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A GENERALIZED a–WRIGHT CONVEXITY AND RELATED FUNCTIONAL EQUATION

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Abstract. Let I be an interval and $M, N : I \times I \to I$ some means with the strict internality property. Suppose that $\varphi : I \to \mathbb{R}$ is a non–constant and continuous solution of the functional equation

$$\varphi(M(x,y)) + \varphi(N(x,y)) = \varphi(x) + \varphi(y).$$

Then φ is one–to–one; moreover for every lower semicontinuous function $f : I \to \mathbb{R}$ satisfying the inequality

$$f(M(x,y)) + f(N(x,y)) \leq f(x) + f(y),$$

the function $f \circ \varphi^{-1}$ is convex on $\varphi(I)$. This is a generalization of an earlier result of Zs. Páles. An application to the a–Wright convex function is given.

1. Introduction

Let $I \subset \mathbb{R}$ be an interval and $a \in (0,1)$ a fixed number. A function $f : I \to \mathbb{R}$ is said to be a–Wright convex if, for all $x, y \in I$,

$$f(ax + (1-a)y) + f((1-a)x + ay) \leq f(x) + f(y). \tag{1}$$

It is shown in [3] that every lower semicontinuous a–Wright convex function is convex.

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Clearly, every linear function f converts (1) into equality. In this connection let us note that Zs. Páles [4] found a close relation between the more general functional inequality

$$f(M(x,y)) + f(N(x,y)) \leq f(x) + f(y), \qquad x, y \in I, \tag{2}$$

and the corresponding functional equation

$$\varphi(M(x,y)) + \varphi(N(x,y)) = \varphi(x) + \varphi(y), \qquad x, y \in I, \tag{3}$$

where $M, N : I \times I \to I$ are continuous functions satisfying the following strict internality condition

$$x, y \in I,\ x \neq y \Rightarrow M(x,y), N(x,y) \in (\min(x,y), \max(x,y)), \tag{4}$$

(in particular, M and N are means on I). He proved that: *if there exists a continuous strictly monotonic solution $\varphi : I \to \mathbb{R}$ of (3), then a continuous function $f : I \to \mathbb{R}$ satisfies (2) if, and only if, $f \circ \varphi^{-1}$ is a convex function on $\varphi(I)$*. In this note we show that this result remains true if φ is non–constant and continuous, and f lower semicontinuous.

2. Main result

The following result improves the result of Páles [4]

THEOREM. *Let $M, N : I \times I \to I$ be continuous functions satisfying condition (4), and suppose that $\varphi : I \to \mathbb{R}$ is a non–constant and continuous solution of equation (3). Then φ is one–to–one, and for every lower semicontinuous function $f : I \to \mathbb{R}$ satisfying inequality (2), the function $f \circ \varphi^{-1}$ is convex on $\varphi(I)$.*

PROOF. Put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Define $M_k, N_k : I \times I \to I,\ k \in \mathbb{N}_0$, by

$$M_0(x,y) := M(x,y), \qquad N_0(x,y) := N(x,y),$$

$$M_{k+1}(x,y) := M(M_k(x,y), N_k(x,y)),$$
$$N_{k+1}(x,y) := N(M_k(x,y), N_k(x,y)),$$

and $m_k, n_k : I \times I \to I,\ k \in \mathbb{N}_0$,

$$m_k(x,y) := \min((M_k(x,y)), (N_k(x,y)),$$
$$n_k(x,y) := \max((M_k(x,y)), (N_k(x,y)).$$

Of course all the functions M_k, N_k, m_k, n_k are continuous. As M and N are means we have

$$m_0(x,y) \leq m_1(x,y) \leq \ldots \leq m_k(x,y) \leq n_k(x,y) \\ \leq \ldots \leq n_1(x,y) \leq n_0(x,y),$$

and

$$M_k(x,y), N_k(x,y) \in [m_k(x,y), n_k(x,y)], \tag{6}$$

for all $k \in \mathbb{N}_0$ and $x, y \in I$. It follows that the sequences (m_k) and (n_k) converge on $I \times I$. Thus there exist $m_\infty, n_\infty : I \times I \to I$ such that

$$\lim_{k\to\infty} m_k(x,y) =: m_\infty(x,y) \leq n_\infty(x,y) := \lim_{k\to\infty} n_k(x,y),$$

for all $x, y \in I$. Since the functions of both sequences are continuous, (m_k) is increasing and (n_k) is decreasing, the function m_∞ is lower semicontinuous, and n_∞ is upper semicontinuous on $I \times I$. Suppose that there are $x, y \in I$ such that $m_\infty(x,y) < n_\infty(x,y)$. Hence, as M and N are the strict means, we would get

$$M(m_\infty(x,y), n_\infty(x,y)),\ N(m_\infty(x,y), n_\infty(x,y)), \in (m_\infty(x,y), n_\infty(x,y)).$$

