Annales Mathematicae Silesianae 11. Katowice 1997, 89 – 99

Prace Naukowe Uniwersytetu Śląskiego nr 1665

ON THE CAUCHY EQUATION ON SPHERES

ROMAN GER AND JUSTYNA SIKORSKA

Abstract. We deal with a conditional functional equation

$$\varphi(x) = \varphi(y) \quad \text{implies} \quad f(x+y) = f(x) + f(y).$$

Under some assumptions imposed on function φ, the domains and ranges of φ and f we derive the additivity of f. As a consequence we obtain two Hyers–Ulam stability results related to the equation considered.

C. Alsina and J. L. Garcia-Roig [1] considered a conditional functional equation

$$\|x\| = \|y\| \quad \text{implies} \quad f(x+y) = f(x) + f(y), \tag{1}$$

where $f : X \to Y$ is a continuous function mapping a real inner product space $(X, (\cdot|\cdot))$ with $\dim X \geqslant 2$ into a real topological vector space Y. They obtained the linearity of such a function f in the case where $Y = \mathbb{R}^n$. Their results are contained in the following two theorems.

THEOREM A. *Let* $(X, (\cdot|\cdot))$ *be a real inner product space with* $\dim X \geqslant 2$ *and let* Y *be a real linear topological space. A continuous mapping* $f : X \to Y$ *satisfies* (1) *if and only if* f *is a continuous linear transformation.*

THEOREM B. *A mapping* f *from a real inner product space* $(X, (\cdot|\cdot))$ *with* $\dim X \geqslant 2$ *and with values in* $\mathbb{R}^n$ *satisfies* (1) *if and only if* f *is additive.*

On the other hand, Gy. Szabó [3] has obtained the following

AMS Classification: Primary 39B52, 39B72. Secondary 41A45.

THEOREM C. *Suppose that* $(X, \|\cdot\|)$ *is a real normed linear space with* $\dim X \geqslant 3$ *and* $(Y, +)$ *is an abelian group. Then a mapping* $f : X \to Y$ *satisfies the conditional Cauchy equation* (1) *if and only if* f *is additive.*

The method of the proof of this nice result is by no means elementary. In particular, some sophisticated connectivity results on intersection of spheres of equal radii in normed linear spaces are involved. Moreover, the method spoken of requires the dimension of the space considered to be greater than or equal 3.

In what follows we proceed with the study of equation (1) with the norm replaced by an abstract function fulfilling suitable conditions. In a natural way this motivates us to deal simultaneously with more general structures than those considered in Theorems A, B and C. Our first result reads as follows.

THEOREM 1. *Let* X *be a real linear space with* $\dim X \geqslant 2$, $(Y, +)$ *be an abelian group,* Z *be a given nonempty set and let* $\varphi : X \to Z$ *be an even mapping such that*

(i) *for any two linearly independent vectors* $x, y \in X$ *there exist linearly independent vectors* $u, v \in \mathrm{Lin}\,\{x, y\}$ *such that* $\varphi(u+v) = \varphi(u-v)$;
(ii) *if* $x, y \in X$, $\varphi(x+y) = \varphi(x-y)$, *then* $\varphi(\alpha x + y) = \varphi(\alpha x - y)$ *for all* $\alpha \in \mathbb{R}$;
(iii) *for all* $x \in X$ *and* $\lambda \in \mathbb{R}_+ := (0, \infty)$ *there exists a* $y \in X$ *such that* $\varphi(x+y) = \varphi(x-y)$ *and* $\varphi((\lambda+1)x) = \varphi((\lambda-1)x - 2y)$.

If $f : X \to Y$ *satisfies the condition*

$$\text{(2)} \qquad \varphi(x) = \varphi(y) \quad \textit{implies} \quad f(x+y) = f(x) + f(y), \quad x, y \in X.$$

then f *is additive.*

PROOF. From condition (2) we have $f(2x) = 2f(x)$ for all $x \in X$ whereas the evenness of φ jointly with the equation imply the oddness of f.

