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DIFFERENTIAL EQUATIONS OF THE SECOND ORDER WITH MEASURES AS COEFFICIENTS

WŁADYSŁAW KIERAT AND URSZULA SZTABA

Abstract. The note deals with differential equations of the second order with Borel measures as coefficients. The problem of existence and uniqueness of solutions is discussed. The Ritz–Galerkin method is used for determining of approximate solutions

1. We shall consider the boundary value problem

$$\begin{aligned} -u'' + \mu_1 u &= \mu_2 \\ u(a) = u(b) &= 0, \end{aligned} \tag{1}$$

where μ_1 and μ_2 are real Borel measures, $\mu_1 \geq 0$.

If μ_1 and μ_2 are integrable functions with respect to the Lebesgue measure, then the Ritz–Galerkin method is often used to investigate Problem (1). Here we shall show that this method may be applied to solving Problem (1) under the above assumptions, too. We are looking for a continuous function u vanishing at the end points a, b and fulfilling Equation (1) in the weak (distributional) sense. This means that

$$-\int_a^b u\varphi'' dx + \int_a^b u\varphi d_{\mu_1}(x) = \int_a^b \varphi d_{\mu_2}(x) \qquad \text{for} \quad \varphi \in D(a,b), \tag{2}$$

where $\int_a^b \varphi d_{\mu_2}(x) := \int_a^b \varphi q d_{|\mu_2|}(x)$, $|\mu_2|$ is the variation of μ_2 and $|q(x)| = 1$ a.e. ([3], p. 137) and $D(a,b)$ denotes the Schwartz space of the test functions with support contained in (a,b).

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We shall start with presenting some facts concerning the Sobolew space ([1], p.22).

DEFINITION 1.

$$W^{1,2}(a,b) = \{u \in L^2(a,b) :\ Du \in L^2(a,b), \quad \text{where } Du \text{ is the distributional derivative}\}$$

The natural norm of u in $W^{1,2}(a,b)$ is $\|u\| := \left(\|u\|^2_{\alpha^2} + \|Du\|^2_{\alpha^2}\right)^{\frac{1}{2}}$. The space $W^{1,2}(a,b)$ is complete with respect to this norm.

DEFINITION 2. $W^{1,2}_0(a,b)$ is the closure of $D(a,b)$ in $W^{1,2}(a,b)$ with respect to the norm $\|\cdot\|$.

One can show that

$$W^{1,2}_0(a,b) = \{u :\ u \quad \text{is absolutely continuous,} \quad u(a) = u(b) = 0 \text{ and } Du \in L^2(a,b)\}.$$

For simplicity of notation we put

$$\alpha(\varphi,\psi) := \int_a^b D\varphi D\psi dx + \int_a^b \varphi\psi d_{\mu_1}(x),$$

$$\beta(\varphi) := \int_a^b \varphi d_{\mu_2}(x),$$

for $\varphi, \psi \in W^{1,2}_0(a,b)$. It is easy to check that α is a bilinear symmetric positive definite form on $W^{1,2}_0(a,b)$ and β is a linear form on $W^{1,2}_0(a,b)$. If we are looking for a solution u in $W^{1,2}_0(a,b)$, then Equation (2) is equivalent to the equation

$$\int_a^b Du\varphi' dx + \int_a^b u\varphi d_{\mu_1}(x) = \int_a^b \varphi d_{\mu_2}(x), \qquad \varphi \in D(a,b). \tag{3}$$

or, using the above notation, too

$$\alpha(u,\varphi) = \beta(\varphi), \qquad \varphi \in D(a,b).$$

In the sequel we shall need the following two norms

$$\|u\|_\alpha := [\alpha(u,u)]^{\frac{1}{2}} \quad \text{and} \quad \|u\|_D := \|Du\|_{L^2}$$

for u in $W_0^{1,2}(a,b)$.

Now we are in a position to state

THEOREM 1. *The following norms* $\|\cdot\|, \|\cdot\|_D$ *and* $\|\cdot\|_\alpha$ *are equivalent on* $W_0^{1,2}(a,b)$.

