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ABSTRACT BOCHNER AND MCSHANE INTEGRALS

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Abstract. A short description of the classical Bochner integral is presented together with the McShane concept of integration based on Riemann type integral sums. The corresponding classes are compared and it will be shown that the situation is different for finite– and infinite – dimensional valued vector functions.

Preliminaries

Assume that $[a,b] \subset \mathbb{R}$ is given and that μ is a (nonnegative) measure on $[a,b]$. Let Y be a Banach space with the norm $\|\cdot\|_Y$.

A function $f : [a,b] \to Y$ is called *simple* if there is a finite sequence $I_m \subset [a,b]$, $m = 1, \ldots, p$ of measurable sets such that

$$I_m \cap I_l = \emptyset \quad \text{for } m \neq l$$

and

$$[a,b] = \bigcup_{m=1}^{p} I_m$$

where

$$f(t) = y_m \in Y \quad \text{for } t \in I_m,$$

i.e. f is constant on the measurable set I_m.

Denote by $\mathcal{J}(\mu, Y) = \mathcal{J}(\mu) = \mathcal{J}$ the set of all simple mappings defined on $[a,b]$.

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Clearly $\mathcal{J}$ is a linear space and if f is a simple function then also $\|f\|$: $[a,b] \to \mathbb{R}$ is a simple function.

The abstract Bochner integral

We define the *integral of a simple function* $f : [a,b] \to Y$ in the following natural way

$$\int_a^b f d\mu = \sum_{m=1}^{p} y_m \mu(I_m). \tag{1}$$

If $A \subset [a,b]$ is measurable and the function f is simple, then we define

$$f_A(t) = f(t) \text{ if } t \in A$$

and

$$f_A(t) = 0 \text{ if } t \notin A.$$

The function f_A is simple and we set

$$\int_A f d\mu = \int_a^b f_A d\mu.$$

The integral of simple functions $f \in \mathcal{J}$ defined in this way is evidently a linear mapping $\int : \mathcal{J} \to Y$.

If A, B are disjoint measurable sets then from the linearity of the integral and from the obvious identity $f_{A\cup B} = f_A + f_B$ we have

$$\int_{A\cup B} f d\mu = \int_A f d\mu + \int_B f d\mu. \tag{2}$$

If $Y = \mathbb{R}$ and $f \leq g$ where $f, g \in \mathcal{J}$, then

$$\int f d\mu \leq \int g d\mu. \tag{3}$$

If $f \geq 0$ and $A \subset B$, then

$$\int_A f d\mu \leq \int_B f d\mu. \tag{4}$$

For the integral of a function $f \in \mathcal{J}$ we have

$$\|\int_A f d\mu\| \leq \int_A \|f\| d\mu \leq \sup_{t\in[a,b]} \|f(t)\|\mu(A) \tag{5}$$

because

$$\left\| \int_A f d\mu \right\| = \left\| \sum_{m=1}^{p} y_m \mu(A \cap I_m) \right\| \leq \sum_{m=1}^{p} \|y_m\| \mu(A \cap I_m)$$

$$\leq \max_m \|y_m\| \sum_{m=1}^{p} \mu(A \cap I_m) = \sup_{t \in [a,b]} \|f(t)\| \mu(A)$$

and $\bigcup_{m=1}^{p} (A \cap I_m) = A$.

For a given $f \in \mathcal{J}$ let us define

$$\|f\|_1 = \int_a^b \|f\| d\mu. \tag{6}$$

For the mapping $\|\cdot\|_1 : \mathcal{J} \to \mathbb{R}$ the following holds:

(a)

$$\|f\|_1 \geq 0 \text{ for every } f \in \mathcal{J},$$

(b)

$$\|af\|_1 = |a| \|f\|_1 \text{ for every } f \in \mathcal{J} \text{ and } a \in \mathbb{R},$$

(c)

$$\|f + g\|_1 \leq \|f\|_1 + \|g\|_1 \text{ for every } f, g \in \mathcal{J}.$$

By $\|\cdot\|_1$ a seminorm on $\mathcal{J}$ is given; the implication $\|f\|_1 = 0 \Rightarrow f = 0$ does not hold, it suffices to take $A \subset [a,b]$ such that $\mu(A) = 0$ and a function f for which $f(t) = 0$ provided $t \notin A$.

The triangle inequality (c) can be shown in such a way that a decomposition of the interval $[a,b]$ into measurable sets is produced with respect to which each of the functions f and g is simple, i.e. f and g have constant values at each measurable component of the decomposition, and the inequality results from the triangle inequality in the Banach space Y.

The seminorm $\|\cdot\|_1$ given above for elements of $\mathcal{J}$ is sometimes called the *L^1– seminorm.*

We will consider the completion of the linear space $\mathcal{J}$ of simple functions on $[a,b]$ with respect to the L^1–seminorm.

1. Definition. The sequence $(f_q) = (f_q)_{q=1}^{\infty}, f_q \in \mathcal{J}$, $q = 1, 2, \ldots$ is called an L^1– Cauchy sequence if for every $\varepsilon > 0$ there is an $N = N_\varepsilon \in \mathbb{N}$ such that

$$\|f_q - f_r\|_1 < \varepsilon \text{ for } q, r \geq N_\varepsilon..$$

The sequence $(f_q), f_q \in \mathcal{J}$, $q = 1, 2, \ldots$ is called an L^1– zero sequence if

$$\lim_{q \to \infty} \|f_q\|_1 = 0.$$

The completion of $\mathcal{J}$ is given as the space of equivalence classes of L^1 Cauchy sequences of functions from $\mathcal{J}$, where two L^1- Cauchy sequences are equivalent if their difference is an L^1- zero sequence.

Let us denote by $\mathcal{L} = \mathcal{L}(\mu)$ the set of all functions $f : [a, b] \to Y$ for which there is an L^1- Cauchy sequence f_q, $q = 1, 2, \dots$ of simple functions which converge to f $\mu-$ almost everywhere in $[a, b]$, i.e.

$$\lim_{q\to\infty} \|f_q(t) - f(t)\|_Y = 0$$

for $\mu-$ almost all $t \in [a, b]$.

$\mathcal{L}$ has the structure of a linear space, i.e. if (f_q) and (g_q) are L^1- Cauchy sequences of simple functions which converge $\mu-$ almost everywhere to f and g, respectively, then $(f_q + g_q)$ and (af_q) are L^1- Cauchy sequences of simple functions converging $\mu-$ almost everywhere to $f + g$ and af, respectively ($a \in \mathbb{R}$ is an arbitrary number).

2. FUNDAMENTAL LEMMA. *Let (f_q) be an L^1- Cauchy sequence of simple functions defined on $[a, b]$. Then there is a subsequence, which converges pointwise $\mu-$ almost everywhere and for every $\varepsilon > 0$ there is a $Z \subset [a, b]$ with $\mu(Z) < \varepsilon$ such that this subsequence converges absolutely and uniformly on $[a, b] \setminus Z$.*

PROOF. Since the sequence (f_q) is L^1- Cauchy, for every $k \in \mathbb{N}$ there is $N_k \in \mathbb{N}$ such that if $q, r \geq N_k$, then

$$\|f_q - f_r\|_1 < \frac{1}{2^{2k}}.$$

It can be assumed that $N_k < N_{k+1}$. We set

$$g_k = f_{N_k};$$

then

$$\|g_m - g_n\|_1 = \|f_{N_m} - f_{N_n}\|_1 < \frac{1}{2^{2n}}$$

for $m \geq n$.

We will show that the series

$$g_1(t) + \sum_{k=1}^{\infty} (g_{k+1}(t) - g_k(t))$$

converges absolutely for $\mu-$ almost all $t \in [a, b]$ to an element in Y and that this convergence is uniform except a set with arbitrarily small $\mu-$ measure.

Denote

$$M_n = \{t \in [a,b]; \|g_{n+1}(t) - g_n(t)\|_Y \geq \frac{1}{2^n}\}.$$

Then we have

$$\frac{1}{2^n}.\mu(M_n) = \int_{M_n} \frac{1}{2^n} d\mu \leq \int_{M_n} \|g_{n+1}(t) - g_n(t)\|_Y d\mu$$
$$\leq \int_a^b \|g_{n+1}(t) - g_n(t)\|_Y d\mu = \|g_{n+1} - g_n\|_1 < \frac{1}{2^{2n}}$$

and this yields

$$\mu(M_n) < \frac{1}{2^n}.$$

Let us define

$$Z_n = M_n \cup M_{n+1} \cup \dots .$$

Then $Z_{n+1} \subset Z_n$ and

$$\mu(Z_n) \leq \sum_{j=1}^{\infty} \mu(M_j) < \sum_{j=1}^{\infty} \frac{1}{2^j} = \frac{1}{2^{n+1}}.$$

For $t \notin Z_n$ and $k \geq n$ we have

$$\|g_{k+1}(t) - g_k(t)\|_Y < \frac{1}{2^k}$$

and therefore the series $\sum_{k=n}^{\infty}(g_{k+1}(t) - g_k(t))$ converges absolutely and uniformly for $t \notin Z_n$.

Putting $Z = Z_k$, we have for sufficiently large k

$$\mu(Z) = \mu(Z_k) < \frac{1}{2^{k-1}} < \varepsilon$$

and this leads to the assertion on the absolute and uniform convergence.

If we take $M = \cap Z_n$, then evidently $\mu(M) = 0$ and if $t \notin M$, then $t \notin Z_n$ for some n. Therefore the series $g_1(t) + \sum_{k=1}^{\infty}(g_{k+1}(t) - g_k(t))$ converges for $t \notin M$ and this means that $\lim_{k\to\infty} g_k(t) = \lim_{k\to\infty} f_{N_k}(t)$ exists for μ–almost all $t \in [a,b]$.

