SECOND-ORDER DIFFERENTIAL SYSTEMS AND A REGULARIZATION OPERATOR

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Abstract. Sufficient conditions for the existence of solutions to the boundary value problems with a Carathéodory right side for the second order ordinary differential systems are established by means of a continuous approximations.

1. Introduction

In this paper there are proved theorems of existence of a solution to the differential system

$$(1.1) \qquad x'' = f(t, x, x')$$

satisfying the boundary condition

$$(1.2) \qquad V(x) = o,$$

where V is a continuous operator of boundary conditions and o is a zero point of the space $\mathbb{R}^{2n}$, $o = \overbrace{(0, 0, \ldots, 0)}^{(2n \text{ times})}$.

My results have been motivated by the fact that many methods used for ordinary differential systems haven't the same results when the function on the right side of (1.1) is Carathéodory or continuous one. The problem (1.1), (1.2) with the L^∞-Carathéodory function, the most similar to the continuous one, has been approximated here by a sequence of problems with

a continuous right side. The existence of a solution to the problem (1.1), (1.2) will be proved as a consequence of the existence of solutions to the approximated problems.

Let $-\infty < a < b < \infty$, $I = [a, b]$, $\mathbb{R} = (-\infty, \infty)$, n, k natural numbers. $\mathbb{R}^n$ denotes as usual Euclidean n-space and $\|x\|$ denotes the Euclidean norm. $C_n^k = C^k([a, b], \mathbb{R}^n)$ is the Banach space of functions u such that $u^{(k)}$ is continuous on I with the norm

$$\|u\|_k = \max\left\{\|u\|, \|u'\|, \|u''\|, \ldots, \|u^{(k)}\|\right\},$$

where

$$\|u\| = \max\left\{\|u(t)\|, t \in I\right\}.$$

Let C_n denote C_n^0. $C_{nO}^\infty = C^\infty(\mathbb{R}, \mathbb{R}^n)$ is the space of functions ϕ such that for each $k \in \{1, 2, \ldots\}$ there exists continuous on $\mathbb{R}$ function $\phi^{(k)}$ and a support of function ϕ is a bounded closed set, $\operatorname{supp} \phi = \overline{\{x \in \mathbb{R}; \|\phi(x)\| > 0\}}$. Finally let $L_n^\infty = L_n^\infty((a, b), \mathbb{R}^n)$ be as usually a space of measurable functions with a finite norm

$$|u|_\infty = \inf_{M \in \mathcal{M}}\left\{\sup_{t \in I - M}\left\{\|u(t)\|\right\}\right\},$$

where $\mathcal{M}$ is a set of all measurable subsets of an interval I with a measure zero.

DEFINITION 1.1. A function $f : I \times \mathbb{R}^{2n} \to \mathbb{R}^n$ is a L^∞-Carathéodory function provided: if $f = f(t, u, p)$
 (i) the map $(u, p) \mapsto f(t, u, p)$ is continuous for almost every $t \in I$,
 (ii) the map $t \mapsto f(t, u, p)$ is measurable for all $(u, p) \in \mathbb{R}^n \times \mathbb{R}^n$,
(iii) for each bounded subset $B \subset \mathbb{R}^n \times \mathbb{R}^n$ the function $\sup\{\|f(t, u, p)\|, (u, p) \in B\} \in L^\infty(I)$.

DEFINITION 1.2. A function $w : I \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function provided: if $w = w(t, \delta)$
 (i) the map $\delta \mapsto w(t, \delta)$ is continuous for almost every $t \in I$,
 (ii) the map $t \mapsto w(t, \delta)$ is measurable for all $\delta \in \mathbb{R}$,
(iii) for each bounded subset $B \subset \mathbb{R}$ the function $\sup\{|w(t, \delta)|, \delta \in B\}$ is Lebesgue integrable function on interval I.