PROOF. It is easy to see that $|\varphi(x)| \le (b-a)^{\frac{1}{2}}\|\varphi\|_D$ for $x \in [a,b]$ and $\varphi \in W_0^{1,2}(a,b)$. Since

$$\|\varphi\|_{L^\infty} \le (b-a)^{\frac{1}{2}}\|\varphi\|_D \tag{4}$$

and

$$\|\varphi\|_{L^2} \le (b-a)\|\varphi\|_D \tag{5}$$

it follows that the norm $\|\cdot\|$ and $\|\cdot\|_D$ are equivalent on the space $W_0^{1,2}(a,b)$. Note that

$$\|\varphi\|_\alpha \le \|\varphi\|_D + \|\varphi\|_{L^2\mu_1} \qquad \text{for} \quad \varphi \in W_0^{1,2}(a,b).$$

Therefore we have

$$\|\varphi\|_D \le \|\varphi\|_\alpha \le \|\varphi\|_D + \|\varphi\|_{L^2\mu_1} \qquad \text{for} \quad \varphi \in W_0^{1,2}(a,b).$$

By (4) we obtain

$$\|\varphi\|^2_{L^2\mu_1} \le \|\varphi\|^2_{L^\infty}\mu_1([a,b]) \le (b-a)\|\varphi\|^2_D\ \mu_1([a,b]).$$

Finally we get

$$\|\varphi\|_D \le \|\varphi\|_\alpha \le \left[1 + ((b-a)\mu_1([a,b]))^{\frac{1}{2}}\right]\|\varphi\|_D. \tag{6}$$

Thus the proof of our theorem is finished.

Let $(\cdot,\cdot)_{L^2}$ and $(\cdot,\cdot)_{L^2\mu_1}$ denote the ordinary inner product on the space $L^2(a,b)$ and $L^2\mu_1(a,b)$. We set

$$(\varphi,\psi)_D := (D\varphi, D\psi)_{L^2}, \qquad \varphi,\psi \in W_0^{1,2}(a,b);$$
$$(\varphi,\psi) := (\varphi,\psi)_D + (\varphi,\psi)_{L^2}, \qquad \varphi,\psi \in W_0^{1,2}(a,b),$$

and thus we have

$$\alpha(\varphi,\psi) = (\varphi,\psi)_D + (\varphi,\psi)_{L^2_{\mu_1}}$$

for $\varphi,\psi \in W_0^{1,2}(a,b)$.

We know that $(W_0^{1,2}(a,b),(\cdot,\cdot))$ is a Hibbert space.

COROLLARY 1. The spaces $(W_0^{1,2}(a,b),(\cdot,\cdot)_D)$ and $(W_0^{1,2}(a,b),\alpha(\cdot,\cdot))$ are Hilbert spaces, too.

Now, we are in a position to prove the main

THEOREM 2. *Problem (1) has exactly one solution in* $W_0^{1,2}(a,b)$.

PROOF. By the definition of $W_0^{1,2}(a,b)$, the set $D(a,b)$ is dense in $W_0^{1,2}(a,b)$ so there exists at most one solution of Problem (1) in $W_0^{1,2}(a,b)$. Since the space $(W_0^{1,2}(a,b),(\cdot,\cdot))$ is a Hilbert space and β is a continuous linear form on $W_0^{1,2}(a,b)$ there exists a function u in $W_0^{1,2}(a,b)$ such that (3) holds. This finishes the proof.

In general there exist no more regular solutions of Problem (1), apart from those belonging to $W_0^{1,2}(a,b)$.

EXAMPLE 1. Let us consider the defferential equation

$$-x'' + \delta_{\frac{1}{2}} x = f$$

with the boundary condition

$$x(0) = x(1) = 0,$$

where $\delta_{\frac{1}{2}}$ is the Dirac measure concentred at the point $t = \frac{1}{2}$ and $f \in L^1(0,1)$.

It is easy to see that this problem has no classical solutions (belonging to $W_0^{2,2}(0,1)$).

2. In this section we use the Ritz–Galerkin method to determine approximate solutions of Problem (1). We begin with a formulation of the Ritz theorem.

Let E be a real vector space and $\alpha : E \times E \to R$ be a bilinear symmetric positive definite form. Moreover, let $\beta : E \to R$ be a linear form. Let us consider the quadratic form

$$F(x) := \frac{1}{2}\alpha(x,x) - \beta(x).$$

THEOREM 3. (Ritz) ([2], p. 21). *The following conditions are equivalent:*

$$\text{(I)} \qquad \alpha(x,y) = \beta(y) \qquad \text{for} \quad y \in E$$
$$\text{(II)} \qquad F(x) = \inf_{y\in E} F(y).$$

We introduce the norm $\|x\|_\alpha = [\alpha(x,x)]^{\frac{1}{2}}$ in the space E. If we assume that $(E, \|\cdot\|_\alpha)$ is complete, then $(E, \alpha(\cdot,\cdot))$ is a Hilbert space. Let $\{x_n : n = 1,2,\dots\}$ be a sequence of elements x_n belonging to E such that

$$\text{(7)} \qquad \operatorname{cl}(\operatorname{lin}\{x_n : n = 1,2,\dots\}) = E,$$

where $\operatorname{lin}\{x_n : n = 1,2,\dots\}$ denotes the vector space spanned by the elements x_n. Let E_n be the space spanned by elements $x_1 \dots x_n$. Let x_n^* be an element in E_n such that $F(x_n^*) = \inf_{y\in E_n} F(y)$. It is known that $\|x_n^* - x\|_\alpha$ tends to zero, when $F(x) = \inf_{y\in E} F(y)$.

