

$$(3.16)\qquad x'(t) = A(t)x(t),$$

$$(3.17)\qquad x(0) = x(\omega).$$

THEOREM 3.1. *We assume conditions* (3.0), $(3.4)_{(b)}$. *Moreover, we assume that the trivial solution is the unique solution of problem* (3.16)–(3.17) *in* $\mathcal{G}^n(\mathbb{R})$. *Then problem* (1.0)–(1.1) *has exactly one solution* $x \in \mathcal{G}^n(\mathbb{R})$.

THEOREM 3.2. *We assume conditions* (3.0), (3.3), $(3.4)_{(a)-(b)}$. *Moreover we assume that the trivial solution is the unique* ω*-periodic solution of system* (3.16) *in* $\mathcal{G}^n(\mathbb{R})$. *Then there exists exactly one* ω*-periodic solution of system* (1.0) *in* $\mathcal{G}^n(\mathbb{R})$.

REMARK 3.1. If A and f have properties (3.0), $(3.4)_{(b)}$, then the problem

$$x'(t) = A(t)x(t) + f(t), \tag{3.18}$$

$$x(t_0) = x_0, \quad t_0 \in \mathbb{R}, \quad x_0 \in \overline{\mathbb{R}}^n \tag{3.19}$$

has exactly one solution $x \in \mathcal{G}^n(\mathbb{R})$ (see [11]). Besides, every solution x of system (3.18) has a representation

$$x(t) = Z(t)c + Q(t), \tag{3.20}$$

where Z is a solution of the problem

$$Z'(t) = A(t)Z(t), \tag{3.21}$$

$$Z(t_0) = I, \quad t_0 \in \mathbb{R}, \tag{3.22}$$

$c = (c_1, \dots, c_n)^T$, c_j are generalized constant functions on $\mathbb{R}$ for $j = 1, \dots, n$, I denotes the identity matrix and Q is a particular solution of system (3.18). The solution x is the class of solutions of the problems

$$x'(t) = R_A(\varphi, t)x(t) + R_f(\varphi, t), \tag{3.23}$$

$$x(t_0) = R_{x_0}(\varphi), \quad \varphi \in \mathcal{A}_1 \quad \text{(see [11])}. \tag{3.24}$$

EXAMPLE 3.1. Let δ denotes the generalized function (delta Dirac's generalized function), which admits as the representative the functions $R_\delta(\varphi, t) = \varphi(-t)$, where $\varphi \in \mathcal{A}_1$. Then δ has property $(3.4)_{(b)}$. It is not difficult to show that the problem

$$x'(t) = (\delta^2(t))'x(t), \tag{3.25}$$

$$x(-1) = 1 \tag{3.26}$$

has not any solutions in $\mathcal{G}(\mathbb{R})$ (see [11]).

REMARK 3.2. If $p \in L^1_{loc}(\mathbb{R})$, then we put

$$R_p(\varphi_\varepsilon, t) = \int_{-\infty}^{\infty} p(t + \varepsilon u)\varphi(u)du = (p * \varphi_\varepsilon)(t), \tag{3.27}$$

where $\varphi \in \mathcal{A}_1$ (see [2]). Hence

$$p * \varphi_\varepsilon \to p \quad \text{in} \quad L^1_{loc}(\mathbb{R}) \qquad \text{(see [1])} \tag{3.28}$$

and R_p has property $(3.4)_{(b)}$.

REMARK 3.3. It is known that the algebra $\varepsilon_f(\mathbb{R})$ of all piecewise continuous functions on $\mathbb{R}$ is not a subalgebra of $\mathcal{G}(\mathbb{R})$ (see [2]). If $g_1, g_2 \in C^\infty(\mathbb{R})$, then the classical product and the product in $\mathcal{G}(\mathbb{R})$ give rise to the same element of $\mathcal{G}(\mathbb{R})$. If necessary we denote the product in $\mathcal{G}$ by $\odot$ to avoid confusion with the classical product.