3. LEMMA. *Assume that (f_q) and (g_q) are L^1– Cauchy sequences of simple functions, which converge μ– almost everywhere to a function $f : [a,b] \to Y$. Then the limits $\lim_{q\to\infty} \int_a^b f_q d\mu$ and $\lim_{q\to\infty} \int_a^b g_q d\mu$ exist and*

$$\lim_{q\to\infty} \int_a^b f_q d\mu = \lim_{q\to\infty} \int_a^b g_q d\mu. \tag{7}$$

PROOF. It is easy to show the existence of the limits. Indeed, for simple functions f_q we have

$$\begin{aligned} \| \int_a^b f_q d\mu - \int_a^b f_r d\mu \|_Y &= \| \int_a^b (f_q - f_r) d\mu \|_Y \\ &\leq \int_a^b \|f_q - f_r\|_Y d\mu = \|f_q - f_r\|_1. \end{aligned}$$

This means that the sequence of integrals $\int_a^b f_q d\mu$ is a Cauchy sequence in the Banach space Y, and therefore it is convergent, i.e. the limit $\lim_{q\to\infty} \int_a^b f_q d\mu$ exists and similarly also for $\lim_{q\to\infty} \int_a^b g_q d\mu$.

Let us set $h_q = f_q - g_q$. The sequence h_q is L^1- Cauchy and $\lim_{q\to\infty} h_q(t) = 0$ for μ- almost all $t \in [a,b]$. This implies that the sequence of integrals $\int_a^b h_q d\mu$ is convergent. It remains to show that

$$\lim_{q\to\infty} \int_a^b h_q d\mu = 0.$$

To a given $\varepsilon > 0$ choose $N \in \mathbb{N}$ so that for $r, q \geq N$ we have

$$\|h_q - h_r\|_1 < \varepsilon.$$

Define

$$M = \{t \in [a,b]; h_N(t) \neq 0\} \subset [a,b].$$

For $q \geq N$ we have

$$\begin{aligned} \int_{[a,b]\setminus M} \|h_q\|_Y d\mu &= \int_{[a,b]\setminus M} \|h_q - h_N\|_Y d\mu \\ &\leq \int_a^b \|h_q - h_N\|_Y d\mu = \|h_q - h_N\|_1 < \varepsilon \end{aligned}$$

because $h_N(t) = 0$ for $t \in [a,b] \setminus M$. By the Fundamental Lemma 2 there exists a subset $Z \subset M$ with

$$\mu(Z) < \frac{\varepsilon}{\sup\limits_{t\in[a,b]} \|h_N(t)\|_Y + 1}$$

and a subsequence h_{q_s} which converges to zero uniformly on the set $M \setminus Z$. Hence there is an $s_0 \in \mathbb{N}, s_0 \geq N$ such that for $s \geq s_0$ anf for $t \in M \setminus Z$ we have

$$\|h_{q_s}(t)\| < \frac{\varepsilon}{\mu([a,b])}.$$

Therefore

$$\int_{M\setminus Z} \|h_{q_s}(t)\| d\mu < \frac{\varepsilon\mu(M\setminus Z)}{\mu([a,b])} \leq \varepsilon$$

provided $s \geq s_0$. For $s \geq s_0$ we also have

$$\begin{aligned}\int_Z \|h_{q_s}(t)\| d\mu \leq & \int_Z \|h_{q_s}(t) - h_N(t)\| d\mu + \int_Z \|h_N(t)\| d\mu \\ \leq & \|h_{q_s} - h_N\|_1 + \sup_{t\in[a,b]} \|h_N(t)\| \mu(Z) \\ < & \varepsilon + \frac{\varepsilon}{\sup_{t\in[a,b]} \|h_N(t)\|_Y + 1} \sup_{t\in[a,b]} \|h_N(t)\|_Y < 2\varepsilon.\end{aligned}$$

Hence

$$\begin{aligned}\|h_{q_s}\|_1 = & \int_a^b \|h_{q_s}(t)\| d\mu \\ = & \int_{[a,b]\setminus M} \|h_{q_s}(t)\| d\mu + \int_{M\setminus Z} \|h_{q_s}(t)\| d\mu + \int_Z \|h_{q_s}(t)\| d\mu \\ < & \varepsilon + \varepsilon + 2\varepsilon = 4\varepsilon,\end{aligned}$$

i.e. $\lim_{s\to\infty} \int_a^b h_{q_s}(t) d\mu = 0$ and therefore also $\lim_{q\to\infty} \int_a^b h_q(t) d\mu = 0$.

4. Definition. For $f \in \mathcal{L}$ we define

$$\int_a^b f d\mu = \lim_{q\to\infty} \int_a^b f_q d\mu \tag{8}$$

where (f_q) is an arbitrary L^1– Cauchy sequence of simple functions which converge μ– almost everywhere in $[a,b]$ to $f \in \mathcal{L}$.

The value $\int_a^b f d\mu$ given by (8) is called the *Bochner integral of the function* f. In some cases the more extensive notation $(\mathcal{L}) \int_a^b f d\mu$ will be used for this concept of integral.

By (1) the integral was defined in a very natural way for simple functions. By (8) this integral is extended to functions $f \in \mathcal{L}^1$.

The correctness of this definition is clear by Lemma 3 because by this Lemma the integral of a function $f \in \mathcal{L}$ defined by (8) does not depend on the choice of the L^1– Cauchy sequence of simple functions which converge μ– almost everywhere in $[a,b]$ to the function f.

5. LEMMA. *If $f \in \mathcal{L}$ and (f_q) is the L^1- Cauchy sequence of simple functions which corresponds to f, then $\|f\|_Y$ is integrable and the sequence $(\|f_q\|_Y)$ approximates $\|f\|_Y$. In this case we have*

$$\int_a^b \|f\|_Y d\mu = \lim_{q\to\infty} \int_a^b \|f_q\|_Y d\mu = \lim_{q\to\infty} \|f_q\|_1. \tag{9}$$

Moreover

$$\| \int_a^b f d\mu \|_Y \le \int_a^b \|f\|_Y d\mu. \tag{10}$$

PROOF. Since

$$|\|f_q(t)\|_Y - \|f_r(t)\|_Y| \le \|f_q(t) - f_r(t)\|_Y,$$

we get

$$\begin{aligned} \|\|f_q\|_Y - \|f_r\|_Y\|_1 &= \int_a^b |\|f_q(t)\|_Y - \|f_r(t)\|_Y| d\mu \\ &\le \int_a^b \|f_q(t) - f_r(t)\|_Y d\mu = \|f_q - f_r\|_1 \end{aligned}$$

and this means that the sequence $\|f_q\|_Y$ of real – valued simple functions is L^1- Cauchy. Moreover

$$\lim_{q\to\infty} \|f_q(t)\|_Y = \|f(t)\|_Y$$

for $\mu-$ almost all $t \in [a, b]$ and consequently $\|f\|_Y$ is integrable.
Since by (5) for $f_q \in \mathcal{J}$ we have

$$\| \int_A f_q d\mu \|_Y \le \int_A \|f_q\|_Y,$$

(8) and (9) can be used for obtaining (10) by passing to the limits with $q \to \infty$ on both sides of this inequality.

From Lemma 3 we know that $\lim_{q\to\infty} \|f_q\|_1$ does not depend on the choice of the sequence (f_q) which approximates f; therefore the seminorm defined for simple functions $f \in \mathcal{J}$ can be extended to functions $f \in \mathcal{L}$ by the relation

$$\|f\|_1 = \int_a^b \|f(t)\|_Y d\mu = \lim_{q\to\infty} \|f_q\|_1.$$

6. LEMMA. *If $f \in \mathcal{L}$ then for every $\varepsilon > 0$ there is a simple function $g_\varepsilon \in \mathcal{J}$ such that*

$$\|f - g_\varepsilon\|_1 < \varepsilon, \tag{11}$$

i.e. the set $\mathcal{J}$ of simple functions is dense in $\mathcal{L}$ with respect to the seminorm $\|\cdot\|_1$.

PROOF. Since $f \in \mathcal{L}$ there is an L^1– Cauchy sequence (f_q) of elements $f_q \in \mathcal{J}$ which converges μ– almost everywhere to f, i.e. given $\varepsilon > 0$ there is $N_\varepsilon \in \mathbb{N}$ such that

$$\|f_r - f_q\|_1 < \varepsilon \tag{12}$$

provided $r, q > N_\varepsilon$. Let us fix $r > N_\varepsilon$ and put $g_\varepsilon = f_r \in \mathcal{J}$. Then (g_q) where $g_q = f_q - f_r = f_q - g_\varepsilon \in \mathcal{J}$ is L^1– Cauchy and $g_q \to f - f_r = f - g_\varepsilon$ μ– almost everywhere in $[a, b]$. Hence by (12) we have

$$\|f - g_\varepsilon\|_1 = \|f - f_r\|_1 = \lim_{q\to\infty} \|g_q\|_1 = \lim_{q\to\infty} \|f_q - f_r\|_1 < \varepsilon$$

and (11) is satisfied.

7. LEMMA. *The space $\mathcal{L}$ equipped with the seminorm $\|\cdot\|_1$ is complete.*

PROOF. Assume that (g_q) is a Cauchy sequence with respect to the seminorm $\|\cdot\|_1$. By Lemma 6 for every $q \in \mathbb{N}$ there exists a simple function $f_q \in \mathcal{J}$ such that

$$\|g_q - f_q\|_1 < \frac{1}{q}.$$

Then

$$\|f_q - f_r\|_1 \le \|f_q - g_q\|_1 + \|g_q - g_r\|_1 + \|g_r - f_r\|_1 < \frac{1}{q} + \frac{1}{r} + \|g_q - g_r\|_1$$

and therefore the sequence (f_q) is L^1– Cauchy. By the Fundamental Lemma 2 the sequence (f_q) contains a subsequence (f_{q_s}) which converges μ– almost everywhere in $[a, b]$ to a certain function $f \in \mathcal{L}$. For this subsequence (f_{q_s}) we have

$$\|g_{q_s} - f\|_1 \le \|g_{q_s} - f_{q_s}\|_1 + \|f_{q_s} - f\|_1$$

and this means that the subsequence (g_{q_s}) of (g_q) converges in the seminorm $\|\cdot\|_1$ to f. This implies that also the original sequence (g_q) converges in this seminorm to $f \in \mathcal{L}$ an henceforth $\mathcal{L}$ is complete.

Partitions, systems and gauges

Let an interval $[a,b] \subset \mathbb{R}$, $-\infty < a < b < +\infty$ be given. A pair (τ, J) of a point $\tau \in \mathbb{R}$ and a compact interval $J \subset \mathbb{R}$ is called a *tagged interval*, τ is the *tag* of J.