LEMMA 1.1. *Let $f : I \times \mathbb{R}^{2n} \to \mathbb{R}^n$ be a L^∞-Carathéodory function and B a bounded subset of $\mathbb{R}^n \times \mathbb{R}^n$.*
Then there exists a constant $K \in \mathbb{R}$ and a set $M \in \mathcal{M}$ such that

$$\|f(t, u, p)\| \leq K \quad \text{for } t \in I - M, \ (u, p) \in B.$$

PROOF. It is a trivial consequence of the definition L^∞ and the definition 1.1. $\qquad\square$

In the whole paper assume $f : I \times \mathbb{R}^{2n} \to \mathbb{R}^n$ is the L^∞-Carathéodory function and $V : C_n^1 \to \mathbb{R}^{2n}$ is a continuous operator.

If f is continuous, by a solutions to the equation (1.1) we mean a classical solution with a continuous 2nd derivative, while if f is a Carathéodory function, a solution will mean a function x which has an absolutely continuous 1st derivative such that x fulfills the equality $x''(t) = f(t, x(t), x'(t))$ for almost every $t \in I$.

By xy in $\mathbb{R}^n$ we mean a scalar product of two vectors from $\mathbb{R}^n$.

2. Regularization operator

Let ϕ be in $C_{1,O}^\infty$ such that

$$\phi(t) \geq 0 \ \forall t \in \mathbb{R}, \quad \operatorname{supp}\phi = [-1,1], \quad \int_{-1}^{1} \phi(t)\,dt = 1.$$

For an example of such function see [3] page 26.

Instead of problem (1.1), (1.2) we will consider the equation

$$(2.1_\epsilon) \qquad\qquad x'' = f_\epsilon(t, x, x')$$

with the boundary condition (1.2), where ϵ is a positive real number and for $\forall (u,p) \in \mathbb{R}^n \times \mathbb{R}^n$

$$f_\epsilon(t, u, p) = \frac{1}{\epsilon} \int_a^b \phi\left(\frac{t - \eta}{\epsilon}\right) f(\eta, u, p)\, d\eta$$

or equivalently

$$f_\epsilon(t, u, p) = \int_{-1}^{1} \overline{f}(t - \epsilon\eta, u, p)\phi(\eta)\, d\eta,$$

where $\overline{f}(t, u, p) = \begin{cases} f(t, u, p) & t \in [a, b] \\ 0 & t \notin [a, b] \end{cases}$.

LEMMA 2.1. Let B be a bounded subset $\mathbb{R}^n \times \mathbb{R}^n$ and $\Phi = \max\{\phi(t); t \in [-1, 1]\}$. Then the function $f_\epsilon(t, u, p)$ is continuous on $I \times B$ and for every $\epsilon > 0$, $t \in I$ and $(u, p) \in B$

$$\|f_\epsilon(t, u, p)\| \leq 2K\sqrt{n}\Phi.$$

PROOF. Continuity of f_ϵ follows from the theorem on continuous dependence of the integral on a parameter.

Boundeness of f_ϵ follows from the inequalities bellow where we use lemma 1.1.

$$|f_\epsilon(t,u,p)| = \left| \int_{-1}^{1} \overline{f}(t - \epsilon\eta, u, p)\phi(\eta)\, d\eta \right|$$

$$\leq \sqrt{n} \int_{-1}^{1} |\overline{f}(t - \epsilon\eta, u, p)|\phi(\eta)\, d\eta \leq 2K\sqrt{n}\Phi$$

$\square$

DEFINITION 2.1. Let $w : I \times [0, \infty) \to [0, \infty)$ be a Caratéodory function. We say $w \in M(I \times [0, \infty); [0, \infty))$ if there will be satisfied this conditions:

(i) For almost every $t \in I$ and for every $d_1, d_2 \in \mathbb{R}$, $d_1 < d_2$

$$w(t, d_1) \leq w(t, d_2).$$

(ii) For almost every $t \in I$ $w(t, 0) = 0$.