Taking into account the continuous dependence of x on coefficients A_{kj} and f_j, $(k, j = 1, \ldots, n)$ we have

THEOREM 3.3. *We assume that*

$$A_{kj}, f_j \in L^1_{loc}(\mathbb{R}) \qquad \textit{for} \quad k, j = 1, \ldots, n; \tag{3.29}$$

$$A_{kj}, f_j \;\; \textit{are} \;\; \omega \;\; \textit{-periodic functions} \;\; (k, j = 1, \ldots, n); \tag{3.30}$$

(3.31) *the trivial solution is the unique ω-periodic solution of system* (3.16) *in the Caratheodory sense.*

Then $x = (0, \ldots, 0)^T$ is the unique ω-periodic solution of the system

$$x'(t) = A(t) \odot x(t) \tag{3.32}$$

in $\mathcal{G}^n(\mathbb{R})$.

THEOREM 3.4. *We assume conditions* $(3.4)_{(a)-(b)}$, (3.7). *Then $x = (0, \ldots, 0)^T$ is the unique ω-periodic solution of system* (3.16) *in $\mathcal{G}^n(\mathbb{R})$.*

COROLLARY 3.1. *If the matrix A has properties* $(3.4)_{(a)-(b)}$, (3.5), (3.6), *then $x = (0,\dots,0)^T$ is the unique ω-periodic solution of system* (3.16) *in* $\mathcal{G}^n(\mathbb{R})$.

REMARK 3.4. It is known that if $\varphi \in \mathcal{A}_1$ and if $d(\varphi)$ denotes the diameter of the support of φ (i.e. $d(\varphi) = \sup\limits_{x,y\in \operatorname{supp}\varphi} |x-y|$), then

$$R_1(\varphi,t) = \left(\exp\frac{1}{d(\varphi)}\right)^{-1} \in \mathcal{N}[\mathbb{R}] \qquad \text{(see [2])}.$$

Let

$$R_{p_1}(\varphi,t) = \begin{cases} d(\varphi), & \text{if } \varphi \in \mathcal{A}_{2k-1}\setminus\mathcal{A}_{2k},\ k=1,2,\dots \\ R_1(\varphi,t), & \text{otherwise} \end{cases}$$

and let

$$R_x(\varphi,t) = \begin{cases} R_1(\varphi,t), & \text{if } \varphi \in \mathcal{A}_{2k-1}\setminus\mathcal{A}_{2k},\ k=1,2,\dots \\ d(\varphi), & \text{otherwise.} \end{cases}$$

Then

$$R_x(\varphi,t)R_{p_1}(\varphi,t) \in \mathcal{N}[\mathbb{R}],\quad R_x(\varphi,t) \notin \mathcal{N}[\mathbb{R}],$$

$D_1R_x(\varphi,t) = 0$, $R_{p_1}(\varphi_\varepsilon,t) > 0$ and $R_{p_1}(\varphi_\varepsilon,t) \to 0$ as $\varepsilon \to 0$ (for fixed φ). Thus $x = [R_x(\varphi,t)] \neq 0$ is an ω-periodic solution in $\mathcal{G}(\mathbb{R})$ of the equation

$$x'(t) = p_1(t)x(t),$$

where $p_1 = [R_{p_1}(\varphi,t)]$.

REMARK 3.5. If conditions (3.1); (3.7) are satisfied, then the quadratic form $(u^T, R_A(\varphi_\varepsilon,t)u)$ is positive definite.

Now we shall consider the equations

$$x^{(n)}(t) + p_1(t)x^{(n-1)}(t) + \dots + p_n(t)x(t) = 0, \tag{3.33}$$

$$x^{(n)}(t) + p_1(t)x^{(n-1)}(t) + \dots + p_n(t)x(t) = r(t). \tag{3.34}$$

THEOREM 3.5. *We assume conditions* (3.9); (3.14). *Then $x = 0$ is the unique ω-periodic solution of equation* (3.33) *in* $\mathcal{G}(\mathbb{R})$.

Corollary 3.2. *If conditions* (3.8), (3.9), (3.14) *are satisfied, then* $x = 0$ *is the unique* ω*-periodic solution of equation* (3.34) *in* $\mathcal{G}(\mathbb{R})$.

Theorem 3.6. *We assume conditions* (3.9); (3.15). *Then* $x = 0$ *is the unique* ω*-periodic solution of the equation*

$$x^{(n)}(t) + p_n(t)x(t) = 0 \tag{3.35}$$

in $\mathcal{G}(\mathbb{R})$.

Corollary 3.3. *If conditions* (3.8), (3.9) *and* (3.15) *are satisfied, then* $x = 0$ *is the unique* ω*-periodic solution of equation* (3.35) *in* $\mathcal{G}(\mathbb{R})$.