A finite collection $\{(\tau_j, J_j),\ j = 1, \dots, p\}$ of tagged intervals is called an *L - system* on $[a,b]$ if

$$Int(J_i) \cap Int(J_j) = \emptyset \text{ for } i \neq j.$$

($Int(J)$ denotes the interior of an interval J.)

A finite collection $\{(\tau_j, J_j),\ j = 1, \dots, k\}$ of tagged intervals is called an *L - partition* of $[a,b]$ if

$$Int(J_i) \cap Int(J_j) = \emptyset \text{ for } i \neq j$$

and

$$\bigcup_{j=1}^{k} J_j = [a,b].$$

($Int(J)$ denotes the interior of an interval J.)

An L – partition $\{(\tau_j, J_j),\ j = 1, \dots, k\}$ for which

$$\tau_j \in J_j,\ j = 1, \dots, k$$

is called a *P - partition* of $[a,b]$.

Clearly every P – partition of $[a,b]$ is also an L – partition of $[a,b]$.

Sometimes it is useful to denote

$$J_i = [\alpha_{i-1}, \alpha_i], \quad i = 1, \dots, k$$

for a given L – partition of $[a,b]$, where

$$a = \alpha_0 < \alpha_1 < \dots < \alpha_k = b.$$

In other words we will assume in the sequel that the partition $\{(\tau_i, J_i),$ $i = 1, \dots, k\}$ is ordered in such a way that

$$\sup J_i = \inf J_{i+1}, \quad i = 1, \dots, k-1.$$

Given a positive function $\delta : [a,b] \to (0, +\infty)$ called a *gauge* on $[a,b]$, a tagged interval (τ, J) with $\tau \in [a,b]$ is said to be *δ–fine* if

$$J \subset [\tau - \delta(\tau), \tau + \delta(\tau)].$$

Using this concept we can speak about *δ-fine L - partitions (or systems)* and *δ-fine P - partitions* $\{(\tau_j, J_j),\ j = 1, \ldots, k\}$ of the interval $[a, b]$ whenever (τ_j, J_j) is δ-fine for every $j = 1, \ldots, k$.

It is a well-known fact that given a gauge $\delta : [a, b] \to (0, +\infty)$ there exists a δ-fine P - partition of $[a, b]$.

This result is called *Cousin's lemma,* see e.g. [13, Theorem on p. 119].

8. LEMMA. *Assume that $f \in \mathcal{L}$ and $\varepsilon > 0$. Then there is a gauge $\omega : [a, b] \to (0, +\infty)$ and $\eta \in (0, \varepsilon)$ such that the following statement holds. If*

$$\{(H_m, t_m), m = 1, \ldots, p\}$$

is an ω- fine L- system for which

$$\sum_{m=1}^{p} \mu(H_m) < \eta,$$

then

$$\sum_{m=1}^{p} \|f(t_m)\|_Y \mu(H_m) < \varepsilon.$$

PROOF. For $j = 1, 2, \ldots$ let us set

$$E_j = \{t \in [a, b]; \quad j - 1 \leq \|f(t)\|_Y < j\}.$$

Since $\|f\|_Y$ is integrable by Lemma 4, the sets E_j are measurable and we have $E_i \cap E_j = \emptyset$ for $i \neq j$ and

$$\bigcup_{j=1}^{\infty} E_j = [a, b].$$

We also have

$$\sum_{j=1}^{\infty} (j - 1)\mu(E_j) \leq \int_a^b \|f(t)\| d\mu < \sum_{j=1}^{\infty} j\mu(E_j)$$

and therefore

$$\sum_{j=1}^{\infty} j\mu(E_j) \leq \int_a^b \|f(t)\| d\mu + \sum_{j=1}^{\infty} \mu(E_j) = \int_a^b \|f(t)\| d\mu + \mu([a, b]) < \infty.$$

Assume that $\varepsilon_0 > 0$ is given. For $j = 1, 2, \dots$ there is an open set $G_j \subset [a, b]$ for which $E_j \subset G_j$ and

$$\mu(G_j) < \mu(E_j) + \frac{1}{2^j}$$

and this together with the inequality given above yields

$$\sum_{j=1}^{\infty} j\mu(G_j) < \sum_{j=1}^{\infty} j\mu(E_j) + \sum_{j=1}^{\infty} \frac{j}{2^j} < \infty.$$

Hence there is an $r \in \mathbb{N}$ such that

$$\sum_{j=r+1}^{\infty} j\mu(G_j) < \varepsilon_0.$$

If $t \in [a, b]$ then there is exactly one $j \in \mathbb{N}$ such that $t \in E_j$. For a given $t \in [a, b]$ let us choose the gauge ω such that

$$[a, b] \cap (t - \omega(t), t + \omega(t)) \subset G_j.$$

If now $\{(H_m, t_m), m = 1, \dots, p\}$ is an $\omega-$ fine $L-$ system, then we have $t_m \in E_{j_m}$,

$$H_m \subset (t_m - \omega(t_m), t_m + \omega(t_m)) \subset G_{j_m}$$

and

$$\|f(t_m)\|_Y < j_m$$

for $m = 1, \dots, p$. Hence

$$\begin{aligned}\sum_{m=1}^{p} \|f(t_m)\|_Y \mu(H_m) &\le \sum_{\substack{m=1\\ j_m \le r}}^{p} j_m \mu(H_m) + \sum_{\substack{m=1\\ j_m > r}}^{p} j_m \mu(H_m)\\ &\le r \sum_{\substack{m=1\\ j_m \le r}}^{p} \mu(H_m) + \sum_{\substack{m=1\\ j_m > r}}^{p} j_m \mu(G_{j_m}) < r\eta + \varepsilon_0.\end{aligned}$$

Taking $\varepsilon_0 < \dfrac{\varepsilon}{2}$ and $\eta < \dfrac{\varepsilon}{2r+1}$ we obtain the desired result.

McShane integral, the classes $\mathcal{S}^*$ and $\mathcal{S}$

9. DEFINITION. By $\mathcal{S}^* = \mathcal{S}^*([a, b]; Y)$ the set of functions $f : [a, b] \to Y$ is denoted for which to every $\varepsilon > 0$ there is a gauge δ on $[a, b]$ such that

$$\sum_{i=1}^{k} \sum_{j=1}^{l} \|f(t_i) - f(s_j)\|_Y \mu(J_i \cap L_j) < \varepsilon \tag{13}$$

for every δ–fine L – partitions $\{(t_i, J_i),\ i = 1, \dots, k\}$ and $\{(s_j, L_j),\ j = 1, \dots, l\}$ of $[a, b]$.

By $\mathcal{S} = \mathcal{S}([a, b]; Y)$ we denote the set of functions $f : [a, b] \to Y$ for which to every $\varepsilon > 0$ there is a gauge δ on $[a, b]$ such that

$$\|\sum_{i=1}^{k} f(t_i)\mu(J_i) - \sum_{j=1}^{l} f(s_j)\mu(L_j)\|_Y < \varepsilon \tag{14}$$

for every δ–fine L – partitions $\{(t_i, J_i),\ i = 1, \dots, k\}$ and $\{(s_j, L_j),\ j = 1, \dots, l\}$ of $[a, b]$.

Functions $f \in \mathcal{S}$ are called *McShane integrable* while functions $f \in \mathcal{S}^\star$ are called *absolutely McShane integrable.*

10. LEMMA. *If $f \in \mathcal{S}^\star$ then $f \in \mathcal{S}$, i.e. $\mathcal{S}^\star \subset \mathcal{S}$.*

PROOF. If $\{(t_i, J_i),\ i = 1, \dots, k\}$ and $\{(s_j, L_j),\ j = 1, \dots, l\}$ are δ–fine L – partitions of $[a, b]$ we have

$$\mu(J_i) = \sum_{j=1}^{l} \mu(J_i \cap L_j)$$

and

$$\mu(L_j) = \sum_{i=1}^{k} \mu(J_i \cap L_j).$$

Hence

$$\begin{aligned}
&\|\sum_{i=1}^{k} f(t_i)\mu(J_i) - \sum_{j=1}^{l} f(s_j)\mu(L_j)\|_Y \\
&\quad = \|\sum_{j=1}^{l}\sum_{i=1}^{k} f(t_i)\mu(J_i \cap L_j) - \sum_{i=1}^{k}\sum_{j=1}^{l} f(s_j)\mu(J_i \cap L_j)\|_Y \\
&\quad = \|\sum_{j=1}^{l}\sum_{i=1}^{k} (f(t_i) - f(s_j))\mu(J_i \cap L_j)\|_Y \\
&\quad \le \sum_{j=1}^{l}\sum_{i=1}^{k} \|f(t_i) - f(s_j)\|_Y \mu(J_i \cap L_j)
\end{aligned}$$

and by Definition 9 this yields the statement.

11. PROPOSITION. *If $f \in \mathcal{S}$ then there is an element $S_f \in Y$ such that for every $\varepsilon > 0$ there exists a gauge δ on $[a,b]$ such that*

$$\| \sum_{i=1}^{k} f(t_i)\mu(J_i) - S_f \|_Y < \varepsilon \tag{15}$$

for every δ–fine L – partition $\{(t_i, J_i),\ i = 1,\dots,k\}$ of $[a,b]$.

PROOF. Let $\varepsilon > 0$ be given and assume that δ is the gauge which corresponds to $\frac{\varepsilon}{2}$ by the definition of the class of functions $\mathcal{S}$.

Denote

$$S(\varepsilon) = \{S(f,D) = \sum_{i=1}^{k} f(t_i)\mu(J_i); \quad D = \{(t_i, J_i),\ i = 1,\dots,k\}$$

$$\text{an arbitrary } \delta\text{– fine} L \text{ partition of } [a,b]\}.$$

The set $S(\varepsilon) \subset Y$ is nonempty because by Cousin's lemma there exists a δ–fine L– partition $\{(t_i, J_i),\ i,k\}$ of $[a,b]$. Since by definition of $\mathcal{S}$ we have

$$\| \sum_{i=1}^{k} f(t_i)\mu(J_i) - \sum_{j=1}^{l} f(s_j)\mu(L_j) \|_Y < \frac{\varepsilon}{2}$$

for every δ–fine L – partitions $\{(t_i, J_i),\ i = 1,\dots,k\}$ and $\{(s_j, L_j),\ j = 1,\dots,l\}$ of $[a,b]$, we have also

$$\operatorname{diam} S(\varepsilon) < \frac{\varepsilon}{2}$$

(by diam $S(\varepsilon)$ the diameter of the set $S(\varepsilon)$ is denoted). Further evidently

$$S(\varepsilon_1) \subset S(\varepsilon_2),$$

provided $\varepsilon_1 < \varepsilon_2$. Hence the set

$$\bigcap_{\varepsilon>0} \overline{S(\varepsilon)} = S_f \in Y$$

consists of a single point because the space Y is complete (by $\overline{S(\varepsilon)}$ the closure of the set $S(\varepsilon)$ in Y is denoted).