The above information will be used to determining approximate solutions of Problem (1) in the space $W_0^{1,2}(0,1)$.

Now, we construct a sequence $\{f_n\}$ in the space $W_0^{1,2}(0,1)$ which has property (7). For $m = 2$ we put $f_2(t) = t$ for $0 \le t \le \frac{1}{2}$, $f_2(t) = 1 - t$ for $\frac{1}{2} < t \le 1$. In the general case we take

$$f_m(t) = \begin{cases} 2^{\frac{n}{2}} t - \frac{2k-2}{2^{\frac{n}{2}+1}} & \text{for } \frac{2k-2}{2^{n+1}} \le t < \frac{2k-1}{2^{n+1}} \\ -2^{\frac{n}{2}} t + \frac{2k}{2^{\frac{n}{2}+1}} & \text{for } \frac{2k-1}{2^{n+1}} \le t \le \frac{2k}{2^{n+1}} \\ 0 & \text{for other } \; t \text{ in } [0,1], \end{cases}$$

where $m = 2^n + k$, $1 \le k \le 2^n$, $n = 1,2,\dots$

THEOREM 4. *The functions f_n, $n = 2,3,\dots$ constitute a complete orthonormal system in $(W_0^{1,2}(0,1), (\cdot,\cdot)_D)$.*

PROOF. Note that the Haar function χ_n is the distributional derivative of f_n. For $f \in W_0^{1,2}(0,1)$ we have

$$f(x) = \int_0^x g(t)dt$$

for some $g \in L^2(0,1)$, $x \in [0,1]$. The function g has the Fourier representation

$$\text{(8)} \qquad g = \int_0^1 g(x)dx + \sum_{n=2}^{\infty} c_n \chi_n$$

with respect to the Haar functions. It is clear that $\int_0^1 g(x)dx = 0$. Since the space $(W_0^{1,2}(0,1), (\cdot,\cdot)_D)$ is complete therefore there exists a function $\tilde{f}$ in $W_0^{1,2}(0,1)$ such that

$$\sum_{n=2}^{\infty} c_n f_n = \tilde{f}. \tag{9}$$

Series (9) converges in the distributional sense. Hence we have

$$\sum_{n=2}^{\infty} c_n \chi_n = D\tilde{f}. \tag{10}$$

It is known that series (10) converges to $D\tilde{f}$ in the space $L^2(0,1)$ (also a.e.). This implies that $D\tilde{f} = g$ a.e. on $[0,1]$. From this we obtain that $f(x) = \tilde{f}(x)$ for each $x \in [0,1]$. Finally we have

$$f(x) = \sum_{n=2}^{\infty} (f, f_n)_D f_n.$$

This completes the proof of the theorem.

COROLLARY 2. cl(lin $\{f_n : n = 2,3,\dots\}) = W_0^{1,2}(0,1)$ with respect to the norm $\|\cdot\|_\alpha$.

Let E_n be the vector space spanned by the functions $f_2, \dots, f_n$. The quadratic form F takes the following form

$$\begin{aligned} F(y) &= G(\lambda_2, \dots, \lambda_n) \\ &= \frac{1}{2}\left(\sum_{i=2}^{n}\sum_{j=2}^{n} \lambda_i\lambda_j \int_0^1 Df_i Df_j dx + \sum_{i=2}^{n}\sum_{j=2}^{n} \lambda_i\lambda_j \int_0^1 f_i f_j d_{\mu_1}(x)\right) \\ &\quad - \sum_{i=2}^{n} \int_0^1 f_i d_{\mu_2}(x), \end{aligned} \tag{11}$$

where $y = \lambda_2 f_2 + \dots + \lambda_n f_n$.
Formula (11) may be rewritten in matrix form

$$G(\Lambda) = \frac{1}{2}\Lambda^T(\Gamma + \Delta)\Lambda - \Lambda^T P,$$

where

$$\Lambda = \begin{vmatrix} \lambda_2 \\ \vdots \\ \lambda_n \end{vmatrix}, \qquad \Gamma = \begin{vmatrix} (f_2, f_2)_D & \cdots & (f_2, f_n)_D \\ \vdots & & \vdots \\ (f_n, f_2)_D & \cdots & (f_n, f_n)_D \end{vmatrix},$$

$$\Delta = \begin{vmatrix} \int_0^1 f_2 f_2 d_{\mu_1}(x) & \cdots & \int_0^1 f_2 f_n d_{\mu_1}(x) \\ \vdots & & \vdots \\ \int_0^1 f_n f_2 d_{\mu_1}(x) & \cdots & \int_0^1 f_n f_n d_{\mu_1}(x) \end{vmatrix}, \qquad P = \begin{vmatrix} \int_0^1 f_2 d_{\mu_2}(x) \\ \vdots \\ \int_0^1 f_n d_{\mu_2}(x) \end{vmatrix}.$$