Remark 3.6. If conditions (3.8), (3.10) are satisfied, then $x = 0$ is the unique ω–periodic solution of the equation

$$x''(t) + p_2(t)x(t) = 0 \tag{3.36}$$

in $\mathcal{G}(\mathbb{R})$ (see [12]).

Remark 3.7. If conditions (3.8)–(3.9), (3.11)–(3.12) are satisfied, then $x = 0$ is the unique ω–periodic solution of equation (3.36) in $\mathcal{G}(\mathbb{R})$ (see [12]).

Remark 3.8. If conditions (3.13) are satisfied, then equation (3.36) has only the trivial ω–periodic solution in the Caratheodory sense (see [8]).

4. Proofs

Proof of Theorem 3.1. To this purpose we consider the following systems of equations

$$Hc = b, \tag{4.1}$$

and

$$Hc = 0, \tag{4.2}$$

where

$$H = Z(0) - Z(\omega), \qquad c = (c_1, \dots, c_n)^T, \tag{4.3}$$

$$b = Q(\omega) - Q(0), \tag{4.4}$$

Z is solution of problem (3.21)–(3.22) and Q is defined by (3.20). From assumptions of Theorem 3.1 and from [13] we infer that $\det(H)$ is an invertible element of $\overline{\mathbb{R}}$. This proves the Theorem 3.1.

PROOF OF THEOREM 3.2. The uniqueness of the ω–periodic solution of equation (1.0) follows from assumptions of Theorem 3.2. It is sufficient to show existence of the ω–periodic solution of equation (1.0). First, we shall prove that if $R_A(\varphi,t)$ and $R_f(\varphi,t)$ are ω–periodic representatives of A and f respectively, then there is $N \in \mathbf{N}$ such that for every $\varphi \in \mathcal{A}_N$ there is a constant $\varepsilon_0 > 0$ such that: for every $0 < \varepsilon < \varepsilon_0$ the equation

$$x'(t) = R_A(\varphi_\varepsilon,t)x(t) + R_f(\varphi_\varepsilon,t) \tag{4.5}$$

has exactly one ω–periodic solution. To this purpose we examine the problem

$$Z'(t) = R_A(\varphi_\varepsilon,t)Z(t), \qquad Z(0) = I. \tag{4.6}$$

Let $R_Z(\varphi_\varepsilon,t)$ be solution of problem (4.6). Then every solution $R_x(\varphi_\varepsilon,t)$ of equation (4.5) has the following representation

$$R_x(\varphi_\varepsilon,t) = R_Z(\varphi_\varepsilon,t)c(\varphi_\varepsilon) + Q(\varphi_\varepsilon,t) \tag{4.7}$$

where $c(\varphi_\varepsilon) = (c_1(\varphi_\varepsilon),\dots,c_n(\varphi_\varepsilon))^T$, $\quad c_i(\varphi_\varepsilon) \in \mathbf{R}$ for $i = 1,\dots,n$ and

$$Q(\varphi_\varepsilon,t) = R_Z(\varphi_\varepsilon,t)\int_0^t R_Z^{-1}(\varphi_\varepsilon,s)R_f(\varphi_\varepsilon,s)ds. \tag{4.8}$$

We consider equation (4.5) with conditions

$$R_x(\varphi_\varepsilon,0) = R_x(\varphi_\varepsilon,\omega). \tag{4.9}$$

By (4.5) and (4.9) we obtain the system of equations

$$\overline{H}(\varphi_\varepsilon)\overline{c}(\varphi_\varepsilon) = \overline{b}(\varphi_\varepsilon), \tag{4.10}$$

where

$$\overline{H}(\varphi_\varepsilon) = R_Z(\varphi_\varepsilon,0) - R_Z(\varphi_\varepsilon,\omega) \tag{4.11}$$

and

$$\overline{b}(\varphi_\varepsilon) = Q(\varphi_\varepsilon(0) - Q(\varphi_\varepsilon,\omega)). \tag{4.12}$$

Taking into account assumptions of Theorem 3.2 and invertibility of the matrix $\overline{H}$ (see [13]) we conclude that there is $N \in \mathbf{N}$ such that: for every $\varphi \in \mathcal{A}_N$ there are constants $c > 0$ and $\varepsilon_0 > 0$ such that

$$(4.13) \qquad |\det \overline{H}(\varphi_\varepsilon)| \geqslant c\varepsilon^N \qquad \text{for} \quad 0 < \varepsilon < \varepsilon_0.$$