DEFINITION 2.2. Let B be a compact subset of $\mathbb{R}^{2n}$, $\tau \in \mathbb{R}$ and $\delta \in [0, \infty)$. Let us denote by $\omega(\tau, \delta)$ a function

$$\omega(\tau, \delta) = \max\{|\overline{f}(\tau, u, p) - \overline{f}(\tau, u', p')|; (u, p), (u', p') \in B,$$
$$|u - u'|, |p - p'| \leq \delta\}$$

and by $\omega_\epsilon(\tau, \delta)$ a function

$$\omega_\epsilon(\tau, \delta) = \frac{1}{\epsilon} \int_a^b \phi\left(\frac{\tau - \eta}{\epsilon}\right) \omega(\eta, \delta)\, d\eta$$

or equivalently

$$\omega_\epsilon(\tau, \delta) = \int_{-1}^{1} \omega(\tau - \epsilon\eta, \delta)\phi(\eta)\, d\eta.$$

LEMMA 2.2. *Let B be a bounded closed subset of $\mathbb{R}^n \times \mathbb{R}^n$. Then for every $\epsilon > 0$:*

(i) $\omega, \omega_\epsilon \in M(I \times [0, \infty); [0, \infty))$.

(ii) *For every $(u, p) \in B$, for every $\delta \geq 0$ and for almost every $t \in I$*

$$f_\epsilon(t, u, p) \to f(t, u, p) \qquad \text{for } \epsilon \to 0,$$

$$\omega_\epsilon(t,\delta) \to \omega(t,\delta) \qquad \text{for } \epsilon \to 0.$$

(iii) For every $(u,p) \in B$ and for almost every $t \in I$

$$|f_\epsilon(t,u,p) - f_\epsilon(t,u',p') - f(t,u,p) + f(t,u',p')|$$
$$\leq \sqrt{n}\,\omega_\epsilon(t, \max\{|u-u'|, |p-p'|\})$$
$$+ \omega(t, \max\{|u-u'|, |p-p'|\}).$$

(iv) For every $(u,p) \in B$ and for almost every $t \in I$

$$\int_a^t f_\epsilon\left((\tau,u,p) - f(\tau,u,p)\right)\, d\tau$$

converges uniformly to 0 for $\epsilon \to 0$ on the set $I \times B$.

PROOF.

(i) Since $f(\tau,.,.)$ is L^∞-Carathéodory and B is a compact set then for almost every $\tau \in I$ $0 \leq \omega(\tau,\delta) \leq 2K$, $\omega(\tau,.)$ is nondecreasing and continuous, $\omega(.,\delta)$ is measurable and

$$\lim_{\delta \to 0+} \omega(\tau,\delta) = 0.$$

It means, that $\omega(\tau,0) = 0$ for almost every $t \in I$. Therefore we can see that $\omega \in M(I \times [0,\infty); [0,\infty))$.

From the theorem on continuous dependence of the integral on a parameter there follows that ω_ϵ is for arbitrary $\epsilon > 0$ continuous function. Therefore ω_ϵ is Carathéodory function such that $\omega_\epsilon(\tau,0) = 0$ for almost every $\tau \in I$. From inequalities for $\delta_1 < \delta_2$, $\tau \in I$

$$(2.2) \qquad 0 \leq \omega(\tau,\delta_1) \leq \omega(\tau,\delta_2)$$

follows for almost every $\eta \in I$

$$0 \leq \frac{1}{\epsilon}\phi\left(\frac{\tau-\eta}{\epsilon}\right)\omega(\eta,\delta_1) \leq \frac{1}{\epsilon}\phi\left(\frac{\tau-\eta}{\epsilon}\right)\omega(\eta,\delta_2)$$

and therefore

$$(2.3) \qquad 0 \leq \omega_\epsilon(\tau,\delta_1) \leq \omega_\epsilon(\tau,\delta_2).$$

It means that $\omega_\epsilon \in M(I \times [0,\infty); [0,\infty))$

(ii) This statement is a consequence to [2] theorem 2.5.3 which assert that on our assumption there hold for every $\delta > 0$, $(u,p) \in B$ and $i = 1,2,\dots,n$

$$\lim_{\epsilon \to 0+} \int_{-1}^{1} |\omega_\epsilon(\tau,\delta) - \omega(\tau,\delta)|\, d\tau = 0,$$

$$\lim_{\epsilon \to 0+} \int_{-1}^{1} |f_{\epsilon i}(\tau, u, p) - f_i(\tau, u, p)| \, d\tau = 0,$$

where f_i, $f_{\epsilon i}$ is i-th component of the function f, f_ϵ respectively.