It is easy to check that

$$G(\Lambda^*) = \inf_{\Lambda \in R^{n-1}} G(\Lambda),$$

when

$$(\Gamma + \Delta)\Lambda^* = P. \tag{12}$$

Obviously Γ is a diagonal matrix. For the differential equation

$$\begin{aligned} - x'' + \delta_{\frac{1}{2}} x &= 1 \\ x(0) = x(1) &= 0 \end{aligned} \tag{13}$$

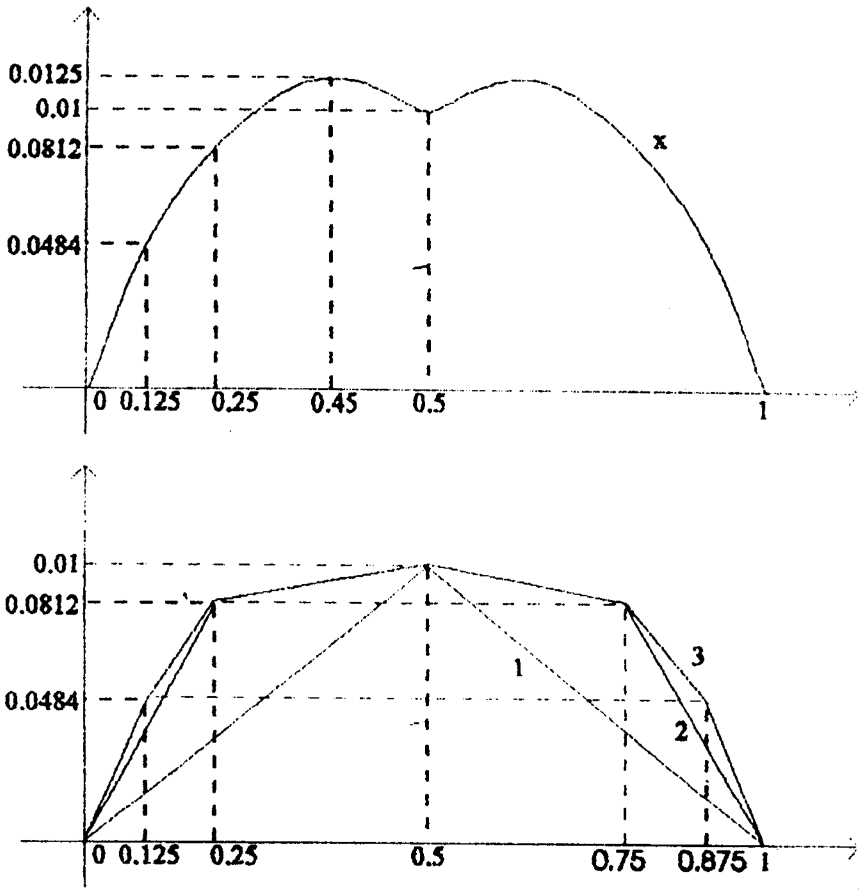
we obtain the following matrix equation

$$\begin{vmatrix} \frac{5}{4} & 0 & \cdots & \cdots & 0 \\ 0 & 1 & & & \vdots \\ \vdots & 0 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{vmatrix} * \begin{vmatrix} \lambda_2 \\ \vdots \\ \vdots \\ \vdots \\ \lambda_{2^m+1} \end{vmatrix} = \begin{vmatrix} a_2 \\ \vdots \\ \vdots \\ \vdots \\ a_{2^m+1} \end{vmatrix},$$

where $a_2 = \frac{1}{4}$, $a_{2^k+l} = 2^{-\frac{3k}{2}-2}$ for $l = 1, 2, \ldots, 2^k$, $k = 1, \ldots, m$. The exact solution of (13) is

$$x(t) = \begin{cases} \frac{t^2}{2} + \frac{9}{20}t & \text{for } 0 \le t \le \frac{1}{2} \\ \frac{t^2}{2} + \frac{11}{20}t - \frac{1}{20} & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$

The following graphs compare x and x_n^* for $n = 2, 4, 8$.



$1 - x_2^*$

$2 - x_4^*$

$3 - x_8^*$

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Uniwersytet Śląski
Instytut Matematyki
ul. Bankowa 14
40 007 Katowice