Using (4.8)–(4.13) we deduce that equation (4.5) has exactly one ω–periodic solution $x(\varphi_\varepsilon, t)$ (for $\varphi \in \mathcal{A}_q$; $q \geqslant N$ and $0 < \varepsilon < \varepsilon_0$). Applying (4.10) and (4.13), we get

$$(4.14) \qquad \overline{c}(\varphi_\varepsilon) = \left(\overline{H}(\varphi_\varepsilon)\right)^{-1} \overline{b}(\varphi_\varepsilon)$$

(for $\varphi \in \mathcal{A}_q$ and $0 < \varepsilon < \varepsilon_0$).
Hence (we put $\overline{c}(\varphi_\varepsilon) = (0, \dots, 0)^T$ if $\det \overline{H}(\varphi_\varepsilon) = 0$)

$$(4.15) \qquad \overline{c}(\varphi) \in \mathcal{E}^n_M[\mathbb{R}].$$

On the other hand

$$(4.16) \qquad R_Z(\varphi, t) \in \mathcal{E}^{n\times n}_M[\mathbb{R}], \qquad R_Z^{-1}(\varphi, t) \in \mathcal{E}^{n\times n}_M[\mathbb{R}]$$

(see [11]), therefore

$$(4.17) \qquad R_x(\varphi, t) \in \mathcal{E}^n_M[\mathbb{R}],$$

which completes the proof of Theorem 3.2.

Proof of Theorem 3.3. If x is a nontrivial ω–periodic solution of system (3.32) in $\mathcal{G}^n(\mathbb{R})$, then

$$(4.18) \qquad R_{x'}(\varphi_\varepsilon, t) = R_A(\varphi_\varepsilon, t) R_x(\varphi_\varepsilon, t) + \eta(\varphi_\varepsilon, t)$$

where

$$(4.19) \qquad R_A(\varphi_\varepsilon, t) = ((R_{A_{kj}} * \varphi_\varepsilon)(t)),$$

$$(4.20) \qquad \eta \in \mathcal{N}^n[\mathbb{R}]$$

and $R_x(\varphi, t)$ is an ω–periodic representative of x. On the other hand $R_x(\varphi, t)$ has the representation (4.7), where

$$(4.21) \qquad Q(\varphi_\varepsilon, t) = R_Z(\varphi_\varepsilon, t) \int_0^t (R_Z(\varphi_\varepsilon, s))^{-1} \eta(\varphi_\varepsilon, s) ds.$$

By (4.16) and (4.21) we get

$$Q(\varphi,t)\in\mathcal{N}^n[\mathbb{R}]. \tag{4.22}$$

In view of (3.27)–(3.28) we have

$$\lim_{\varepsilon\to 0} R_{Z_{ij}}(\varphi_\varepsilon,t)=z_{ij}(t) \tag{4.23}$$

(almost uniformly, for every fixed $\varphi\in\mathcal{A}_N$ and for sufficiently large N), where

$$R_Z(\varphi_\varepsilon,t)=(R_{Z_{ij}}(\varphi_\varepsilon,t)),\quad Z(t)=(z_{ij}(t)),$$

$i,j=1,\dots,n$; $R_Z(\varphi_\varepsilon,t)$ and Z are solutions of problems (4.6) and (3.22) respectively (in the Caratheodory sense). Relations (4.22)–(4.23), (4.7), (4.12), (4.14), (4.16) and (3.31) yield (we put $\overline{c}(\varphi_\varepsilon)=(0,\dots,0)^T$ if $\det\overline{H}(\varphi_\varepsilon)=0$)

$$c(\varphi)\in\mathcal{N}^n[\mathbb{R}] \tag{4.24}$$

and this completes the proof of Theorem 3.3.