Fix arbitrarily $x, y \in X$ such that $\varphi(x+y) = \varphi(x-y)$. From the evenness of φ we have $\varphi(x+y) = \varphi(y-x)$ as well. Thus

$$f(2x) = f((x+y) + (x-y)) = f(x+y) + f(x-y),$$
$$f(2y) = f((x+y) + (y-x)) = f(x+y) + f(y-x).$$

Summing these two equations side by side and applying the oddness of f we get

$$f(x) + f(y) = f(x+y)\,.$$

Consequently, we infer that

$$\varphi(x+y)=\varphi(x-y) \quad \text{implies} \quad f(x+y)=f(x)+f(y), \quad x,y\in X.$$

This is a substantial generalization of the so called isosceles orthogonal additivity equation in normed spaces, corresponding to the case where $\varphi = \|\cdot\|$ (see e.g. R. C. James [2] and Gy. Szabó [3]). To solve such an equation we first show that f has to be additive on parallel arguments. Assume that $x \in X$ and $\lambda \in \mathbb{R}_+$. By condition (iii) there exists a $y \in X$ such that $\varphi(x+y)=\varphi(x-y)$ and $\varphi((\lambda+1)x)=\varphi((\lambda-1)x-2y)$, whence by means of (ii) and the oddness of f we have

$$\begin{aligned} f(x+\lambda x) &= f(x+y+\lambda x-y) = f(x+y)+f(\lambda x-y) \\ &= f(x)+f(y)+f(\lambda x)+f(-y) = f(x)+f(\lambda x)\,. \end{aligned}$$

To get a similar relationship for nonpositive λ's let us distinguish three cases:

1. $\lambda = 0$;
2. $\lambda \in (-1,0)$;
3. $\lambda \in (-\infty,-1]$.

The case 1. is trivial. Assuming 2., on account of the previous considerations, having $1+\lambda>0$ and $-\frac{\lambda}{1+\lambda}>0$, we deduce that

$$\begin{aligned} f(x) &= f((1+\lambda)x-\lambda x) = f\left((1+\lambda)x+\left(-\frac{\lambda}{1+\lambda}\right)(1+\lambda)x\right) \\ &= f((1+\lambda)x)-f(\lambda x), \end{aligned}$$

whence

$$f(x+\lambda x)=f(x)+f(\lambda x).$$

Finally, assuming 3. we have $-1-\lambda \geqslant 0$ and, consequently,

$$\begin{aligned} f(\lambda x) &= f(-x+(1+\lambda)x) = f(-x+(-1-\lambda)(-x)) \\ &= f(-x)+f((1+\lambda)x) = -f(x)+f((1+\lambda)x), \end{aligned}$$

whence

$$f(x+\lambda x)=f(x)+f(\lambda x).$$

Observe that, on account of hypothesis (ii), the relationship $\varphi(s+t) = \varphi(s-t)$ implies the equality $\varphi(\alpha s+\beta t) = \varphi(\alpha s-\beta t)$ whenever $\alpha,\beta\in\mathbb{R}$ and $s,t\in X$.

Now, let $x,y\in X$ be arbitrary linearly independent vectors. On account of (i) there exist linearly independent vectors $u,v\in \operatorname{Lin}\{x,y\}$ such that $\varphi(u+v)=\varphi(u-v)$. Clearly, x and y can be represented in the form $x=$

$\alpha u + \beta v$ and $y = \gamma u + \delta v$ with some uniquely determined real coefficients α, β, γ and δ. Hence

$$\begin{aligned} f(x+y) &= f((\alpha+\gamma)u + (\beta+\delta)v) = f((\alpha+\gamma)u) + f((\beta+\delta)v) \\ &= f(\alpha u) + f(\gamma u) + f(\beta v) + f(\delta v) \\ &= f(\alpha u + \beta v) + f(\gamma u + \delta v) = f(x) + f(y), \end{aligned}$$

which completes the proof. □

EXAMPLE 1. Let $(X, (\cdot|\cdot))$ be a real inner product space with $\dim X \geqslant 2$, $Z = \mathbb{R}$ and $\varphi(x) := \|x\|$, $x \in X$. Such a function φ satisfies all the hypotheses of Theorem 1; therefore, we get Szabó's result from [3] in an inner product space, including the dimension 2.

EXAMPLE 2. Let X, Z be real linear spaces and let $A : X^2 \to Z$ be a bilinear and symmetric mapping such that

a) for every positive λ and every $x \in X$ there exists a $y \in X$ fulfilling the conditions

$$A(x, y) = 0 \quad \text{and} \quad A(y, y) = \lambda A(x, x);$$

b) for each two linearly independent vectors $x, y \in X$ there exist linearly independent vectors $u, v \in \text{Lin}\,\{x, y\}$ such that $A(u, v) = 0$.

