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A LAGRANGE-TYPE INCREMENT INEQUALITY

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Abstract. We prove an extension of Lagrange's increment inequality without using Lagrange's mean value theorem and the Hahn-Banach theorems.

1. Introduction

Let X and Y be normed spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and for $a, b \in X$ define

$$[a, b] = \{ \lambda a + (1 - \lambda) b : \quad 0 \leq \lambda \leq 1 \}.$$

Because of [1, p. 23], it is certainly well-known that the following theorem can be proved directly without using Lagrange's mean value theorem and the Hahn-Banach theorems.

THEOREM 1. *If f is a function from a subset D of X into Y and $a, b \in D$, with $a \neq b$, such that $[a, b] \subset D^\circ$ and f is differentiable at each point of $[a, b]$, then*

$$\frac{| f(b) - f(a) |}{| b - a |} \leq \sup_{x \in [a, b]} | f'(x) | .$$

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However, it seems to be overlooked that, by introducing the absolute infimum derivative

$$f^{\#}(x,y) = \inf_{t\in D\cap]x,y]} \frac{|f(t)-f(x)|}{|t-x|},$$

a slight modification of the same direct proof can be used to prove a much more general theorem.

2. The absolute infimum derivative

To get rid of the differentiability condition in Theorem 1, it seems convenient to introduce the following

DEFINITION. If f is a function from a subset D of X into Y, and $x \in D$ and $y \in X$, then the extended real number

$$f^{\#}(x,y) = \inf_{t\in D\cap]x,y]} \frac{|f(t)-f(x)|}{|t-x|},$$

where $]x,y] = [x,y]\setminus\{x\}$, will be called the absolute infimum derivative of f at x relative to $[x,y]$.

REMARK 1. Recall that $\inf \emptyset = +\infty$, and therefore $f^{\#}(x,y) = +\infty$ if $D\cap]x,y] = \emptyset$.

The relationship of the absolute infimum derivative with the directional and total derivatives can be cleared up by the next

PROPOSITION 1. *If f is a function from a subset D of X into Y and $x \in D$ and $y \in X\setminus\{x\}$ such that f is differentiable at x in the direction $y-x$, then*

$$f^{\#}(x,y) \le \frac{1}{|y-x|}\,|f'_{y-x}(x)|.$$

PROOF. If $z = y-x$ and $t_\lambda = x+\lambda z$ for $\lambda > 0$, then

$$f'_z(x) = \lim_{\lambda\to 0} \lambda^{-1}\big(f(t_\lambda)-f(x)\big).$$

Therefore, for each $\varepsilon > 0$, there exists a $\lambda \in]0,1]$ such that

$$\big|f(t_\lambda) - f(x) - \lambda f'_z(x)\big| < \varepsilon\lambda.$$

And hence, by the triangle inequality, it follows that

$$|f(t_\lambda)-f(x)| < \big(|f'_z(x)|+\varepsilon\big)\lambda .$$

Now, since $\lambda=|z|^{-1}|t_\lambda-x|$ and $t_\lambda \in]x, y]$, it is clear that

$$f^{\#}(x,y) \le \big(|f'_z(x)|+\varepsilon\big)|z|^{-1}.$$

Therefore, the inequality

$$f^{\#}(x,y) \le |f'_z(x)||z|^{-1}$$

is also true.

Now, as a useful consequence of this proposition, we can also state

PROPOSITION 2. *If f is a function from a subset D of X into Y and $x \in D^\circ$ such that f is differentiable at x, then*

$$f^{\#}(x,y) \le |f'(x)|$$

for all $y \in X \setminus \{x\}$.

PROOF. Recall that now we have

$$|f'_{y-x}(x)| = |f'(x)(y-x)| \le |f'(x)||y-x|$$

for all $y \in X \setminus \{x\}$, and thus Proposition 1 can be applied.

REMARK 2. Note that, according to the ideas of [7], the condition $x \in D^\circ$ should be weakened.

3. A Lagrange-type inequality

Now, as a substantial generalization of Theorem 1, we can also prove

THEOREM 2. *If f is a function from a subset D of X into Y and $a, b \in D$, with $a \neq b$, such that*

(1) $x \in \big(D\cap]a,x[\big)'$ *implies* $x \in D$ *for all* $x \in]a,b[$,

(2) $|f(x)-f(a)| \le \varliminf_{\substack{t\to x\\ t\in D\cap]a,x[}} |f(t)-f(a)|$ *for all* $x \in]a,b]$ *with*

$x \in (D\cap]a,x[)'$, *then*

$$\frac{|f(b)-f(a)|}{|b-a|} \leq \sup_{x\in D\cap[a,b[} f^{\#}(x,b)\,.$$

PROOF. Assume that

$$M = \sup_{x\in D\cap[a,b[} f^{\#}(x,b) < +\infty\,,$$

and for $\varepsilon > 0$ define

$$A = \left\{ x \in D \cap [a,b] \;:\; |f(x)-f(a)| \leq (M+\varepsilon)\,|x-a| \right\}.$$

Then, it is clear that $\{a\} \subset A \subset [a,b]$.

Moreover, we can also show that there exists a $c \in A$ such that

$$|c-b| = d(A,b) = \inf_{x\in A} |x-b|\,.$$

Namely, if this is not the case, then $|x-b| > d(A,b)$ for all $x \in A$. Therefore, by induction, we can find a sequence (a_n) in A such that

$$d(A,b) < |a_n - b| < \min\left\{ |a_{n-1}-b|,\, d(A,b)+n^{-1} \right\}$$

for all $n > 1$. Hence, it is clear that the sequence $(|a_n - b|)$ strictly decreasingly converges to $d(A,b)$.