PROOF OF THEOREM 3.4. Let x be a nontrivial ω–periodic solution of system (3.32) in $\mathcal{G}^n(\mathbb{R})$. We examine equality (4.18), where $R_A(\varphi,t)$ and $R_x(\varphi,t)$ are ω–periodic representatives of A and x respectively. By (4.18) and (3.7) we get

$$\begin{aligned}((R_x(\varphi_\varepsilon,t))^T,R_{x'}(\varphi_\varepsilon,t))&=\Big((R_x(\varphi_\varepsilon,t))^T,R_A(\varphi_\varepsilon,t)R_x(\varphi_\varepsilon,t)\Big)\\&\quad+(R_x(\varphi_\varepsilon,t),\eta(\varphi_\varepsilon,t))\geqslant\gamma_0(R_x(\varphi_\varepsilon,t),R_x(\varphi_\varepsilon,t))+\overline{\eta}(\varphi_\varepsilon,t)\\&=\gamma_0\|R_x(\varphi_\varepsilon,t)\|^2+\overline{\eta}(\varphi_\varepsilon,t),\end{aligned} \tag{4.25}$$

where

$$\overline{\eta}(\varphi,t)=(R_x(\varphi,t),\eta(\varphi,t))\in\mathcal{N}[\mathbb{R}]. \tag{4.26}$$

Hence

$$\begin{aligned}0&\geqslant\gamma_0\int_0^\omega\|R_x(\varphi_\varepsilon,t)\|^2dt+\eta^*(\varphi_\varepsilon)=\gamma_0\omega\|R_x(\varphi_\varepsilon,\tau_\varepsilon)\|^2+\eta^*(\varphi_\varepsilon)\\&\geqslant\gamma_0\omega R^2_{x_i}(\varphi_\varepsilon,\tau_\varepsilon)+\eta^*(\varphi_\varepsilon),\end{aligned} \tag{4.27}$$

where

$$\eta^*(\varphi)\in\mathcal{N},\qquad \tau_\varepsilon\in[0,\omega]\quad\text{and}\quad i=1,\dots,n. \tag{4.28}$$

The last inequalities imply the relations

$$|R_{x_i}(\varphi_\varepsilon, \tau_\varepsilon)| \leqslant c\varepsilon^{\alpha(q)-N_0}, \quad i = 1, \dots, n \tag{4.29}$$

(for $q \geqslant N_0'$, $\varphi \in \mathcal{A}_q$ and $0 < \varepsilon < \varepsilon_0'$).
On the other hand (by (4.18)) we have

$$R_x(\varphi_\varepsilon, t) = R_x(\varphi_\varepsilon, \tau_\varepsilon) + \int_{\tau_\varepsilon}^{t} R_A(\varphi_\varepsilon, s) R_x(\varphi_\varepsilon, s) ds + \int_{\tau_\varepsilon}^{t} \eta(\varphi_\varepsilon, s) ds. \tag{4.30}$$

Applying relations (3.4), (4.29)–(4.30) and the Gronwall inequality we deduce that

$$\|R_x(\varphi_\varepsilon, t)\|_{[0,\omega]} \leqslant c_0 \varepsilon^{\alpha(q)-N_0} \tag{4.31}$$

and

$$\left\| \frac{d}{dt^r} R_x(\varphi_\varepsilon, t) \right\|_{[0,\omega]} \leqslant c_r \varepsilon^{\alpha(q)-N_r} \tag{4.32}$$

(for $q \geqslant N_r$, $\varphi \in \mathcal{A}_q$ and $0 < \varepsilon < \varepsilon_0^*$).
Therefore

$$R_x(\varphi, t) \in \mathcal{N}^n[\mathbb{R}] \tag{4.33}$$

which completes the proof of Theorem 3.4.

Proof of Theorem 3.5. Let $x = [R_x(\varphi, t)]$ be a nonzero ω–periodic solution of equation (3.33). Then we consider two cases:

(4.34) there exists $t_0 \in [0, \omega]$ such that $x(t_0) = 0$,

(4.35) $x(t) \neq 0$ for all $t \in \mathbb{R}$.

If $n = 1$ and $x(t_0) = 0$, then Theorem 3.5 is obvious (by Remark 3.1).
We assume that $n > 1$ and $x(t_0) = 0$. Then

(4.36)
$$R_{x^{(n)}}(\varphi_\varepsilon, t) + R_{p_1}(\varphi_\varepsilon, t) R_{x^{(n-1)}}(\varphi_\varepsilon, t) \dots + R_{p_n}(\varphi_\varepsilon, t) R_x(\varphi_\varepsilon, t) = \eta(\varphi_\varepsilon, t),$$

where $\eta(\varphi, t) \in \mathcal{N}[\mathbb{R}]$ and $R_{p_i}(\varphi, t), R_{x^{(n-i)}}(\varphi, t)$ are ω–periodic representatives of p_i and $x^{(n-i)}$ respectively for $i = 1, \dots, n$.