For the integral sum $S(f,D)$ we get

$$\| \sum_{i=1}^{k} f(t_i)\mu(J_i) - S_f \|_Y < \frac{\varepsilon}{2},$$

whenever $D = \{(t_i, J_i),\ i = 1, \dots, k\}$ is an arbitrary $\delta-$ fine $L-$ partition of $[a,b]$.

12. DEFINITION. The value S_f given by Proposition 11 for a function $f \in \mathcal{S}$ will be denoted by $(\mathcal{S}) \int_a^b f d\mu$ and called the McShane integral of the function f.

REMARK. It is easy to see that if $f \in \mathcal{S}^\star$ then by Lemma 10 it is also $f \in \mathcal{S}$ and in this case we have a $S_f \in Y$ such that (14) holds, i.e. the McShane integral $(\mathcal{S}) \int_a^b f d\mu$ can be defined for functions f belonging to $\mathcal{S}^\star$.

It is easy to show that the McShane integral has the usual properties, e.g. *If for $f, g : [a,b] \to Y$ the integrals $(\mathcal{S}) \int_a^b f d\mu$ and $(\mathcal{S}) \int_a^b g d\mu$ exist then for $c_1, c_2 \in \mathbb{R}$ the integral $(\mathcal{S}) \int_a^b (c_1 f + c_2 g) d\mu$ exists and*

$$(\mathcal{S}) \int_a^b (c_1 f + c_2 g) d\mu = c_1\, (\mathcal{S}) \int_a^b f d\mu + c_2\, (\mathcal{S}) \int_a^b g d\mu.$$

If $(\mathcal{S}) \int_a^b f d\mu$ exists and $[c,d] \subset [a,b]$ then also the integral $(\mathcal{S}) \int_c^d f d\mu$ exists.

In integration theory based on integral sums like the McShane integral the following lemma is useful.

13. LEMMA (SAKS – HENSTOCK). *Assume that $f \in \mathcal{S}$. Given $\varepsilon > 0$ assume that the gauge δ on $[a,b]$ is such that*

$$\| \sum_{i=1}^k f(t_i)\mu(J_i) - (\mathcal{S}) \int_a^b f d\mu \|_Y < \varepsilon$$

for every $\delta-$ fine $L-$ partition $\{(t_i, J_i), i = 1, \dots, k\}$ of $[a,b]$.

Then if $\{(r_j, K_j), j = 1, \dots, m\}$ is an arbitrary $\delta-$ fine $L-$ system we have

$$\| \sum_{j=1}^m \left(f(r_j)\mu(K_j) - (\mathcal{S}) \int_{K_j} f d\mu \right) \|_Y \leq \varepsilon.$$

PROOF. Since $\{(r_j, K_j), j = 1, \dots, m\}$ is a $\delta-$ fine $L-$ system the complement $[a,b] \setminus \bigcup_{j=1}^m \operatorname{Int} K_j$ consists of a finite system M_m, $m = 1, \dots, p$ of intervals in $[a,b]$. The function f belongs to $\mathcal{S}$ and therefore the integrals

$(\mathcal{S})\int_{M_m} f d\mu$ exist and by definition for any $\eta > 0$ there is a gauge δ_m on M_m with $\delta_m(t) < \delta(t)$ for $t \in M_m$ such that for every $m = 1, \dots, p$ we have

$$\|\sum_{i=1}^{k_m} f(s_i^m)\mu(J_i^m) - (\mathcal{S})\int_{M_m} f d\mu\|_Y < \frac{\eta}{p+1}$$

provided $\{(s_i^m, J_i^m), i = 1, \dots, k_m\}$ is a δ_m- fine $L-$ partition of the interval M_m. The sum

$$\sum_{j=1}^{m} f(r_j)\mu(K_j) + \sum_{m=1}^{p} \sum_{i=1}^{k_f(s_i^m)} \mu(J_i^m)$$

represents an integral sum which corresponds to a certain $\delta-$ fine $L-$ partition of $[a,b]$ and consequently by the assumption we have

$$\|\sum_{j=1}^{m} f(r_j)\mu(K_j) + \sum_{m=1}^{p} \sum_{i=1}^{k_m} f(s_i^m)\mu(J_i^m) - (\mathcal{S})\int_a^b f d\mu\|_Y < \varepsilon.$$

Hence

$$\|\sum_{j=1}^{m} f(r_j)\mu(K_j) - (\mathcal{S})\int_{K_j} f d\mu\|_Y$$

$$\leq \|\sum_{j=1}^{m} f(r_j)\mu(K_j) + \sum_{m=1}^{p} \sum_{i=f(s_i^m)} \mu(J_i^m) - (\mathcal{S})\int_a^b f d\mu\|_Y$$

$$+ \sum_{m=1}^{p} \|\sum_{i=1}^{k_m} f(s_i^m)\mu(J_i^m) - (\mathcal{S})\int_{M_m} f d\mu\|_Y < \varepsilon + p\frac{\eta}{p+1} < \varepsilon + \eta$$

Since this inequality holds for every $\eta > 0$ we obtain immediately the statement of the lemma.

14. Corollary. *If $f \in \mathcal{S}$ and the Banach space Y is finite–dimensional and if given $\varepsilon > 0$ the gauge δ on $[a,b]$ is such that*

$$\|\sum_{i=1}^{k} f(t_i)\mu(J_i) - (\mathcal{S})\int_a^b f d\mu\|_Y < \varepsilon$$

for every $\delta-$ fine $L-$ partition $\{(t_i, J_i), i = 1, \dots, k\}$ of $[a,b]$, then we have

$$\sum_{j=1}^{m} \|f(r_j)\mu(K_j) - (\mathcal{S})\int_{K_j} f d\mu\|_Y \leq K\varepsilon$$

for an arbitrary δ– fine L– system $\{(r_j, K_j), j = 1, \dots, m\}$. K is a constant which depends on the dimension of the Banach space Y only.

PROOF. It is easy to see that there is no restriction in assuming $\dim Y = 1$. The more – dimensional case can be treated componentwise.

Assume therefore that $f : [a,b] \to \mathbb{R}$. Define M_+ as the set of indices $j = 1, \dots, m$ for which

$$f(r_j)\mu(K_j) - (\mathcal{S}) \int_{K_j} f d\mu \geq 0$$

and M_- as the set of indices $j = 1, \dots, m$ for which

$$f(r_j)\mu(K_j) - (\mathcal{S}) \int_{K_j} f d\mu < 0.$$

Then by the Saks – Henstock lemma 13 we have

$$\sum_{j \in M_+} (f(r_j)\mu(K_j) - (\mathcal{S}) \int_{K_j} f d\mu) = | \sum_{j \in M_+} (f(r_j)\mu(K_j) - (\mathcal{S}) \int_{K_j} f d\mu)| \leq \varepsilon$$

and

$$-\sum_{j \in M_+} (f(r_j)\mu(K_j) - (\mathcal{S}) \int_{K_j} f d\mu) = | \sum_{j \in M_-} (f(r_j)\mu(K_j) - (\mathcal{S}) \int_{K_j} f d\mu)| \leq \varepsilon.$$

Hence

$$\begin{aligned}
&\sum_{j=1}^{m} |f(r_j)\mu(K_j) - (\mathcal{S}) \int_{K_j} f d\mu| \\
&\quad = \sum_{j \in M_+} (f(r_j)\mu(K_j) - (\mathcal{S}) \int_{K_j} f d\mu) - \sum_{j \in M_+} (f(r_j)\mu(K_j) \\
&\quad - (\mathcal{S}) \int_{K_j} f d\mu) \leq 2\varepsilon.
\end{aligned}$$

The constant K for the general case comes from the relation between the given norm $\|\cdot\|_Y$ on Y and the norm given for example as the sum of absolute values of the coordinates of a point in Y.

Comparison of Bochner and McShane integrals

Our aim now is to compare the concept of Bochner and McShane integral described above.

15. PROPOSITION. *If $f \in \mathcal{L}$ then also $f \in \mathcal{S}^*$ and*

$$(\mathcal{L}) \int_a^b f d\mu = (\mathcal{S}) \int_a^b f d\mu \tag{16}$$

PROOF. Assume that $f \in \mathcal{L}$ and that $\varepsilon > 0$ is given.

Let f_q, $q = 1, 2, \dots$ be an L^1– Cauchy sequence of simple functions which converges to f μ– almost everywhere in $[a, b]$, i.e.

$$\lim_{q \to \infty} \|f_q(t) - f(t)\|_Y = 0$$

for μ– almost all $t \in [a, b]$.

Let $\eta \in (0, \varepsilon)$ and the gauge $\omega : [a, b] \to (0, \infty)$ be given by Lemma 8. Take $\alpha \in (0, \frac{\eta}{2})$. By the Fundamental Lemma 2 the sequence f_q, $q = 1, 2, \dots$ can be chosen in such a way that there exists a set $Z_\alpha \subset [a, b]$ with $\mu(Z_\alpha) < \frac{\alpha}{2}$ such that the sequence f_q converges to the function f uniformly on $[a, b] \setminus Z_\alpha$. The μ– measurable set Z_α can be approximated from above by an open set G_α in such a way that $Z_\alpha \subset G_\alpha$ and $\mu(G_\alpha) < \alpha$. Let us define the closed set

$$F_\alpha = [a, b] \setminus G_\alpha.$$

Concluding we have the following result. To $\alpha > 0$ there exists a closed set $F_\alpha \subset [a, b]$ such that

$$\mu(F_\alpha \subset [a, b]) = \mu(G_\alpha) < \alpha$$

and there is an $n_\alpha \in \mathbb{N}$ such that

$$\|f_q(t) - f(t)\|_Y < \alpha$$

for $q \geq n_\alpha$ and $t \in F_\alpha$.