(iii) Obviously for $\|u - u'\|, \|p - p'\| \le \delta$

$$\|f_\epsilon(t, u, p) - f_\epsilon(t, u', p')\| = \left\| \int_{-1}^{1} \phi(\eta) \left(\overline{f}(t - \epsilon\eta, u, p) - \overline{f}(t - \epsilon\eta, u', p') \right) d\eta \right\|$$

$$\le \sqrt{n} \int_{-1}^{1} \|\overline{f}(t - \epsilon\eta, u, v) - \overline{f}(t - \epsilon\eta, u', v')\| \phi(\eta) \, d\eta$$

$$\le \sqrt{n} \int_{-1}^{1} \omega(t - \epsilon\eta, \delta) \phi(\eta) \, d\eta = \sqrt{n} \omega_\epsilon(t, \delta)$$

Now it is easy to see that statement (iii) of this lemma holds.

(iv) Firstly we will prove that for every $(t, u, p) \in I \times B$ and every $e > 0$ there exists $\epsilon_0 > 0$ and neighbourhood $O_{(t,u,p)}$ of (t, u, p) in the set $I \times B$ such that for every $0 < \epsilon < \epsilon_0$ and for every $(t', u', p') \in O_{(t,u,p)}$

$$\left\| \int_{a}^{t'} (f_\epsilon(\tau, u', p') - f(\tau, u', p')) \, d\tau \right\| < e.$$

By (ii) and by Lebesgue dominated convergence theorem there exists $\epsilon_1 > 0$ such that for every $0 < \epsilon < \epsilon_1$

$$\int_{a}^{b} \|f_\epsilon(\tau, u, p) - f(\tau, u, p)\| \, d\tau < \frac{e}{4\sqrt{n}}.$$

Since $\omega \in M(I \times [0, \infty); [0, \infty))$ there exists such $\delta > 0$ that

$$\int_{a}^{b} \omega(\tau, \delta) \, d\tau < \frac{e}{4n}.$$

By (ii) and by Lebesgue dominated convergence theorem there exists $\epsilon_2 > 0$ such that for every $0 < \epsilon < \epsilon_2$

$$\int_{a}^{b} \omega_\epsilon(\tau, \delta) \, d\tau < \frac{e}{2n}.$$

Let us denote $O_{(t,u,p)} = \{(t', u', p') \in I \times B; \|u - u'\| < \delta, \|p - p'\| < \delta\}$ and

$\epsilon_0 = \min\{\epsilon_1, \epsilon_2\}$. Now for every $0 < \epsilon < \epsilon_0$ and for every $(t', u', p') \in O_{(t,u,p)}$

$$\| \int_a^{t'} (f_\epsilon(\tau, u', p') - f(\tau, u', p')) \, d\tau\| \leq \| \int_a^{t'} (f_\epsilon(\tau, u, p) - f(\tau, u, p)) \, d\tau\|$$

$$+ \| \int_a^{t'} (f_\epsilon(\tau, u, p) - f_\epsilon(\tau, u', p') - f(\tau, u, p) + f(\tau, u', p')) \, d\tau\|$$

$$\leq \sqrt{n} \int_a^b \|f_\epsilon(\tau, u, p) - f(\tau, u, p)\| \, d\tau + \sqrt{n} \int_a^b \sqrt{n}\omega_\epsilon(\tau, \delta) + \omega(\tau, \delta) \, d\tau$$

$$< \sqrt{n}\frac{e}{4\sqrt{n}} + n\frac{e}{2n} + \sqrt{n}\frac{e}{4n} \leq e.$$

This means, that the system of the sets $\{O_{(t,u,p)}\}_{(t,u,p) \in I \times B}$ covers the compact set $I \times B$ and therefore there exists a finite subsystem which covers the set $I \times B$ and therefore the statement of the (iv) holds. $\qquad \square$

LEMMA 2.3. *Let $B \subset \mathbb{R}^{2n}$ be a compact set. Let E be a set of $\epsilon > 0$ such that systems of functions $\{x_\epsilon\}_{\epsilon \in E}$, $\{x'_\epsilon\}_{\epsilon \in E}$, $x_\epsilon : I \to \mathbb{R}^n$ are equi-continuous and $0 \in \overline{E}$.*
Then $\int_a^t f_\epsilon(\tau, x_\epsilon(\tau), x'_\epsilon(\tau)) - f(\tau, x_\epsilon(\tau), x'_\epsilon(\tau)) \, d\tau$ converges uniformly to 0 on the set I.