Hence

$$R_x(\varphi_\varepsilon,t) = -\int\limits_{t_0}^{t}\left(\int\limits_{c_\lambda}^{\tau}\left(\sum_{i=1}^{n} R_{p_i}(\varphi_\varepsilon,s)R_{x^{(n-i)}}(\varphi_\varepsilon,s)\right)ds^{n-1}\right)d\tau + R_{x(t_0)}(\varphi_\varepsilon) + \eta_1(\varphi_\varepsilon), \tag{4.37}$$

where

$$R_{x(t_0)}(\varphi),\ \eta_1(\varphi) \in \mathcal{N}.$$

We put

$$M_{i\varepsilon} = \|R_{x^{(n-i)}}(\varphi_\varepsilon,t)\|_{[0,\omega]},\quad M_\varepsilon = \sum_{i=1}^{n} M_{i\varepsilon}, \tag{4.38}$$

$$F^{(\nu)}(\varphi_\varepsilon,t) = -\left(\int\limits_{c_\lambda}^{t}\left(\sum_{i=1}^{n} R_{p_i}(\varphi_\varepsilon,s)R_{x^{(n-i)}}(\varphi_\varepsilon,s)\right)ds^{n-1}\right)^{(\nu)} \tag{4.39}$$

for $\nu = 0,\dots,n-2$.

By [19] we infer that $F^{(\nu)}(\varphi_\varepsilon,t)$ is a hereditarily ω–periodic function (for $\varphi \in \mathcal{A}_N$ and $0 < \varepsilon < \varepsilon_0^{\cdot}$). Relations (4.38)–(4.39) yield

$$\left|\frac{d}{dt^{n-2}}F(\varphi_\varepsilon,t)\right| \leqslant M_\varepsilon \max_{1\leqslant i\leqslant n}\int\limits_{0}^{\omega+3\beta}\left|R_{p_i}(\varphi_\varepsilon,t)\right|dt \tag{4.40}$$

and

$$\left\|\frac{d}{dt^\nu}F(\varphi_\varepsilon,t)\right\|_{[0,\omega]} \leqslant M_\varepsilon(\omega+3\beta)^{n-\nu-2}\max_{1\leqslant i\leqslant n}\int\limits_{0}^{\omega+3\beta}|R_{p_i}(\varphi_\varepsilon,t)|dt$$

(for $\nu = 0,\dots,n-2$).

In view of relations (4.36)–(4.41) we get

$$M_\varepsilon \leqslant M_\varepsilon(a(\omega,\beta,\varepsilon))^{-1}\max_{1\leqslant i\leqslant n}\int\limits_{0}^{\omega+3\beta}|R_{p_i}(\varphi_\varepsilon,t)|dt + c\varepsilon^{\alpha(q)-N_0} \tag{4.42}$$

(for $q \geqslant N_0$, $\varphi \in \mathcal{A}_q$ and $0 < \varepsilon < \varepsilon_0''$).

Taking into account (3.14) we have

$$M_\varepsilon \leqslant c\varepsilon^{\alpha(q)-N_0}. \tag{4.43}$$

Thus

$$R_x(\varphi,t) \in \mathcal{N}[\mathbb{R}]. \tag{4.44}$$

In the case 2°, we conclude that (having integrated by parts the products $R_{p_i}(\varphi_\varepsilon,t)R_{x^{(n-i)}}(\varphi_\varepsilon,t)$ in (4.36) for $n > 1$)

$$\begin{aligned} 0 &= \int_0^\omega R_{x^{(n)}}(\varphi_\varepsilon,t)dt \\ &= \int_0^\omega \left(\sum_{i=1}^n (-1)^{n-i} R_{p_i^{(n-i)}}(\varphi_\varepsilon,t)\right) R_x(\varphi_\varepsilon,t)dt + \eta_2(\varphi_\varepsilon) \\ &= \omega \left(\sum_{i=1}^n (-1)^{n-i} R_{p_i^{(n-i)}}(\varphi_\varepsilon,\tau_\varepsilon)\right) R_x(\varphi_\varepsilon,\tau_\varepsilon) + \eta_2(\varphi_\varepsilon), \end{aligned} \tag{4.45}$$

where $\eta_2 \in \mathcal{N}$ and $\tau_\varepsilon \in [0,\omega]$.