Assume that $q \geq n_\alpha$. Since f_q is a simple function there is a finite sequence $I_{q_m} \subset [a, b]$, $m = 1, \dots, p_q$ of measurable sets such that

$$I_{q_m} \cap I_{q_l} = \emptyset \quad \text{for } m \neq l$$

and

$$[a, b] = \bigcup_{m=1}^{p_q} I_{q_m}$$

where

$$f_q(t) = y_{q_m} \in Y \quad \text{for } t \in I_{q_m}, m = 1, \dots, p_q.$$

By the measurability of the sets I_{q_m} there exist closed sets F_{q_m} with $F_{q_m} \subset I_{q_m}$ and

$$\mu(I_{q_m} \setminus F_{q_m}) < \frac{\eta}{2p_q} \text{ for } m = 1, \ldots, p_q.$$

Hence

$$\mu(\bigcup_{m=1}^{p_q} (I_{q_m} \setminus F_{q_m})) =< \sum_{m=1}^{p_q} \frac{\eta}{2p_q} = \frac{\eta}{2}.$$

Define further

$$A_{q_m} = F_\alpha \cap F_{q_m}, \; m = 1, \ldots, p_q.$$

The set A_{q_m} is closed and $A_{q_m} \cap A_{q_l} = \emptyset$ for $m \neq l$. Therefore the distance of different sets A_{q_m} is positive, i.e. there is a $\rho > 0$ such that if $t \in A_{q_m}$, $s \in A_{q_l}$ and $m \neq l$, then

$$|t - s| > \rho.$$

Further we have

$$\begin{aligned}
[a, b] \setminus \bigcup_{m=1}^{p_q} A_{q_m} &= \bigcup_{m=1}^{p_q} I_{q_m} \setminus \bigcup_{m=1}^{p_q} (F_\alpha \cap F_{q_m}) \\
&\subset \bigcup_{m=1}^{p_q} (I_{q_m} \setminus F_{q_m}) \cup \bigcup_{m=1}^{p_q} (I_{q_m} \setminus F_\alpha) \\
&= \bigcup_{m=1}^{p_q} (I_{q_m} \setminus F_{q_m}) \cup [a, b] \setminus F_\alpha
\end{aligned}$$

and therefore

$$\mu([a, b] \setminus \bigcup_{m=1}^{p_q} A_{q_m}) \leq \sum_{m=1}^{p_q} \mu(I_{q_m} \setminus F_{q_m}) + \mu([a, b] \setminus F_\alpha) < \frac{\eta}{2} + \alpha < \eta.$$

Let us take a gauge δ on $[a, b]$ such that

$$\delta(t) < \min(\omega(t), \frac{\rho}{2}) \text{ for } t \in [a, b]$$

and

$$(t - \delta(t), t + \delta(t)) \cap [a, b] \subset [a, b] \setminus \bigcup_{m=1}^{p_q} A_{q_m}$$

provided $t \in [a, b] \setminus \bigcup_{m=1}^{p_q} A_{q_m}$. This can be done because the set $[a, b] \setminus \bigcup_{m=1}^{p_q} A_{q_m}$ is open.

Assume that $\{(t_i, J_i),\ i = 1, \dots, k\}$ and $\{(s_j, L_j),\ j = 1, \dots, l\}$ are δ–fine L – partitions of $[a, b]$. By the choice of the gauge δ given above we obtain the following properties of a δ–fine L – partition $\{(t_i, J_i),\ i = 1, \dots, k\}$:

If $t_i \in \bigcup_{m=1}^{p_q} A_{q_m}$, *then there is* $r = 1, \dots, p_q$ *such that* $t_i \in A_{q_r}$; *since* $\delta(t_i) < \frac{\rho}{2}$, *we have*

$$(t_i - \delta(t_i), t_i + \delta(t_i)) \cap A_{q_m} = \emptyset$$

provided $m \neq r$ *and therefore also*

$$J_i \cap A_{q_m} = \emptyset$$

for $m \neq r$.

If $t_i \notin \bigcup_{m=1}^{p_q} A_{q_m}$, *then*

$$J_i \subset (t_i - \delta(t_i), t_i + \delta(t_i)) \subset [a, b] \setminus \bigcup_{m=1}^{p_q} A_{q_m}, \tag{17}$$

i.e.

$$J_i \cap A_{q_m} = \emptyset$$

for every $m = 1, \dots, p_q$. *Moreover, since* $\bigcup_{i=1, t_i \notin \bigcup A_{q_m}}^{k} J_i \subset [a, b] \setminus \bigcup_{m=1}^{p_q} A_{q_m}$, *we get*

$$\mu\Big(\bigcup_{i=1, t_i \notin \bigcup A_{q_m}}^{k} J_i\Big) \le \mu\Big([a, b] \setminus \bigcup_{m=1}^{p_q} A_{q_m}\Big) < \eta.$$

Similar properties hold also for the partition $\{(L_j, s_j)\}$.

Assume now that $t_i, s_j \in \bigcup_{m=1}^{p_q} A_{q_m}$. If $J_i \cap L_j = \emptyset$, then necessarily

$$|t_i - s_j| < \rho$$

because dist $(t_i, J_i) < \frac{\rho}{2}$, dist $(s_j, L_j) < \frac{\rho}{2}$ and

$$|t_i - s_j| \le |t_i - a| + |s_j - a| < \rho$$

where $a \in J_i \cap L_j$. In this situation there is an $r = 1, \dots, p_q$ for which $t_i, s_j \in A_{q_r}$. Indeed if t_i and s_j would belong to different A_{q_m}, then we would have $|t_i - s_j| > \rho$ and this contradicts the inequality given above. Hence

$$f_q(t_i) = f_q(s_j) = y_{q_r}$$

because $A_{q_r} \subset I_{q_r}$. At the same time we also have $A_{q_r} \subset F_\alpha$ and therefore

$$\|f_q(t) - f(t)\|_Y < \alpha \text{ for } t \in A_{q_r}.$$

This yields

$$\|f(t_i) - f(s_j)\|_Y \leq \|f_q(t_i) - f(t_i)\|_Y + \|f_q(s_j) - f(s_j)\|_Y < 2\alpha. \tag{18}$$

If at least one of the inclusions $t_i \in \bigcup_{m=1}^{p_q} = A_{q_m}$, $s_j \in \bigcup_{m=1}^{p_q} A_{q_m}$ does not hold, i.e. if we have for example $s_j \in [a,b] \setminus \bigcup_{m=1}^{p_q} = A_{q_m}$ then by (17) we get

$$J_i \cap L_j \subset L_j \subset [a,b] \setminus \bigcup_{m=1}^{p_q=} A_{q_m}$$

and the tagged interval $(J_i \cap L_j, s_j)$ is $\delta-$ fine. Similarly also the tagged interval $(J_i \cap L_j, t_i)$ is $\delta-$ fine. The other possible cases lead to the same conclusion.

For showing $f \in S^\star$ we need an estimate for the sum

$$S = \sum_{i=1}^{k} \sum_{j=1}^{l} \|f(t_i) - f(s_j)\|_Y \mu(J_i \cap L_j).$$

The set

$$M = \{(i,j); i = 1, \ldots, k,\ j = 1, \ldots, l\}$$

can be splitted into

$$M_1 = \{(i,j) \in M;\ t_i, s_j \in \bigcup_{m=1}^{p_q} A_{q_m}\} = \text{ and } M_2 = M \setminus M_1.$$

Then

$$S = \sum_{(i,j)\in M_1} \|f(t_i) - f(s_j)\|_Y \mu(J_i \cap L_j) + \sum_{(i,j)\in M_2} \|f(t_i) - f(s_j)\|_Y \mu(J_i \cap L_j).$$

By (18) we get

$$\sum_{(i,j)\in M_1} \|f(t_i) - f(s_j)\|_Y \mu(J_i \cap L_j) < 2\alpha \sum_{(i,j)\in M} \mu(J_i \cap L_j) = 2\alpha\mu([a,b]).$$

For the other sum $\sum_{(i,j)\in M_2} \|f(t_i) - f(s_j)\|_Y \mu(J_i \cap L_j)$ we know that

$$\bigcup_{(i,j)\in M_2} (J_i \cap L_j) \subset [a,b] \setminus \bigcup_{m=1}^{p_q} A_{q_m},$$

that the intervals $J_i \cap L_j$ with $(i,j) \in M_2$ are nonoverlapping and

$$\sum_{(i,j)\in M_2} \mu(J_i \cap L_j) \leq \mu([a,b] \setminus \bigcup_{m=1}^{p_q} A_{q_m}) < \eta.$$

Hence by Lemma 8 we get

$$\sum_{(i,j)\in M_2} \|f(t_i) - f(s_j)\|_Y \mu(J_i \cap L_j)$$
$$\leq \sum_{(i,j)\in M_2} \|f(t_i)\|_Y \mu(J_i \cap L_j) + \sum_{(i,j)\in M_2} \|f(s_j)\|_Y \mu(J_i \cap L_j) \leq 2\varepsilon.$$

Altogether we obtain

$$S < 2\alpha\mu([a,b]) + 2\varepsilon$$

and this yields $f \in \mathcal{S}^\star$ by definition.

It remains to show that for the integrals the equality (16) holds.

Suppose that $\varepsilon > 0$ is given. Assume that $E \subset [a,b]$ is an arbitrary measurable set. Let us put $F = [a,b] \setminus E$; then evidently $[a,b] = E \cup F$. In this situation there exist open sets G and H such that

$$E \subset G,\ F \subset H$$

and

$$\mu(G) < \mu(E) + \varepsilon,\ \mu(H) < \mu(F) + \varepsilon$$

Let us define a gauge $\delta : [a,b] \to (0,+\infty)$ such that

$$\text{if } t \in E \text{ then } (t - \delta(t), t + \delta(t)) \cap [a,b] \subset G$$

and

$$\text{if } t \in F \text{ then } (t - \delta(t), t + \delta(t)) \cap [a,b] \subset H$$

hold.