PROOF. This proof is a modification to the proof of lemma 3.1 in [5].
Let us denote for $\epsilon \in E$

$$\alpha_\epsilon = \sup \left\{ \| \int_s^t f_\epsilon(\tau, u, p) - f(\tau, u, p) \, d\tau\|; \ a \leq s < t \leq b, \ (u, p) \in B \right\},$$

$$\beta_\epsilon = \max \left\{ \| \int_a^t f_\epsilon(\tau, x_\epsilon(\tau), x'_\epsilon(\tau)) - f(\tau, x_\epsilon(\tau), x'_\epsilon(\tau)) \, d\tau\|; a \leq t \leq b \right\}.$$

By (iv) of lemma 2.2

$$\lim_{\epsilon \to 0} \alpha_\epsilon = 0.$$

We want to prove

$$\lim_{\epsilon \to 0} \beta_\epsilon = 0.$$

Let $e > 0$ be an arbitrary real number. Then there exist by (i) of lemma 2.2 such $\delta > 0$ that

$$\int_a^b \omega(\tau, \delta) \, d\tau < \frac{e}{3n},$$

and by (i), (ii) of lemma 2.2 such $\epsilon_1 > 0$ that for every $\epsilon \in \boldsymbol{E}$, $\epsilon < \epsilon_1$

$$\int_a^b \omega_\epsilon(\tau, \delta)\, d\tau < \frac{2e}{3n}.$$

Since $\{x_\epsilon\}_{\epsilon \in \boldsymbol{E}}$, $\{x'_\epsilon\}_{\epsilon \in \boldsymbol{E}}$ are equi-continuous there exists $\delta_0 > 0$ such that

$$\|x_\epsilon(t) - x_\epsilon(\tau)\| < \delta, \ \|x'_\epsilon(t) - x'_\epsilon(\tau)\| < \delta \ \text{ for } t, \tau \in I, \ |t - \tau| \leq \delta_0, \epsilon \in \boldsymbol{E}.$$

Let k be such integer that $k \leq \frac{b-a}{\delta_0} < k + 1$. Let us denote $t_i = a + i\delta_0$, $\overline{x_\epsilon}(t) = x_\epsilon(t_i)$ and $\overline{x'_\epsilon}(t) = x'_\epsilon(t_i)$ for $t_i \leq t < t_{i+1}$, where $i = 0, 1, \ldots, k$. Then

$$\|x_\epsilon(t) - \overline{x_\epsilon}(t)\| < \delta,$$

$$\|x'_\epsilon(t) - \overline{x'_\epsilon}(t)\| < \delta,$$

for $t \in I$ and $\epsilon \in \boldsymbol{E}$ and

$$\left\| \int_a^t f_\epsilon(\tau, \overline{x_\epsilon}(\tau), \overline{x'_\epsilon}(\tau)) - f(\tau, \overline{x_\epsilon}(\tau), \overline{x'_\epsilon}(\tau))\, d\tau \right\| \leq (k+1)\alpha_\epsilon$$

for $a < t < b$ and $\epsilon < \epsilon_0$, $\epsilon \in \boldsymbol{E}$.