The relations (3.14) and (4.45) yield

$$|R_x(\varphi_\varepsilon,\tau_\varepsilon)| \leqslant c\varepsilon^{\alpha(q)-N_1} \tag{4.46}$$

(for $q \geqslant N_1$, $\varphi \in \mathcal{A}_q$ and $0 < \varepsilon < \bar{\bar{\varepsilon}}$).
On the other hand

$$R_x(\varphi_\varepsilon,t) = R_x(\varphi_\varepsilon,\tau_\varepsilon) - \int_{\tau_\varepsilon}^t F^{(0)}(\varphi_\varepsilon,s)ds + \eta_3(\varphi_\varepsilon) \tag{4.47}$$

where $\eta_3 \in \mathcal{N}$ and $F^{(0)}(\varphi_\varepsilon,s) = F(\varphi_\varepsilon,s)$. Using arguments similar to those in the case (4.34) we obtain relations (4.43)–(4.44), which completes the proof of Theorem 3.6.

Proof of Theorem 3.6 is similar to the proof of Theorem 3.5.

5. Final remarks

REMARK 5.1. If $A_{kj}, x_j \in C^\infty(\mathbb{R})$, $A = (A_{kj})$, $x = (x_1,\dots,x_n)^T$ (for $j,k = 1,\dots,n$), then the classical product $A \cdot x$ and the product $A \odot x$ in $\mathcal{G}^n(\mathbb{R})$ give rise to the same elements of $\mathcal{G}^n(\mathbb{R})$ (see [2]).

Hence we get

THEOREM 5.1. *We assume that*

$$A_{kj}, f_j \in C^\infty(\mathbb{R}) \quad \text{for} \quad j,k = 1,\dots,n; \tag{5.1}$$

(5.2) A_{kj}, f_j *are ω-periodic functions (for $k,j = 1,\dots,n$);*

(5.3) *the trivial solution is the unique ω-periodic solution of system* (3.16) *in the classical sense,*

(5.4) $\tilde{x}$ *is the ω-periodic solution of system* (1.0) *in the classical sense,* $\tilde{\tilde{x}} \in \mathcal{G}^n(\mathbb{R})$ *is the ω-periodic solution of the system*

$$x'(t) = A(t) \odot x(t) + f(t). \tag{5.5}$$

Then $\tilde{x}$ and $\tilde{\tilde{x}}$ give rise to the same element of $\mathcal{G}^n(\mathbb{R})$.

PROOF OF THEOREM 5.1. Let $\tilde{\tilde{x}} = [R_{\tilde{\tilde{x}}}(\varphi, t)]$ be an ω-periodic solution of system (5.5) and let $\tilde{x}$ be an ω-periodic solution of system (1.0). Then

$$R_{\tilde{\tilde{x}}'}(\varphi_\varepsilon, t) = A(t) R_{\tilde{\tilde{x}}}(\varphi_\varepsilon, t) + f(t) + \eta(\varphi_\varepsilon, t), \tag{5.6}$$

where $\eta \in \mathcal{N}^n[\mathbb{R}]$ and $R_{\tilde{\tilde{x}}}(\varphi, t)$ is an ω-periodic representative of $\tilde{\tilde{x}}$ (for $0 < \varepsilon < \varepsilon_0$, $\varphi \in \mathcal{A}_N$ and for sufficiently large N).

Thus

$$R_{x'}(\varphi_\varepsilon, t) = A(t) R_x(\varphi_\varepsilon, t) - \eta(\varphi_\varepsilon, t), \tag{5.7}$$

where

$$R_x(\varphi_\varepsilon, t) = \tilde{x}(t) - R_{\tilde{\tilde{x}}}(\varphi_\varepsilon, t). \tag{5.8}$$

Using arguments similar to those in relations (4.18)–(4.24) we conclude that

$$\tilde{x} - R_{\tilde{\tilde{x}}}(\varphi, t) \in \mathcal{N}^n[\mathbb{R}] \tag{5.9}$$

which completes the proof of Theorem 5.1.

REMARK 5.2. It is known that every distribution is moderate (see [2]). On the other hand L. Schwartz proved in [18] that there does not exist an algebra $\tilde{\mathcal{A}}$ such that: the algebra $C(\mathbb{R})$ of continuous functions on $\mathbb{R}$ is subalgebra of $\tilde{\mathcal{A}}$, the function 1 is unit element od $\tilde{\mathcal{A}}$, elements of $\tilde{\mathcal{A}}$ are "C^∞" with respect to a derivation which coincides with usual one in $C^1(\mathbb{R})$, and such that the usual formula for the derivation of a product holds. As consequence multiplication in $\mathcal{G}(\mathbb{R})$ does not coincide with usual multiplication of continuous functions.