Let $\{(J_i, t_i)\}$ be a δ– fine L– partition of $[a,b]$ and assume that χ_E is the characteristic function of the set E. Then

$$\sum_{i=1}^{k} \chi_E(t_i)\mu(J_i) = \sum_{i=1, t_i\in E}^{k} \mu(J_i) \leq \mu(G) < \mu(E) + \varepsilon \tag{19}$$

and similarly

$$\sum_{i=1}^{k} \chi_F(t_i)\mu(J_i) < \mu(E) + \varepsilon.$$

Further we have

$$\sum_{i=1}^{k} \chi_{[a,b]}(t_i)\mu(J_i) = \sum_{i=1}^{k} \mu(J_i) = \mu([a,b])$$

and also

$$\chi_{[a,b]} = \chi_E + \chi_F.$$

This yields

$$\begin{aligned}\sum_{i=1}^{k} \chi_E(t_i)\mu(J_i) =& \sum_{i=1}^{k} \chi_{[a,b]}(t_i)\mu(J_i) - \sum_{i=1}^{k} \chi_F(t_i)\mu(J_i) \\ >& \mu([a,b]) - (\mu(F) + \varepsilon) = \mu(E) - \varepsilon.\end{aligned}$$

This inequality together with (19) implies

$$|\sum_{i=1}^{k} \chi_E(t_i)\mu(J_i) - \mu(E)| < \varepsilon.$$

Since by definition we have

$$(\mathcal{L}) \int_a^b \chi_E(t) d\mu = \mu(E),$$

we have also

$$|\sum_{i=1}^{k} \chi_E(t_i)\mu(J_i) - (\mathcal{L}) \int_a^b \chi_E(t) d\mu| < \varepsilon. \tag{20}$$

By definition we know that $\chi_E \in \mathcal{L}$ and by the result stated above we have also $\chi_E \in \mathcal{S}^\star$. The last inequality means that

$$(\mathcal{L}) \int_a^b \chi_E(t) d\mu = (\mathcal{S}) \int_a^b \chi_E(t) d\mu$$

If now $y \in Y$, then the function $y\chi_E : [a,b] \to Y$ belongs to $\mathcal{L}$. Therefore $y\chi_E \in \mathcal{S}^\star$ by the previous results and also

$$(\mathcal{L}) \int_a^b y\chi_E(t) d\mu = y\mu(E) = y(\mathcal{L}) \int_a^b \chi_E(t) d\mu.$$

Hence by (20) we get

$$\|\sum_{i=1}^{k} y\chi_E(t_i)\mu(J_i) - (\mathcal{L})\int_a^b y\chi_E(t)d\mu\|_Y$$

$$= \|y\left[\sum_{i=1}^{k} \chi_E(t_i)\mu(J_i) - (\mathcal{L})\int_a^b \chi_E(t)d\mu\right]\|_Y < \|y\|_Y\varepsilon,$$

i.e. we obtain

$$(\mathcal{L})\int_a^b y\chi_E(t)d\mu = (\mathcal{S})\int_a^b y\chi_E(t)d\mu$$

and this immediately implies that

$$(\mathcal{L})\int_a^b g(t)d\mu = (\mathcal{S})\int_a^b g(t)d\mu$$

for an arbitrary simple function $g : [a,b] \to Y$. Without any loss of generality it can be assumed that for the approximating sequence (f_q) of simple functions the inequality

$$\|f_q(t)\|_Y \leq \|f(t)\|_Y + 1$$

holds for $\mu-$ almost all $t \in [a,b]$. (It is possible to define $g_q(t) = f_q(t)$ if $\|f_q(t)\|_Y \leq \|f(t)\|_Y + 1$ and $g_q(t) = 0$ otherwise; g_q is the the desired bounded approximating sequence of simple functions for the function f.)

Since

$$\lim_{q\to\infty} \|(\mathcal{L})\int_a^b f_q d\mu - (\mathcal{L})\int_a^b f d\mu\|_Y = 0,$$

there is a $q \in \mathbb{N}$, $q > n_\alpha$ such that

$$\|(\mathcal{L})\int_a^b f_q d\mu - (\mathcal{L})\int_a^b f d\mu\|_Y < \varepsilon$$

and for the simple function f_q the equality

$$(\mathcal{L})\int_a^b f_q d\mu = (\mathcal{S})\int_a^b f_q d\mu$$

is satisfied.

Assume that δ_1 is a gauge on $[a,b]$ for which $\delta_1(t) < \delta(t)$ if $t \in [a,b]$ (for the gauge δ see p. 14) and

$$\|\sum_{i=1}^{k} f_q(t_i)\mu(J_i) - (\mathcal{S})\int_a^b f_q d\mu\|_Y < \varepsilon$$

for every δ_1– fine L– partition $\{(J_i, t_i), i = 1, \ldots, k\}$.

For such a δ_1– fine L– partition $\{(J_i, t_i), i = 1, \ldots, k\}$ we have

$$\begin{aligned}
&\|\sum_{i=1}^{k} f(t_i)\mu(J_i) - (\mathcal{L})\int_a^b f d\mu\|_Y \\
&\quad \le \|\sum_{i=1}^{k} f(t_i)\mu(J_i) - \sum_{i=1}^{k} f_q(t_i)\mu(J_i)\|_Y \\
&\quad + \|\sum_{i=1}^{k} f_q(t_i)\mu(J_i) - (\mathcal{S})\int_a^b f_q d\mu\|_Y + \|(\mathcal{L})\int_a^b f_q d\mu - (\mathcal{L})\int_a^b f d\mu\|_Y \\
&\quad \le \|\sum_{i=1}^{k} f(t_i)\mu(J_i) - \sum_{i=1}^{k} f_q(t_i)\mu(J_i)\|_Y + 2\varepsilon.
\end{aligned}$$

We need an estimate for the sum on the right hand side of this inequality. We split the sum into two parts, one with $t_i \in \bigcup_{m=1}^{p_q} A_{qm}$ and the other one with $t_i \notin \bigcup_{m=1}^{p_q} A_{qm}$, i.e.

$$\begin{aligned}
\sum_{i=1}^{k}(f(t_i) - f_q(t_i))\mu(J_i) &= \sum_{i=1, t_i \in \bigcup_{m=1}^{p_q} = A_{qm}}^{k} (f(t_i) - f_q(t_i))\mu(J_i) \\
&+ \sum_{i=1, t_i \notin \bigcup_{m=1}^{p_q} A_{qm}}^{k} (f(t_i) - f_q(t_i))\mu(J_i).
\end{aligned}$$

If $t_i \notin \bigcup_{m=1}^{p_q} A_{qm}$, then $\mu(\bigcup_{i=1, t_i \notin \bigcup_{m=1}^{p_q} A_{qm}}^{k} J_i) < \eta$ and

$$\begin{aligned}
&\|\sum_{i=1, t_i \notin \bigcup_{m=1}^{p_q} A_{qm}}^{k} (f(t_i) - f_q(t_i))\mu(J_i)\|_Y \\
&\quad \le \sum_{i=1, t_i \notin \bigcup_{m=1}^{p_q} A_{qm}}^{k} \|f(t_i)\|_Y \mu(J_i) + \sum_{i=1, t_i \notin \bigcup_{m=1}^{p_q} A_{qm}}^{k} \|f_q(t_i)\|_Y \mu(J_i)
\end{aligned}$$

$$\leq \sum_{i=1, t_i \notin \bigcup_{m=1}^{p_q} A_{q_m}}^{k} (2\|f(t_i)\|_Y + 1)\mu(J_i) \leq 2\varepsilon + \eta < 3\varepsilon$$

by Lemma 8 because $\mu(\bigcup_{i=1, t_i \notin \bigcup_{m=1}^{p_q} A_{q_m}}^{k} J_i) < \eta$ in this case.

If $t_i \in \bigcup_{m=1}^{p_q} A_{q_m}$ then

$$\|f(t_i) - f_q(t_i)\|_Y < \alpha$$

and

$$\|\sum_{i=1, t_i \in \bigcup_{m=1}^{p_q} A_{q_m}}^{k} (f(t_i) - f_q(t_i))\mu(J_i)\|_Y < \alpha \sum_{i=1, t_i \in \bigcup_{m=1}^{p_q} A_{q_m}}^{k} \mu(J_i)$$

$$\leq \alpha \sum_{i=1}^{k} \mu(J_i) = \alpha\mu([a,b]) < \varepsilon\mu([a,b]).$$

Putting together all these estimates, we finally obtain

$$\|\sum_{i=1}^{k} f(t_i)\mu(J_i) - (\mathcal{L})\int_a^b f d\mu\|_Y < 2\varepsilon + 3\varepsilon + \varepsilon\mu([a,b]) = \varepsilon(5 + \mu([a,b]))$$

for every δ_1– fine L– partition $\{(J_i, t_i), i = 1, \ldots, k\}$ and this implies

$$(\mathcal{L})\int_a^b f d\mu = (\mathcal{S})\int_a^b f d\mu,$$

i.e. (16) is satisfied.

We have shown that if Y is a general Banach space then $\mathcal{L} \subset \mathcal{S}^\star \subset \mathcal{S}$. On the other hand the following statement holds.

16. Proposition. If $f \in \mathcal{S}^\star$ then also $f \in \mathcal{L}$ and

$$(\mathcal{L})\int_a^b f d\mu = (\mathcal{S})\int_a^b f d\mu$$

PROOF (A SKETCH ONLY). Assume that $f \in S^\star$. Then to every $m = 1, 2, \dots$ there is a gauge δ_m on $[a, b]$ such that

$$\sum_{i=1}^{k} \sum_{j=1}^{l} \|f(t_i) - f(s_j)\|_Y \mu(J_i \cap L_j) < \frac{\mu([a,b])}{4^m}$$

for every δ_m–fine L – partitions $\{(t_i, J_i),\ i = 1, \dots, k\}$ and $\{(s_j, L_j),\ j = 1, \dots, l\}$ of $[a, b]$.

Without loss of generality we can assume that

$$\delta_{m+1}(t) \le \delta_m(t) \text{ for } t \in [a,b],\ m = 1, 2, \dots.$$

Let $\{(t_i^{(m)}, J_i^{(m)}), i = 1, 2, \dots, k_m\}$ be a δ_m– fine L– partition of $[a, b]$. Assume that $\{(t_i^{(m+1)}, J_i^{(m+1)}), i = 1, 2, \dots, k_{m+1}$ is a refinement of the partition

$$\{(t_i^{(m)}, J_i^{(m)}), i = 1, 2, \dots, k_m,$$

i.e. that for $i = 1, 2, \dots, k_{m+1}$ there is a $j \in i = 1, 2, \dots, k_m$ such that $J_i^{(m+1)} \subset J_j^{(m)}$.