Then it is easy to check that a function $\varphi : X \to Z$ defined by the formula $\varphi(x) := A(x, x)$, $x \in X$, satisfies all the requirements spoken of in Theorem 1.

Let Z be a nonempty set. Let us recall that a binary relation $\prec \subset Z \times Z$ is termed *connected* provided that

$$x \prec y \quad \text{or} \quad y \prec x \quad \text{or} \quad x = y$$

for all $x, y \in Z$.

THEOREM 2. *Let $(X, +), (Y, +)$ and $(Z, +)$ be topological groups. Assume that $(X, +)$ and $(Y, +)$ are commutative, $(Y, +)$ has no elements of order 2 and that $(Z, +)$ is equipped with a connected binary relation $\prec \subset Z \times Z$ having the following two properties:*

(a) *for every $x \in Z$ the relationship $0 \prec x$ implies that $-x \prec 0$;*
(b) *the half-lines $\{x \in Z : x \prec 0\}$ and $\{x \in Z : 0 \prec x\}$ are disjoint and open in Z.*

Moreover, let $\varphi : X \to Z$ be a continuous mapping satisfying the condition

$$\text{(3)}\qquad \begin{array}{l}\textit{for every } x, y \in X \textit{ the set } \{t \in X : \ \varphi(x+t) = \varphi(x-t) = \varphi(y)\} \\ \textit{is nonempty and connected provided that } \varphi(x) \prec \varphi(y).\end{array}$$

Then $f : X \to Y$ is a solution to the conditional functional equation (2) *if and only if f is a group homomorphism.*

PROOF. Fix arbitrarily $x, y \in X$ with $\varphi(x) \prec \varphi(y)$ and put

$$K(x,y) := \{t \in X : \ \varphi(x+t) = \varphi(x-t) = \varphi(y)\} + x.$$

Then, setting $L_y := \varphi^{-1}(\{\varphi(y)\})$, we infer that

$$K(x,y) = L_y \cap (2x - L_y)\,.$$

Moreover, $K(x,y)$ is nonempty and connected as a translation of a set with the same properties.

Define a function $\psi : K(x,y) \to Z$ by the formula

$$\text{(4)}\qquad \psi(t) := \varphi(y+t) - \varphi(2x+y-t), \quad t \in K(x,y)\,,$$

and observe that $2x - t \in K(x,y)$ whenever $t \in K(x,y)$. Since φ is continuous, the function ψ is continuous as well. We are going to show that

$$\text{(5)}\qquad \psi(t_0) = 0 \qquad \text{for some} \quad t_0 \in K(x,y).$$

To this end, we may assume that $\psi \neq 0$ (otherwise, each $t \in K(x,y)$ satisfies (5)). Thus, there exists a $t_1 \in K(x,y)$ such that $0 \prec \psi(t_1)$ or $\psi(t_1) \prec 0$. Assume that $0 \prec \psi(t_1)$ (in the remaining case the proof is literally the same). By means of (4) and (a) we obtain

$$\psi(2x - t_1) = -(\varphi(y+t_1) - \varphi(2x+y-t_1)) = -\psi(t_1) \prec 0.$$

If we had $\psi(t) \neq 0$ for all $t \in K(x,y)$ then, in view of (b) and the continuity of ψ already observed, the connected set $\psi(K(x,y))$ would be splitted into two disjoint nonempty (relatively) open parts, which is a contradiction. Therefore, there exists a $t_0 \in K(x,y)$ with $\psi(t_0) = 0$, which says that

$$\text{(6)}\qquad \varphi(y+t_0) = \varphi(2x+y-t_0).$$

On the other hand, $t_0 \in K(x,y)$ implies that $t_0 \in L_y$ and $t_0 \in 2x - L_y$, whence

$$\text{(7)}\qquad \varphi(t_0) = \varphi(2x - t_0) = \varphi(y).$$

Now, putting $u := t_0$ and $v := 2x - t_0$ we get $t_0 = 2x - v$ and $u + v = 2x$ whence, by using (6) and (7), we arrive at

$$\begin{aligned} f(2x+2y) &= f(u+y+v+y) = f(u+y) + f(v+y) \\ &= f(u) + f(y) + f(v) + f(y) = f(u) + f(v) + 2f(y) \\ &= f(u+v) + 2f(y) = f(2x) + 2f(y). \end{aligned}$$

Since equation (2) immediately implies that $f(2t) = 2f(t)$ for all $t \in X$, we finally have

$$f(x+y) = f(x) + f(y),$$

because, by assumption, $(Y, +)$ has no elements of order 2.