EXAMPLE 5.1. Let $g_1(t)$ and $g_2(t)$ be continuous functions defined by

$$g_1(t) = \begin{cases} 0, & \text{if } t \leqslant 0, \\ t, & \text{if } t > 0, \end{cases} \tag{5.10}$$

$$g_2(t) = \begin{cases} t, & \text{if } t \leqslant 0, \\ 0, & \text{if } t > 0. \end{cases} \tag{5.11}$$

Then their classical product in $C(\mathbb{R})$ is 0. Their product in $\mathcal{G}(\mathbb{R})$ is different from 0 (see [2]).

Let us consider the equations

$$x'(t) = g_1(t)x(t) + g_2'(t), \tag{5.12}$$

$$x'(t) = g_1(t) \odot x(t) + g_2'(t). \tag{5.13}$$

It is easy to show that $x = g_2$ is a classical solution of equation (5.12) (in the Caratheodory sense). On the other hand $x = g_2$ is not a solution of equation (5.13) in the Colombeau algebra $\mathcal{G}(\mathbb{R})$, because $g_1 \odot g_2$ is not zero in $\mathcal{G}(\mathbb{R})$ (see [2], [11]).

To "repair" to consistency problem for multiplication we give the definition introduced by J. F. Colombeau (see [2]).

An element U of $\mathcal{G}(\mathbb{R})$ is said to admit a member $W \in \mathcal{D}'(\mathbb{R})$ as the associated distribution, if it has a representative $R_U(\varphi, t)$ with the following property: for every $\psi \in \mathcal{D}(\mathbb{R})$ there is $N \in \mathbf{N}$ such that for every $\varphi \in \mathcal{A}_N$ we have

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} R_U(\varphi_\varepsilon, t)\psi(t)dt = W(\psi). \tag{5.14}$$

THEOREM 5.2. *We assume that*

(5.15) *conditions* (3.29)–(3.31) *are satisfied,*

(5.16) $\overline{x}$ *is an ω-periodic solution of system* (1.0) *in the Caratheodory sense,* $\overline{\overline{x}} \in \mathcal{G}^n(\mathbb{R})$ *is an ω-periodic solution of the system*

$$x'(t) = A(t) \odot x(t) + f(t). \tag{5.17}$$

Then $\overline{\overline{x}}$ admits an associated distribution which equals $\overline{x}$.

PROOF OF THEOREM 5.2. Let $R_x(\varphi_\varepsilon, t)$ be an ω-periodic solution of system (4.5) (for $\varphi \in \mathcal{A}_N$, $0 < \varepsilon < \overline{\overline{\varepsilon}}$ and for sufficiently large N), where

(5.18)
$$R_A(\varphi_\varepsilon, t) = ((R_{A_{kj}} * \varphi_\varepsilon)(t)),$$
$$R_f(\varphi_\varepsilon, t) = ((R_{f_1} * \varphi_\varepsilon)(t), \dots, (R_{f_n} * \varphi_\varepsilon)(t))^T.$$

Then, by virtue of relations (4.7)–(4.17), (4.23) and (3.28) we have

(5.19)
$$\lim_{\varepsilon \to 0} R_{x_i}(\varphi_\varepsilon, t) = \overline{x_i}(t)$$

(almost uniformly), where $i = 1, \dots, n$; $\varphi \in \mathcal{A}_N$ and $\overline{x} = (\overline{x_1}, \dots, \overline{x_n})^T$. On the other hand $\overline{\overline{x}} = [R_x(\varphi_\varepsilon, t)]$ is an ω-periodic solution of system (5.17) (we put $R_x(\varphi_\varepsilon, t) = (0, \dots, 0)^T$ if $\det \overline{H}(\varphi_\varepsilon) = 0$). This proves of the theorem.

REMARK 5.3. Generalized solutions of ordinary differential equations with additional conditions can be considered on the other way (for example: [3]–[7], [9]–[10], [14]–[17], [20]).

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UNIWERSYTET ŚLĄSKI
INSTYTUT MATEMATYKI
UL. BANKOWA 14
40–007 KATOWICE