Define

$$f_m(t) = f(t_i^{(m)}) \text{ for } t \in \text{ Int } J_i^{(m)},$$

$$f_m(t) = 0 \text{ otherwise.}$$

The function $f_m : [a, b] \to Y$ is evidently simple.

Denote

$$W(m, i) = \{j \in \{1, 2, \dots, k_{m+1}\}; J_j^{(m+1)} \subset J_i^{(m)}\}.$$

Then

$$\int_a^b \|f_{m+1}(t) - f_m(t)\|_Y d\mu$$
$$= \sum_{i=1}^{k_m} \sum_{j \in W(m,i)} \|f(t_j^{(m+1)}) - f(t_i^{(m)})\|_Y \mu(J_j^{(m+1)}) < \frac{\mu([a,b])}{4^m}.$$

Let $\varepsilon > 0$ be given. Let us take $N \in \mathbb{N}$ such that

$$\sum_{m=N}^{\infty} \frac{1}{4^m} < \frac{\varepsilon}{\mu([a,b])}.$$

If $q, r \geq N$, $r < q$ then

$$\begin{aligned}\|f_q - f_r\|_Y \leq & \|f_q - f_{q-1}\|_Y + \cdots + \|f_{r+1} - f_r\|_Y \\ \leq & \mu([a,b]) \left(\frac{1}{4^r} + \frac{1}{4^{r+1}} + \cdots + \frac{1}{4^{q-1}} \right) < \varepsilon.\end{aligned}$$

This implies that the sequence (f_m) is L^1– Cauchy and by the Fundamental Lemma 2 it contains a subsequence (we denote it again (f_m)) which converges pointwise μ– almost everywhere to a certain function $g : [a,b] \to Y$. By the results given above we have $g \in \mathcal{L} \subset \mathcal{S}^\star$ and

$$(\mathcal{S}) \int_a^b g d\mu = (\mathcal{L}) \int_a^b g d\mu = \lim_{m\to\infty} (\mathcal{S}) \int_a^b f_m d\mu = \lim_{m\to\infty} (\mathcal{L}) \int_a^b f_m d\mu.$$

To finish the proof the following facts have to be shown:

$$\lim_{m\to\infty} (\mathcal{S}) \int_J f_m d\mu = (\mathcal{S}) \int_J g d\mu$$

and

$$\lim_{m\to\infty} (\mathcal{S}) \int_J f_m d\mu = (\mathcal{S}) \int_J f d\mu$$

for every interval $J \subset [a,b]$. Then

$$(\mathcal{S}) \int_J (g - f) d\mu = 0$$

for every interval $J \subset [a,b]$. From this then it is possible to show that $f(t) = g(t)$ for μ– almost all $t \in [a,b]$ and consequently $f \in \mathcal{L}$ because $g \in \mathcal{L}$.

The finite dimensional case

Now we will show that the following statement holds.

17. Proposition. If Y is a finite dimensional Banach space, then $\mathcal{S}^\star = \mathcal{S}$.

Proof. Since Y is finite dimensional, we can assume without loss of generality that $\dim Y = 1$. Otherwise it is possible to work componentwise. So assume that $f : [a,b] \to \mathbb{R}$ and $f \in \mathcal{S}$.

Let $\varepsilon > 0$ be given. By definition there is a gauge δ on $[a,b]$ such that

$$|\sum_{i=1}^{k} f(t_i)\mu(J_i) - \sum_{j=1}^{l} = f(s_i)\mu(L_j)| < \frac{\varepsilon}{4}$$

for every $\delta-$ fine $L-$ partitions $\{(t_i, J_i), i = 1, \dots, k\}$ and $\{(s_j, L_j), j = 1, \dots, l\}$. Clearly $\{(s_j, J_i \cap L_j), i = 1, \dots, k, j = 1, \dots, l\}$ and $\{(t_i, J_i \cap L_j), i = 1, \dots, k, j = 1, \dots, l\}$ are also $\delta-$ fine $L-$ partitions of $[a, b]$.

Further we have

$$\sum_{i=1}^{k} f(t_i)\mu(J_i) - \sum_{j=1}^{l} f(s_i)\mu = (L_i) = \sum_{i=1}^{k}\sum_{j=1}^{l}(f(t_i) - f(s_i))\mu(J_i = \cap L_j).$$

Denote by M_+ the set of indices (i, j), $i = 1, \dots, k, j = 1, \dots, l$ for which

$$f(t_i) \geq f(s_j)$$

and by M_- the set of indices (i, j), $i = 1, \dots, k, j = 1, \dots, l$ for which

$$f(t_i) < f(s_j).$$

By the Saks – Henstock lemma 13 we get

$$|\sum_{(i,j)\in M_+} (f(t_i) - f(s_i))\mu(J_i \cap L_j)| = \sum_{(i,j)\in M_+} |f(t_i) - f(s_i)|\mu(J_i \cap L_j) \leq \frac{\varepsilon}{4}$$

and similarly also

$$|\sum_{(i,j)\in M_-} (f(t_i) - f(s_i))\mu(J_i \cap L_j)| = \sum_{(i,j)\in M_+} (f(s_i) - f(t_i))\mu(J_i \cap L_j)$$
$$= \sum_{(i,j)\in M_+} |f(t_i) - f(s_i)|\mu(J_i \cap L_j) \leq \frac{\varepsilon}{4}.$$

Hence

$$\sum_{i=1}^{k}\sum_{j=1}^{l} |f(t_i) - f(s_i)|\mu(J_i \cap L_j) = \sum_{(i,j)\in M_+} |f(t_i) - f(s_i)|\mu(J_i \cap L_j)$$
$$+ \sum_{(i,j)\in M_-} |f(t_i) - f(s_i)|\mu(J_i \cap L_j) \leq \frac{\varepsilon}{2} < \varepsilon$$

and $f \in \mathcal{S}^\star$.

The Dvoretzky and Rogers theorem

18. DEFINITION. Let z_i, $i = 1, 2, \dots$ be a sequence of elements of the Banach space Y. The series $\sum_{i=1}^{\infty} z_i$ is called *unconditionally convergent* if for

an arbitrary permutation of its summands it converges to the same element $z \in Y$.

The series $\sum_{i=1}^{\infty} z_i$ is called *absolutely convergent* if $\sum_{i=1}^{\infty} \|z_i\|_Y < \infty$.

REMARK. The Bolzano – Cauchy condition corresponding to the concept of unconditional convergence of a series $\sum_{i=1}^{\infty} z_i$ reads as follows:

The series $\sum_{i=1}^{\infty} z_i$ is unconditionally convergent if and only if to every $\varepsilon > 0$ there is a $k \in \mathbb{N}$ such that

$$\|\sum_{i \in Q} z_i\|_Y < \varepsilon$$

for every finite set $Q \subset \{k+1, k+2, \dots\}$.

It is easy to see that if $\sum_{i=1}^{\infty} z_i$ is absolutely convergent then it is unconditionally convergent.

In [3] Dvoretzky and Rogers proved the following theorem

19. THEOREM. *In an infinite – dimensional Banach space Y for every sequence $c_i > 0$, $i \in \mathbb{N}$ for which $\sum_{i=1}^{\infty} c_i^2 < \infty$, there is an unconditionally convergent series $\sum_{i=1}^{\infty} z_i$, for which $\|z_i\|_Y = c_i$.*

REMARK. The choice $c_i = \dfrac{1}{n}$ gives an example of an unconditionally convergent series $\sum_{i=1}^{\infty} z_i$ for which $\sum_{i=1}^{\infty} \|z_i\|_Y = \sum_{i=1}^{\infty} \frac{1}{i} = \infty$.

Dvoretzky and Rogers proved also that the unconditional convergence of the series $\sum_{i=1}^{\infty} z_i$ is equivalent to the absolute convergence of $\sum_{i=1}^{\infty} z_i$ if and only if the dimension of the Banach space Y is finite.

The infinite dimensional case

In the sequel we will use these results of Dvoretzky and Rogers to show that the result of Proposition 15 does not hold for the case of an infinite – dimensional Banach space Y, i.e. that we have $S^\star \subset S$ in this case.

20. LEMMA. *Suppose that $z_i \in Y, \lambda_i \in [0,1]$ for $i = 1, \dots, k$. Assume*

$$\|\sum_{i \in Q_l} z_i\|_Y < 1$$

for any part Q_l of elements of $\{1,2,\dots,k\}$ with l elements where $l \le k$. Then

$$\|\sum_{j=1}^{k} \lambda_j z_j\|_Y < \max_j \lambda_j \le 1.$$

PROOF. Without loss of generality assume that

$$0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le 1.$$

Then

$$\begin{aligned}
\sum_{j=1}^{k} \lambda_j z_j &= \lambda_1(z_1 + \cdots + z_k) - \lambda_1(z_2 + \cdots + z_k) + \sum_{j=2}^{k} \lambda_j z_j \\
&= \lambda_1(z_1 + \cdots + z_k) + \lambda_2(z_2 + \cdots + z_k) - \lambda_1(z_2 + \cdots + z_k) \\
&\quad - \lambda_2(z_3 + \cdots + z_k) + \sum_{j=3}^{k} \lambda_j z_j \\
&= \lambda_1(z_1 + \cdots + z_k) + (\lambda_2 - \lambda_1)(z_2 + \cdots + z_k) \\
&\quad + (\lambda_3 - \lambda_2)(z_3 + \cdots + z_k) + \cdots + (\lambda_k - \lambda_{k-1}) z_k
\end{aligned}$$

and therefore

$$\begin{aligned}
&\|\sum_{j=1}^{k} \lambda_j z_j\|_Y \\
&\quad \le \lambda_1 \|\sum_{j=1}^{k} z_j\|_Y + (\lambda_2 - \lambda_1)\|\sum_{j=2}^{k} z_j\|_Y + \cdots + (\lambda_k - \lambda_{k-1})\|z_k\|_Y \\
&\quad \le \lambda_1 + (\lambda_2 - \lambda_1) + (\lambda_3 - \lambda_2) + \cdots + (\lambda_k - \lambda_{k-1}) \\
&\quad = \lambda_k = \max_j \lambda_j \le 1.
\end{aligned}$$

Assume now that the dimension of the Banach space Y is infinite and that $\sum\limits_{j=1}^{\infty} z_j$ is an unconditionally convergent series for which $\sum\limits_{j=1}^{\infty} \|z_j\|_Y = +\infty$. Such a series exists by the above mentioned result of Dvoretzky and Rogers.