Assume now that the condition $\varphi(x) \prec \varphi(y)$ is invalid. Then either $\varphi(y) \prec \varphi(x)$ or $\varphi(x) = \varphi(y)$. In the former case the same reasoning applies after interchanging the roles of x and y. In the latter case the desired additivity relationship is forced directly by equation (2). This completes the proof. □

REMARK 1. Let $(X, \|\cdot\|)$ be a real normed space with $\dim X \geqslant 3$. If $Z = \mathbb{R}$ and $\prec$ stands for the usual inequality $<$ and if $\varphi : X \to \mathbb{R}$ is defined as $\varphi(x) := \|x\|$ for all $x \in X$, we get Szabó's result [3]. Nevertheless, we have to stress once more that the verification of (3) in that case is slightly sophisticated (cf. [3]).

REMARK 2. A careful inspection of the proof of Theorem 2 shows that the assumption of the connectivity of the set $\{t \in X : \varphi(x+t) = \varphi(x-t) = \varphi(y)\}$ and the continuity of function φ were used exclusively to get a solution t_0 of the following system of equations

$$(8) \qquad \begin{cases} \varphi(t_0) = \varphi(y) \\ \varphi(2x - t_0) = \varphi(y) \\ \varphi(y + t_0) = \varphi(2x + y - t_0), \end{cases}$$

where x and y are arbitrarily given.

In many instances we may get the existence of such a t_0 in a direct way whenever $\varphi(x) \prec \varphi(y)$. To visualize this let us consider the following situation:

EXAMPLE 3. Let X be a real linear space. Assume that $(H, (\cdot|\cdot))$ is a real inner product space with $\dim H \geqslant 3$, and $L : X \to H$ is a linear surjection. Define a function $\varphi : X \to \mathbb{R}$ by the following formula:

$$\varphi(x) := \|L(x)\|, \quad x \in X.$$

For $x, y \in X$ arbitrarily fixed, system (8) reads now as follows:

$$(9) \qquad \begin{cases} \|L(t_0)\| = \|L(y)\| \\ (L(x)|L(x)) = (L(x)|L(t_0)) \\ (L(y)|L(t_0)) = (L(y)|L(x)). \end{cases}$$

For a fixed $u \in H$ define function $\phi_u : H \to \mathbb{R}$ by the formula

$$\phi_u := (u|\cdot).$$

Then (9) may be rewritten in the form

$$\begin{cases} L(t_0) \in S(0, \|L(y)\|) \\ L(t_0) - L(x) \in \ker \phi_{L(x)} \\ L(t_0) - L(x) \in \ker \phi_{L(y)}, \end{cases}$$

where $S(a, r)$ stands for a sphere centered at $a \in H$ with radius $r \geqslant 0$. This means that

$$L(t_0) \in S(0, \|L(y)\|) \cap (L(x) + (\ker \phi_{L(x)} \cap \ker \phi_{L(y)})) =: S_0.$$

Plainly, in the present case we have $Z = \mathbb{R}$ and $\prec = <$ whereas the assumption $\varphi(x) \prec \varphi(y)$ implies that S_0 is nonempty which jointly with surjectivity of L gives the existence of such a t_0.

We continue our considerations with two Hyers-Ulam type stability results regarding the conditional functional equation (2).

THEOREM 3. *Suppose that either*

(α) *X is a real linear space with dim $X \geqslant 2$, Z is a given nonempty set and $\varphi : X \to Z$ is an even mapping satisfying conditions* (i), (ii) *and* (iii),

or

(β) *$(X, +)$, $(Z, +)$ are topological groups, $(X, +)$ is commutative, $(Z, +)$ is equipped with a connected binary relation $\prec \subset Z \times Z$ having properties* (a) *and* (b) *and $\varphi : X \to Z$ is a continuous mapping satisfying* (3).