Therefore by (iii) of lemma 3.5 we obtain

$$\left\| \int_a^t \big(f_\epsilon(\tau, x_\epsilon(\tau), x'_\epsilon(\tau)) - f(\tau, x_\epsilon(\tau), x'_\epsilon(\tau)) \big)\, d\tau \right\|$$

$$\leq \sqrt{n} \int_a^t \| f_\epsilon(\tau, x_\epsilon(\tau), x'_\epsilon(\tau)) - f(\tau, x_\epsilon(\tau), x'_\epsilon(\tau)) - f_\epsilon(\tau, \overline{x_\epsilon}(\tau), \overline{x'_\epsilon}(\tau))$$

$$+ f(\tau, \overline{x_\epsilon}(\tau), \overline{x'_\epsilon}(\tau)) \|\, d\tau$$

$$+ \left\| \int_a^t \big(f_\epsilon(\tau, \overline{x_\epsilon}(\tau), \overline{x'_\epsilon}(\tau)) - f(\tau, \overline{x_\epsilon}(\tau), \overline{x'_\epsilon}(\tau)) \big)\, d\tau \right\|$$

$$\leq \sqrt{n} \int_a^b \big(\sqrt{n}\omega_\epsilon(\tau, \delta) + \omega(\tau, \delta) \big)\, d\tau + (k+1)\alpha_\epsilon < e + (k+1)\alpha_\epsilon$$

for $t \in I$, $\epsilon < \epsilon_1$, $\epsilon \in \boldsymbol{E}$.

Therefore $\beta_\epsilon < e + (k+1)\alpha_\epsilon$ for $\epsilon < \epsilon_1$, $\epsilon \in \boldsymbol{E}$. Since $\lim_{\epsilon \to 0} \alpha_\epsilon = 0$ and e is arbitrary then $\lim_{\epsilon \to 0} \beta_\epsilon = 0$. $\square$

THEOREM 2.1. *Let $f : I \times \mathbb{R}^{2n} \to \mathbb{R}^n$ be the L^∞-Carathéodory function. Denote by $\boldsymbol{E}$ the set of positive ϵ such that for $\forall \epsilon \in \boldsymbol{E}$ there exists a solution x_ϵ to the problem (2.1_ϵ), (1.2). Suppose that $0 \in \overline{\boldsymbol{E}}$ and that there exists a*

compact subset $B \subset \mathbb{R}^{2n}$ independent on ϵ such that for $\forall \epsilon \in \boldsymbol{E}$ and $\forall t \in I$ $(x_\epsilon(t), x_\epsilon'(t)) \in B$.

Then there exist a sequence $\{\epsilon_k\}$, $\epsilon_k \in \boldsymbol{E}$, $\epsilon_k \to 0$ and a function $x : I \to \mathbb{R}^n$, $(x(t), x'(t)) \in B$ $\forall t \in I$, such that $(x_{\epsilon_k}, x_{\epsilon_k}') \rightrightarrows (x, x')$ and $x_{\epsilon_k}'' \to x''$ pointwise and x is a solution to the problem (1.1), (1.2).

PROOF. At first let us prove that by the conditions of Theorem 2.1 the set $\{x_\epsilon\}_{\epsilon \in \boldsymbol{E}}$ is relatively compact in C_n^1. Really, to be satisfied assumptions Arzelà-Ascoli theorem, it is necessary to prove equi-continuity of the set $\{x_\epsilon'\}_{\epsilon \in \boldsymbol{E}}$.

Suppose $t_1, t_2 \in I$ and compute

$$\|x_\epsilon'(t_1) - x_\epsilon'(t_2)\| = \| \int_{t_1}^{t_2} x_\epsilon'' \, dt \| = \| \int_{t_1}^{t_2} f_\epsilon(t, x_\epsilon(t), x_\epsilon'(t)) \, dt \|$$

$$= \| \int_{t_1}^{t_2} \int_{-1}^{1} \overline{f}(t - \epsilon\eta, x_\epsilon(t), x_\epsilon'(t)) \phi(\eta) \, d\eta \, dt \|$$

$$\leq |n \int_{t_1}^{t_2} \int_{-1}^{1} \|\overline{f}(t - \epsilon\eta, x_\epsilon(t), x_\epsilon'(t))\| \phi(\eta) \, d\eta \, dt|.$$