Let $K_j \subset [a,b]$, $j = 1,2,\dots$ be open intervals such that $K_j \cap K_i = \emptyset$ for $i \ne j$. We have

$$\sum_{j=1}^{\infty} \mu(K_j) \le \mu([a,b]) < \infty.$$

Denote

$$K = \bigcup_{j=1}^{\infty} K_j,\ C = [a,b] \setminus K.$$

Let us set

$$y_j = \frac{z_j}{\mu(K_j)} \text{ for } j = 1, 2, \dots .$$

The series $\sum_{j=1}^{\infty} y_j \mu(K_j)$ unconditionally converges to a sum $s \in Y$ while

$$\sum_{j=1}^{\infty} \|y_j\|_Y \mu(K_j) = +\infty.$$

Let $\varepsilon > 0$. Take $m \in \mathbb{N}$ such that

(a) $\| \sum_{j=1}^{m} y_j \mu(K_j) - s\|_Y < \frac{\varepsilon}{3}$,

(b) $\| \sum_{j \in Q}^{m} y_j \mu(K_j) - s\|_Y < \frac{\varepsilon}{3}$ for any finite set $Q \subset \{m+1, m+2, \dots\}$

and define

$$f(t) = 0 \text{ for } t \in C,$$

$$f(t) = y_j \text{ for } t \in K_j, j = 1, 2, \dots .$$

Assume that $\delta : [a,b] \to (0, \infty)$ is such a gauge on $[a,b]$ that

$$(t - \delta(t), t + \delta(t)) \cap [a,b] \subset K_j$$

for $j = 1, 2, \dots$ and $t \in K_j$. Let

$$0 < \eta < \frac{\varepsilon}{3} \frac{1}{1 + \sum_{j=1}^{m} \|y_j\|_Y}$$

and let $G \subset [a,b]$ be an open set for which

$$C = [a,b] \setminus K \subset G \text{ and } \mu(G) < \mu(C) + \eta.$$

For $t \in C$ assume that

$$(t - \delta(t), t + \delta(t)) \cap [a,b] \subset G.$$

Let $\{(t_i, J_i), i = 1, \dots, k\}$ be a $\delta-$ fine $L-$ partition of $[a,b]$.

Then by (a)

$$\|\sum_{i=1}^{k} f(t_i)\mu(J_i) - s\|_Y < \frac{\varepsilon}{3} + \|\sum_{i=1}^{k} f(t_i)\mu(J_i) - \sum_{j=1}^{m} y_j\mu = (K_j)\|_Y.$$

Denote

$$K_* = \bigcup_{j=1}^{m} K_j \text{ and } K_{**} = \bigcup_{j=m+1}^{\infty} K_j$$

and split the sum

$$\sum_{i=1}^{k} f(t_i)\mu(J_i) = \sum_{i=1, t_i \in K}^{k} f(t_i)\mu(J_i)$$

into two parts

$$\begin{aligned}\sum_{i=1}^{k} f(t_i)\mu(J_i) &= \sum_{i=1, t_i \in K_*}^{k} f(t_i)\mu(J_i) + \sum_{i=1, t_i \in K_{**}}^{k} f(t_i)\mu(J_i) \\ &= \sum_{j=1}^{m} \sum_{i=1, t_i \in K_j}^{k} f(t_i)\mu(J_i) + \sum_{j=m+1}^{\infty} \sum_{i=1, t_i \in K_j}^{k} f(t_i)\mu(J_i) \\ &= \sum_{j=1}^{m} y_j \sum_{i=1, t_i \in K_j}^{k} \mu(J_i) + \sum_{j=m+1}^{\infty} y_j \sum_{i=1, t_i \in K_j}^{k} \mu(J_i).\end{aligned}$$

Then we obtain

$$\begin{aligned}&\|\sum_{i=1}^{k} f(t_i)\mu(J_i) - \sum_{j=1}^{m} y_j\mu(K_j)\|_Y \\ &\quad \leq \|\sum_{j=1}^{m} y_j\Big(\sum_{i=1, t_i \in K_j}^{k} \mu(J_i) - \mu(K_j)\Big)\|_Y + \|\sum_{j=m+1}^{m} y_j\Big(\sum_{i=1, t_i \in K_j}^{k} \mu(J_i)\Big)\|_Y.\end{aligned}$$

The last term in this inequality consists of a finite number of nonzero terms only and we have

$$\sum_{i=1, t_i \in K_j}^{k} \mu(J_i) \leq \mu(K_j),$$

i.e.

$$\sum_{i=1, t_i \in K_j}^{k} \mu(J_i) = \lambda_j \mu(K_j)$$

where $\lambda_j \in [0,1]$. By (b) and by Lemma 18 we get

$$\| \sum_{j=m+1}^{m} y_j(\sum_{i=1, t_i \in K_j}^{k} \mu(J_i))\|_Y < \frac{\varepsilon}{3}.$$

It remains to give an estimate for $\| \sum_{j=1}^{m} y_j(\sum_{i=1,t_i \in K_j}^{k} \mu(J_i) - \mu(K_j))\|_Y$. We have

$$\|\sum_{j=1}^{m} y_j(\sum_{i=1,t_i\in K_j}^{k} \mu(J_i) - \mu(K_j))\|_Y \leq \sum_{j=1}^{m} \|y_j\|_Y(\mu(K_j) - \sum_{i=1,t_i\in K_j}^{k} \mu(J_i))\|_Y$$

and

$$\mu(K \setminus \bigcup_{t_i\in K} J_i) = \mu(K) - \mu(\bigcup_{t_i\in K} J_i) = \mu([a,b]) - \mu(C) - \mu(\bigcup_{t_i\in K} J_i)$$
$$= \mu(\bigcup_{i=1}^{k} J_i) - \mu(C) - \mu(\bigcup_{t_i\in K} J_i) = \mu(\bigcup_{t_i\in C} J_i) - \mu(C) \leq \mu(G) - \mu(C) < \eta.$$

Since

$$K_j \setminus \bigcup_{t_i\in K_j} J_i \subset K \setminus \bigcup_{t_i\in K} J_i$$

we get

$$0 \leq \mu(K_j \setminus \bigcup_{t_i\in K_j} J_i) = \mu(K_j) - \mu(\bigcup_{t_i\in K_j} J_i) \leq \mu(K \setminus \bigcup_{t_i\in K} J_i) < \eta$$

for every $j = 1,\dots,m$.

Therefore

$$\|\sum_{j=1}^{m} y_j(\sum_{i=1,t_i\in K_j}^{k} \mu(J_i) - \mu(K_j))\|_Y \leq \eta \sum_{j=1}^{m} \|y_j\|_Y < \frac{\varepsilon}{3}$$

Finally we obtain

$$\|\sum_{i=1}^{k} f(t_i)\mu(J_i) - s\|_Y < \varepsilon = 7F$$

and this means that the integral $(\mathcal{S})\int_a^b f d\mu$ exists and

$$(\mathcal{S})\int_a^b f d\mu = s = \sum_{j=1}^{\infty} y_j \mu(K_j),$$

i.e. $f \in \mathcal{S}$. on the other hand, since the series $\sum_{j=1}^{\infty} y_j \mu(K_j)$ does not converge absolutely, the Bochner integral $(\mathcal{L}) \int_a^b f d\mu$ does not exist because $(\mathcal{L}) \int_a^b \|f\|_Y d\mu = \sum_{j=1}^{\infty} \|y_j\|_Y \mu(K_j) = \infty$.

This construction leads to the following statement.

21. PROPOSITION. If Y is an infinite – dimensional Banach space then there exists a function $f : [a,b] \to Y$ such that $f \in \mathcal{S}$ and $f \notin \mathcal{S}^\star$.

Finally together with Proposition 17 we obtain:

22. PROPOSITION. Given a Banach space Y then $\mathcal{S}^\star \subset \mathcal{S}$ and $\mathcal{S}^\star = \mathcal{S}$ if and only if the dimension of Y is finite.

REFERENCES

[1] S. S. Cao, *The Henstock integral for Banach valued functions*, SEA Bull. Math. **16** (1992), 35–40.

[2] N. Dunford, J. T. Schwartz, *Linear operators I.*, Interscience Publishers, New York, London, 1958.

[3] A. Dvoretzky, C. A. Rogers, *Absolute and unconditional convergence in normed linear spaces*, Proc. Nat. Acad. Sci. (USA) **36** (1950), 192–197.

[4] D. H. Fremlin, J. Mendoza, *The integration of vector valued functions*, Illinois J. of Math. **38** (1994), 127–147.

[5] R. Gordon, *The McShane Integral of Banach – valued functions*, Illinois J. of Math. **34** (1990), 557–567.

[6] R. Gordon, *Riemann integration in Banach spaces*, Rocky Mountain J. of Math. **21** (1991), 923–949.

[7] R. Henstock, *Lectures on the Theory of Integration*, World Scientific, Singapore, 1988.

[8] R. Henstock, *The General Theory of Integration*, Clarendon Press, Oxford, 1991.

[9] E. Hille, R. S. Phillips, *Functional Analysis and Semi-groups*, American Mathematical Society, Providence, 1957.

[10] Y. Kubota, *A note on McShane's integral*, Math. Japonica **30** (1985), 57–62.

[11] J. Kurzweil, *Nichtabsolut konvergente Integrale*, BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1980.

[12] S. Lang, *Real and Functional Analysis*, Springer – Verlag, New York, 1993.

[13] J. Mawhin, *Analyse. Fondements, techniques, évolution*, De Boeck – Wesmael, Bruxelles, 1992.

[14] E. J. McShane, *Unified Integration*, Academic Press, New York, London, 1983.

[15] J. Mendoza, *On Lebesgue integrability of McShane integrable functions*, Real Analysis Exchange **18** (2) (1992/93), 456–458.

[16] Š. Schwabik, *Generalized Ordinary Differential Equations*, World Scientific, Singapore, 1992.

[17] K. Yosida, *Functional Analysis*, Academic Press, New York, 1965.

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