Moreover, let $(Y, \|\cdot\|)$ be a Banach space. Given an $\varepsilon > 0$, let $f : X \to Y$ be a mapping such that for all $x, y \in X$ one has

$$(10) \qquad \varphi(x) = \varphi(y) \quad \textit{implies} \quad \|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon.$$

If φ admits a solution $\Phi : Z \to Z$ of the functional equation $\varphi(2x) = \Phi(\varphi(x)), x \in X$, then there exists exactly one additive mapping $g : X \to Y$ such that

$$\|f(x) - g(x)\| \leqslant \varepsilon, \quad x \in X. \tag{11}$$

PROOF. By means of (10) we obtain the inequality $\|f(2x) - 2f(x)\| \leqslant \varepsilon$ valid for all $x \in X$. It is well known that the classical Hyers function sequence $(f_n)_{n \in \mathbb{N}}$ defined by the formula

$$f_n(x) := \frac{1}{2^n} f(2^n x), \quad x \in X, n \in \mathbb{N},$$

is then (uniformly) convergent to a function $g : X \to Y$ satisfying (11).

Assume that $\varphi(x) = \varphi(y)$ for some $x, y \in X$. Then, an easy induction shows that

$$\varphi(2^n x) = \Phi^n(\varphi(x)) = \Phi^n(\varphi(y)) = \varphi(2^n y)$$

for all $n \in \mathbb{N}$, where Φ^n denotes the n-th iteration of Φ. Consequently, relation (10) implies that $\|f_n(x + y) - f_n(x) - f_n(y)\| \leqslant \frac{1}{2^n}\varepsilon$ holds true for every $n \in \mathbb{N}$ whence, by letting n tend to infinity, we infer that $g(x + y) = g(x) + g(y)$. In other words, g yields a solution of equation (2). An appeal to Theorem 1 in the case where condition (α) is assumed (resp. to Theorem 2 whenever (β) is satisfied) proves that g has to be additive.

If we had two additive functions $g_1, g_2 : X \to Y$ satisfying the estimations $\|f(x) - g_i(x)\| \leqslant \varepsilon$, $x \in X$, $i = 1, 2$, then for every $n \in \mathbb{N}$ and every $x \in X$ we would get

$$\begin{aligned} n\|g_1(x) - g_2(x)\| &= \|g_1(nx) - g_2(nx)\| \\ &\leqslant \|g_1(nx) - f(nx)\| + \|f(nx) - g_2(nx)\| \leqslant 2\varepsilon, \end{aligned}$$

i.e. $g_1 = g_2$. This finishes the proof. □

As a matter of fact, to get a "pure" stability result on equation (2) regardless of its solutions we do not need any assumptions forcing a solution of (2) to be additive. This remark leads to the following

THEOREM 4. *Let $(S, +)$ be an abelian semigroup and let $(Y, \|\cdot\|)$ be a Banach space. Assume that we are given a nonempty set Z and a function $\varphi : S \to Z$ admitting a solution $\Phi : Z \to Z$ of the functional equation*

$$\varphi(2x) = \Phi(\varphi(x)), \quad x \in S. \tag{12}$$

Given an $\varepsilon > 0$ and a solution $f : S \to Y$ of (10) *there exists exactly one solution $g : S \to Y$ of equation* (2) *such that estimation* (11) *holds true.*

PROOF. The existence may be derived along the same lines as in the first part of the proof of Theorem 3. To show the uniqueness, note that each solution g of (2) satisfies the equality $g(2^n x) = 2^n g(x)$ for all $x \in S$ and all $n \in \mathbb{N}$. Hence we may repeat the reasoning applied in the last part of the preceding proof with n replaced by 2^n, $n \in \mathbb{N}$. □

So, equation (12) turns out to be crucial while dealing with the stability problem in connection with equation (2). We terminate this paper with an observation that equation (12) happens also to be essential in answering the following question. As it has already been mentioned in the proof of Theorem 1, equation (2) can be reduced to a generalized isosceles orthogonal additivity equation. What about the converse? Our last result gives a partial answer to that question.

THEOREM 5. *Suppose that either*

(γ) *X is a real linear space with* $\dim X \geqslant 2$, *$(Y, +)$ is an abelian group, Z is a given nonempty set and $\varphi : X \to Z$ is an even mapping satisfying* (i), (ii) *and* (iii),

or

(δ) *$(X, +), (Y, +)$ and $(Z, +)$ are topological groups, $(X, +)$ and $(Y, +)$ are commutative, $(Y, +)$ has no elements of order* 2 *and that $(Z, +)$ is equipped with a connected binary relation $\prec \subset Z \times Z$ having properties* (a), (b) *and $\varphi : X \to Z$ is a continuous mapping satisfying* (3) *and such that for all $x \in X$ we have $\varphi(0) \prec \varphi(x)$ or $\varphi(0) = \varphi(x)$.*

If φ admits a solution $\Phi : Z \to Z$ of equation (12), *then $f : X \to Y$ is an odd solution to the functional equation*

$$(13)\quad \varphi(x+y) = \varphi(x-y) \quad \textit{implies} \quad f(x+y) = f(x) + f(y)\,, \quad x, y \in X\,,$$

if and only if f is a group homomorphism.