Since the function f is L^∞-Carathéodory then by the Lemma 1.1 there exist a constant K and a set $M \in \mathcal{M}$ such that

$$\|f(t, u, p)\| \leq K \quad \text{for } t \in I - M, \ (u, p) \in B.$$

Since ϕ is a continuous function there exist a constant Φ such that $\phi(t) \leq \Phi$. Now we have

$$\left| \int_{t_1}^{t_2} \int_{-1}^{1} \|\overline{f}(t - \epsilon\eta, x_\epsilon(t), x_\epsilon'(t))\| \phi(\eta) \, d\eta \, dt \right|$$

$$\leq \left| \int_{t_1}^{t_2} \int_{-1}^{1} K\Phi \, d\eta \, dt \right| \leq 2K\Phi |t_2 - t_1|.$$

This means that the set $\{x_\epsilon\}_{\epsilon \in \boldsymbol{E}}$ is relatively compact in C_n^1. Therefore there exist sequence $\{\epsilon_k\}$, $\epsilon_k \in \boldsymbol{E}$, $\epsilon_k \to 0$ and function $x : I \to \mathbb{R}^n$ such that $x(t) \in B$, $\forall t \in I$, $x_{\epsilon_k} \to x$ in C_n^1.

Now, since x_{ϵ_k} is the solution to the equation (2.1_ϵ) for $\epsilon = \epsilon_k$, we have

$$(2.4) \qquad x_{\epsilon_k}'(t) = x_{\epsilon_k}'(a) + \int_a^t f_{\epsilon_k}(\tau, x_{\epsilon_k}(\tau), x_{\epsilon_k}'(\tau)) \, d\tau \quad , \forall t \in I.$$

Using lemma 2.3 we get

$$x'(t) = x'(a) + \int_a^t f(\tau, x(\tau), x'(\tau)) \, d\tau$$

which means, that x is a solution to the equation (1.1).

Since $(x_{\epsilon_k}, x'_{\epsilon_k}) \rightrightarrows (x, x')$, V is a continuous operator $V : C_n^1 \to \mathbb{R}^{2n}$ and x_{ϵ_k} is a solution to the problem (2.1_{ϵ_k}), (1.2), we see that

$$V(x_{\epsilon_k}) = o,$$

and therefore for $\epsilon_k \to 0$ we have

$$V(x) = o.$$

It means that x is a solution to the problem (1.1), (1.2). $\square$

3. An application

As an example how to use theorem 2.1 we may consider the equation (1.1) with four point boundary conditions

(3.1) $$x(0) = x(c), \qquad x(d) = x(1),$$

where $0 < c \leq d < 1$. In [1] we proved the following result.

THEOREM 3.1. *Let* $f : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ *be a continuous function and let us consider the problem (1.1), (3.1). Assume*
  (i) *there is a constant* $M \geq 0$ *such that* $uf(t,u,p) \geq 0$ *for* $\forall t \in [0,1]$, $\forall u \in \mathbb{R}^n$, $\|u\| > M$ *and* $\forall p \in \mathbb{R}^n$, $pu = 0$.
  (ii) *Suppose there exist continuous positive functions* $A_j, B_j, j \in \{1, 2, \ldots, n\}$
$$A_j : [0,1] \times \mathbb{R}^{n+j-1} \to \mathbb{R}, \qquad B_j : [0,1] \times \mathbb{R}^{n+j-1} \to \mathbb{R}$$

such that

$$|f_j(t,u,p)| \leq A_j(t,u,p_1,p_2,\ldots,p_{j-1})p_j^2 + B_j(t,u,p_1,p_2,\ldots,p_{j-1}),$$

where $f = (f_1, f_2, \ldots, f_n)$, $u \in \mathbb{R}^n$, $p \in \mathbb{R}^n$ $p = (p_1, p_2, \ldots, p_n)$ *and for* $j = 1$, A_1 *and* B_1 *are independent of* p *functions.*
Then the problem (1.1), (3.1) has a solution.

REMARK 3.1. From the proof of this theorem and from the topological transversality theorem in [4] it follows, that the solution to the problem (1.1), (3.1) is bounded by a constant $\mathbf{M}$ which is dependent only on M, A_j, B_j.

Now we can extend the results of Theorem 3.1 onto the L^∞-Carathéodory case.