Now the continuity of M and N implies that for sufficiently large k

$$M(M_k(x,y), N_k(x,y)), N(M_k(x,y), N_k(x,y)) \in (m_\infty(x,y), n_\infty(x,y)),$$

i.e.

$$M_{k+1}(x,y), N_{k+1}(x,y) \in (m_\infty(x,y), n_\infty(x,y)).$$

Hence, by the definition of the sequences (m_k) and (n_k),

$$m_{k+1}(x,y), n_{k+1}(x,y) \in (m_\infty(x,y), n_\infty(x,y)),$$

for sufficiently large k which is a contradiction. This proves that for all $x, y \in I$

$$m_\infty(x,y) = n_\infty(x,y).$$

Define $K : I \times I \to I$ by

$$K(x,y) := m_\infty(x,y), \qquad x, y \in I.$$

The function K, being lower and upper semicontinuous, is continuous. The pointwise convergence of the sequences (M_k) and (N_k) to K is a consequence of relation (6). Take $x, y \in I$ and $x \neq y$. Without any loss of generality we can assume that $x < y$. Then

$$x < M(x,y) < y, \qquad x < N(x,y) < y.$$

Since

$$\min(M(x,y), N(x,y)) \leq K(x,y) \leq \max(M(x,y), N(x,y)),$$

we infer that K has strict internality property.

The definitions of $(M_k), (N_k)$, and relation (2) and (3), by an obvious induction imply, that for all $k \in \mathbb{N}$

$$f(M_k(x,y)) + f(N_k(x,y)) \leq f(x) + f(y), \qquad x, y \in I,$$

and

$$\varphi(M_k(x,y)) + \varphi(N_k(x,y)) = \varphi(x) + \varphi(y), \qquad x, y \in I.$$

Letting k tend to the infinity, and making use of the lower semicontinuity of f, the continuity of φ, and the relation

$$\lim_{k \to \infty} M_k(x,y) = K(x,y) = \lim_{k \to \infty} N_k(x,y),$$

which is a consequence of (6), we hence get

$$(7) \qquad 2f(K(x,y)) \leq f(x) + f(y), \qquad x, y \in I,$$

and

$$(8) \qquad 2\varphi(K(x,y)) = \varphi(x) + \varphi(y), \qquad x, y \in I.$$

Suppose that there are $a, b \in I$, $a \neq b$, such that $\varphi(a) = \varphi(b)$, and put

$$C := \{x \in I : \ \varphi(x) = \varphi(a)\}.$$

By the continuity of φ, the set C is closed in I. Note that C is an interval. In the opposite case we could find $a_1, b_1 \in C$, $a_1 < b_1$, such that $\varphi(x) \neq \varphi(a)$ for all $x \in [a_1, b_1]$. Setting in equation (8) $x = a_1, y = b_1$ we would get

$$2\varphi(K(a_1, b_1)) = \varphi(a_1) + \varphi(b_1) = 2\varphi(a),$$

i.e. $\varphi(K(a_1, b_1)) = \varphi(a)$, which according to the choice of the interval $[a_1, b_1]$ is impossible. Now the continuity of K and the property (6) easily imply that $C = I$. This contradiction proves that φ is one–to–one.

From (8) we have

$$K(x,y) = \varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right), \qquad x,y \in I.$$

Substituting this into (7) we get

$$2f\left[\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right)\right] \le f(x)+f(y), \qquad x,y \in I.$$

Setting here $x := \varphi^{-1}(s)$, $y := \varphi^{-1}(t)$, for $s,t \in \varphi(I)$ gives the Jensen convexity of the function $f \circ \varphi^{-1}$ on the interval $\varphi(I)$. This function is lower semicontinuous as the composition of the continuous function φ and lower semicontinuous function f. It follows that $f \circ \varphi^{-1}$ is convex (cf. for instance [1], Chapter I, Cor. 2.5). This completes the proof.

COROLLARY. *Let $I \subset \mathbb{R}$ be an interval and $a \in (0,1)$ a fixed number. If $f : I \to \mathbb{R}$ is lower semicontinuous and*

$$f(ax+(1-a)y) + f((1-a)x+ay) \le f(x)+f(y)$$

for all $x,y \in I$, then f is convex (and continuous).

PROOF. Since the function $\varphi : I \to \mathbb{R}$, $\varphi := \mathrm{id}|_I$ is a non–constant and continuous solution of the functional equation

$$f(ax+(1-a)y) + f(1-a)x+ay) = f(x)+f(y), \qquad x,y \in I,$$

and the functions $M, N : I \times I \to I$, defined by

$$M(x,y) := ax+(1-a)y, \quad N(x,y) := (1-a)x+ay, \qquad x,y \in I,$$

are continuous means with the strict internality property, the result follows from the above theorem.

REMARK. The a–Wright convex functions appear in a natural way in connection with the converse of the Minkowski inequality (cf. [3]). Note that in [2] (answering to the question posed in [3]) it was shown that there exist a–Wright convex functions which are not Jensen convex.

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