PROOF. Clearly, only the "only if" part requires a motivation. Assume that $f : X \to Y$ is an odd solution of (13). First we are going to show that each of the assumptions (γ) and (δ) separately implies the equality

$$(14)\qquad f(2x) = 2f(x) \quad \text{for all} \quad x \in X\,.$$

Indeed, assume (γ) and observe that for an arbitrarily fixed $x \in X$ axiom (iii) applied for $\lambda = 1$ jointly with the evenness of φ ensures the existence of a $y \in X$ such that $\varphi(x+y) = \varphi(x-y)$ and $\varphi(2x) = \varphi(2y)$. Hence, using (13) and the oddness of f, we obtain

$$f(2x) = f((x+y)+(x-y)) = f(x+y) + f(x-y)$$
$$= f(x) + f(y) + f(x) + f(-y) = 2f(x).$$

Assume (δ) and fix an $x \in X$ arbitrarily. If $\varphi(x) = \varphi(0)$ then $\varphi(2x) = \varphi(0)$ as well, because of (12). Hence, setting $y = x$ in (13) we get (14). Suppose now that $\varphi(0) \prec \varphi(x)$. Then assumption (3) applied for $x = 0$ and with y replaced by x, states that the set

$$T_x := \{t \in X : \ \varphi(t) = \varphi(-t) = \varphi(x)\}$$

is nonempty and connected. Obviously, $t \in T_x$ if and only if $-t \in T_x$, and the function $\psi : T_x \to Z$ defined by the formula

$$\psi(t) := \varphi(x+t) - \varphi(x-t), \quad t \in T_x,$$

is odd and continuous. This forces ψ to vanish at some point $y \in T_x$ which means that $\varphi(x+y) = \varphi(x-y)$. Moreover, $\varphi(x) = \varphi(y)$ in virtue of the definition of T_x, whence with the aid of (12) we get $\varphi(2x) = \varphi(2y)$. Now, it suffices to repeat the reasoning applied in case (γ). Thus (14) has been proved.

Now, we are in a position to show that the conditional functional equation (2) is a consequence of (13). In fact, fix $x, y \in X$ with $\varphi(x) = \varphi(y)$ and put $u := x + y$, $v := x - y$. Since, by (12), $\varphi(2x) = \varphi(2y)$, we conclude that $\varphi(u+v) = \varphi(u-v)$ getting

$$2f(x) + 2f(y) = f(2x) + f(2y) = f(u+v) + f(u-v) = f(u) + f(v)$$
$$+ f(u) + f(-v) = 2f(u) = 2f(x+y) = f(2(x+y))$$
$$= f(2x+2y).$$

Now each of the assumptions (γ) and (δ) separately implies the equality

$$f(x+y) = f(x) + f(y).$$

Indeed, if (γ) is assumed, it suffices to observe that the function $X \ni x \longmapsto F(x) := f(2x) \in Y$ is additive on account of Theorem 1. Obviously, so is also the function $X \ni x \longmapsto F(\frac{1}{2}x) = f(x) \in Y$.

Finally, if (δ) holds true, then the assumptions upon $(Y, +)$ jointly with the equality

$$2f(x) + 2f(y) = 2f(x+y)$$

imply the additivity of f and it remains to apply Theorem 2. Thus the proof has been finished. □

Acknowledgement. The research of the first author has been supported by KBN grant No. 2 P03A 049 09 (Poland).

References

[1] C. Alsina, J. L. Garcia-Roig, *On a conditional Cauchy equation on rhombuses*, Functional analysis, approximation theory and numerical analysis **5–7**, World Sci. Publishing, River Edge, NJ, 1994.

[2] R. C. James, *Orthogonality in normed linear spaces*, Duke Math. J. **12** (1945), 291-302.

[3] Gy. Szabó, *A conditional Cauchy-equation on normed spaces*, Publicationes Math., Debrecen **42** (1993), 256-271.

Institute of Mathematics
Silesian University
ul. Bankowa 14
40–007 Katowice
Poland