Moreover, since $A \subset [a,b]$ and $[a,b]$ is compact, there exists a subsequence (x_n) of (a_n) and a point $x_0 \in [a,b]$ such that $x_n \to x_0$. Clearly, the sequence $(|x_n - b|)$, being a subsequence of $(|a_n - b|)$, also strictly decreasingly converges to $d(A,b)$. Moreover, now we also have $|x_n - b| \to |x_0 - b|$. Therefore $d(A,b) = |x_0 - b|$.

Now, to get a contradiction, we we need only show that $x_0 \in A$ also holds. For this, note that the properties x_n, $x_0 \in [a,b]$ and $|x_n - b| > |x_0 - b|$ imply that $x_n \in [a, x_0[$. And the properties $x_n \in D\cap[a,x_0[$ and $x_n \to x_0$ imply that $x_0 \in (D\cap\,]a,x_0[)'$. Therefore, by the conditions (1) and $b \in D$, we have $x_0 \in D$.

Moreover, note that the property $x_n \in A \cap [a, x_0[$ implies that

$$|f(x_n)-f(a)| \leq (M+\varepsilon)\,|x_n - a| \leq (M+\varepsilon)\,|x_0 - a|\,.$$

Hence, since $x_n \to x_0$, it is clear that

$$\inf_{\substack{|t-x_0|<r \\ t\in D\cap\,]a,x_0[}} |f(t)-f(a)| \leq (M+\varepsilon)\,|x_0 - a|$$

for all $r > 0$. Therefore, we also have

$$\varlimsup_{\substack{t \to x_0 \\ t \in D \cap]a, x_0[}} |f(t) - f(a)| \leq (M + \varepsilon)|x_0 - a|.$$

Hence, because of the condition (2), it is clear that $x_0 \in A$. And this contradicts the assumption that $|x - b| > d(A, b)$ for all $x \in A$.

Now, having proved that $d(A, b) = |c - b|$ for some $c \in A$, it is easy to show that necessarily $c = b$ holds.

Namely, if $c \neq b$, then $c \in A \setminus \{b\}$, and hence $c \in D \cap [a, b[$. Therefore

$$f^{\#}(c, b) < M + \varepsilon,$$

and thus there exists an $x \in D \cap]c, b]$ such that

$$|f(x) - f(c)| < (M + \varepsilon)|x - c|.$$

Hence since

$$|f(c) - f(a)| \leq (M + \varepsilon)|c - a|$$

and $|x - c| + |c - a| = |x - a|$, it is clear that

$$|f(x) - f(a)| \leq (M + \varepsilon)|x - a|.$$

Therefore $x \in A$. And this contradicts the fact that $d(A, b) = |c - b|$.

Finally, to complete the proof, we note that if $c = b$, then $b \in A$, and hence

$$|f(b) - f(a)| \leq (M + \varepsilon)|b - a|.$$

Therefore, the inequality

$$|f(b) - f(a)| \leq M|b - a|$$

is also true.

REMARK 3. Note that the conditions (1) and (2) are trivially fulfilled if $D \cap [a, b]$ is closed and the restriction of f to $D \cap]a, b]$ is continuous.

Note that thus, for any $a, b \in X$ with $a \neq b$, D may be a finite subset of $[a, b]$ with $a, b \in D$, and f may be an arbitrary function from D into Y.

4. A partial strengthening of Theorem 2

Besides Theorem 2, it seems to be of some interest to prove the following more particular

THEOREM 3. If in addition to the conditions of Theorem 2, we have

$$\inf_{x \in D \cap]a,b[} |f(x) - f(a)| = 0, \tag{3}$$

then

$$\frac{|f(b) - f(a)|}{|b-a|} \leq \sup_{x \in D \cap]a,b[} f^{\#}(x, b).$$

PROOF. Because of the condition (3), for each $\varepsilon > 0$, there exists a $c \in]a, b[$ such that

$$|f(c) - f(a)| < \varepsilon.$$

Moreover, by using Theorem 2, it is easy to see that

$$\frac{|f(b) - f(c)|}{|b-c|} \leq \sup_{x \in D \cap [c,b[} f^{\#}(x, b) \leq \sup_{x \in D \cap]a,b[} f^{\#}(x, b).$$

And hence, since

$$\frac{|f(b) - f(a)|}{|b-a|} \leq \frac{|f(b) - f(c)|}{|b-c|} + \frac{|f(c) - f(a)|}{|b-a|},$$

it is clear that

$$\frac{|f(b) - f(a)|}{|b-a|} < \sup_{x \in D \cap]a,b[} f^{\#}(x, b) + \frac{\varepsilon}{|b-a|}.$$

Therefore, the stated inequality is also true.

REMARK 4. Note that the additional condition (3) is trivially fulfilled if $a \in (D \cap]a, b[)'$ and the restriction of f to $D \cap [a, b[$ is continuous at a.

Note that now, for any $a, b \in X$ with $a \neq b$, D may be a finite subset of $[a, b]$ with $a, b \in D$, and f may be a function from D into Y such that $f(x) = f(a)$ for some $x \in D \setminus \{a, b\}$.

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