THEOREM 3.2. *Let $f : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ be a L^∞-Carathéodory function and let us consider the problem (1.1), (3.1). Assume*

(i) *there is a constant $M \geq 0$ such that $uf(t,u,p) \geq 0$ for almost every t in $[0,1]$, $\forall u \in \mathbb{R}^n$, $\|u\| > M$ and $\forall p \in \mathbb{R}^n$, $pu = 0$.*

(ii) *Suppose there exist continuous positive functions A_j, B_j, $j \in \{1, 2, \ldots, n\}$*

$$A_j : [0,1] \times \mathbb{R}^{n+j-1} \to \mathbb{R}, \qquad B_j : [0,1] \times \mathbb{R}^{n+j-1} \to \mathbb{R}$$

such that for almost every $t \in [0,1]$

$$|f_j(t,u,p)| \leq A_j(t,u,p_1,p_2,\ldots,p_{j-1})p_j^2 + B_j(t,u,p_1,p_2,\ldots,p_{j-1}),$$

where $f = (f_1, f_2, \ldots, f_n)$, $u \in \mathbb{R}^n$, $p \in \mathbb{R}^n$ $p = (p_1, p_2, \ldots, p_n)$ and for $j = 1$, A_1 and B_1 are independent of p functions.

Then the problem (1.1), (3.1) has a solution.

PROOF. Let f_ϵ be an approximated function as in Part 2. Then

1) for $\forall \epsilon \in (0,1)$, for $\forall t \in [0,1]$, $\forall u \in \mathbb{R}^n$, $\|u\| > M$ and $\forall p \in \mathbb{R}^n$, $pu = 0$

$$f_\epsilon(t,u,p)u = \left(\frac{1}{\epsilon} \int_a^b \phi\left(\frac{t-\eta}{\epsilon}\right) f(\eta,u,p)\,d\eta \right) u$$

$$= \frac{1}{\epsilon} \int_a^b \phi\left(\frac{t-\eta}{\epsilon}\right) f(\eta,u,p)u\,d\eta \geq 0$$

by assumption (i) of this theorem.

2) Let $j \in \{1, 2, \ldots, n\}$, $u \in \mathbb{R}^n$, $p \in \mathbb{R}^n$, $p = (p_1, p_2, \ldots, p_n)$,

$$\mathcal{A}_j(u,p_1,p_2,\ldots,p_{j-1}) = \max_{t \in [0,1]}\{A_j(t,u,p_1,p_2,\ldots,p_{j-1})\}$$

and

$$\mathcal{B}_j(u,p_1,p_2,\ldots,p_{j-1}) = \max_{t \in [0,1]}\{B_j(t,u,p_1,p_2,\ldots,p_{j-1})\}.$$

Since A_j, B_j are continuous functions then $\mathcal{A}_j, \mathcal{B}_j$ are continuous too. Now we have

$$|f_{\epsilon,j}(t,u,p)| = |\int_{-1}^{1} \overline{f_j}(t - \epsilon\eta, u, p)\phi(\eta)\,d\eta| \leq \int_{-1}^{1} |\overline{f_j}(t - \epsilon\eta, u, p)|\phi(\eta)\,d\eta$$

$$\leq \int_{-1}^{1} (\mathcal{A}_j(u, p_1, p_2, \ldots, p_{j-1})p_j^2$$

$$+ \,\mathcal{B}_j(u, p_1, p_2, \ldots, p_{j-1}))\phi(\eta)\,d\eta$$

$$\leq \int_{-1}^{1} \mathcal{A}_j(u, p_1, p_2, \ldots, p_{j-1})p_j^2\phi(\eta)\,d\eta$$

$$+ \int_{-1}^{1} \mathcal{B}_j(u, p_1, p_2, \ldots, p_{j-1})\phi(\eta)\,d\eta$$

$$= \mathcal{A}_j(u, p_1, p_2, \ldots, p_{j-1})p_j^2 + \mathcal{B}_j(u, p_1, p_2, \ldots, p_{j-1}).$$

By the theorem 3.1 and remark 3.1 there exists a solution to the approximated problem (2.1_ϵ), (3.1) for every ϵ and $\|x_\epsilon\|_1 \leq M$.

Now all assumptions theorem 2.1 are fullfiled end therefore there exists the solution to the problem (1.1), (3.1). $\qquad\qquad \square$

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