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TOPOLOGICAL DEGREE METHODS IN BVPS WITH NONLINEAR CONDITIONS

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Abstract. We consider the second order differential equation

$$x'' = f(t, x, x'),$$

where f is a Carathéodory function. We prove the existence of at least one solution of the equation satisfying the nonlinear boundary conditions

$$g_1(x(a), x'(a)) = 0, \ g_2(x(b), x'(b)) = 0.$$

Our methods of proofs are based on the topological degree arguments for auxiliary operator equation.

1. Introduction

We study the nonlinear BVP

$$x'' = f(t, x, x'), \tag{1.1}$$

$$g_1(x(a), x'(a)) = 0, \qquad g_2(x(b), x'(b)) = 0, \tag{1.2}$$

where $\mathbf{J} = [a, b] \subset \mathbb{R}$, $f \in \mathbf{Car}\ (\mathbf{J} \times \mathbb{R}^2)$, $g_1, g_2 \in \mathbf{C}(\mathbb{R}^2)$.

The existence principles for problem (1.1), (1.2) or for similar nonlinear problems were studied earlier in [1], [2], [3] or [6]. In the first three papers

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the authors require monotonicity of g_1, g_2 or growth conditions for f or an appropriate linear part in (1.2). In [6] we need only sign conditions. Let us show the typical result of [6].

THEOREM [6, Theorem 3.1]. *Let $r, R \in (0,\infty)$ be such that for a.e. $t \in J$ and each $x \in [-r,r]$ the conditions*

$$g_1(-r,0)\cdot g_1(r,0) < 0, \quad g_2(-r,0)\cdot g_2(r,0) < 0,$$

$$g_2(x,R)\cdot g_2(x,-R) < 0,$$

$$f(t,-r,0) < 0, \quad f(t,r,0) > 0,$$

$$f(t,x,R) > 0, \quad f(t,x,-R) < 0,$$

are fulfilled.

Then problem (1.1), (1.2) has at least one solution u satisfying

$$-r \le u(t) \le r, \quad -R \le u'(t) \le R \qquad \text{for each} \quad t \in \mathbf{J}.$$

In this paper, our approach has been motivated by [1] and is close to [6]. We introduce auxiliary operators L and N and study the operator equation $Lx = \lambda Nx$ with a real parametr λ. It is important to find a proper form of L and N in this approach. Here, we define L and N by a different way than in [6] and we get dim ker $L = 1$ in contrast to [6], where it was 2. Therefore the application of the Continuation Theorem (see below) is easier and we get results that can be used for differential equations which cannot be solved by the theorems of the above papers.

CONTINUATION THEOREM [1, p.40]. *Let $\mathbf{X}$, $\mathbf{Y}$ be Banach spaces, L : dom $L \subset \mathbf{X} \to \mathbf{Y}$ a Fredholm map of index 0 and $\Omega \subset \mathbf{X}$ an open bounded set. Let $N : \mathbf{X} \to \mathbf{Y}$ be L–compact on $\overline{\Omega}$, $Q : Y \to Y$ a continuous projector with* Ker Q =Im L *and $\mathcal{J}$:* Im $Q \to$ Ker L *an isomorphism. Suppose*

a) *for each $\lambda \in (0,1)$ every solution x of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega$;*
b) *$QNx \neq 0$ for each $x \in$*Ker$L \cap \partial\Omega$ *and*
c) *the Brouwer degree $d[N_0, \Omega \cap$ Ker$L$, $0] \neq 0$, where $N_0 = \mathcal{J}QN$:* Ker$L \to$ Ker$L$.

Then the equation $Lx = Nx$ has at least one solution in dom$L \cap \overline{\Omega}$.

2. Bounded nonlinearity

First we suppose that f is bounded by an integrable function φ and prove the existence of at least one solution.

THEOREM 2.1. *Let $r_1, r_2 \in \mathbb{R}$, $r_1 \le r_2$, $-R_1, R_2 \in [0,\infty)$, $\mu, \nu \in \{-1,1\}$ and $\varphi \in L(J)$ be such that for a.e. $t \in \mathbf{J}$ and each $x, y \in \mathbb{R}$*

$$\mu g_1(r_1,0) \le 0, \quad \mu g_1(r_2,0) \ge 0, \tag{2.1}$$

$$\nu g_2(x,R_1) \le 0, \quad \nu g_2(x,R_2) \ge 0 \tag{2.2}$$

and

$$|f(t,x,y)| \le \varphi(t), \tag{2.3}$$

are satisfied.

Then problem (1.1), (1.2) has a solution u with

$$r_1 \le u(a) \le r_2, \quad R_1 \le u'(b) \le R_2. \tag{2.4}$$

PROOF.

1. *Auxiliary problems.*

Let us set for $n \in \mathbb{N}$

$$\tilde g_{1n}(x,y) = \begin{cases} g_1(r_2,0) + \mu(x - r_2 - 1/n) & \text{for} \quad x \ge r_2 + 1/n \\ g_1(r_2,y) & \\ +[g_1(r_2,0) - g_1(r_2,y)]n(x - r_2) & \text{for} \quad r_2 < x < r_2 + 1/n \\ g_1(x,y) & \text{for} \quad r_1 \le x \le r_2 \\ g_1(r_1,y) & \\ -[g_1(r_1,0) - g_1(r_1,y)]n(x - r_1) & \text{for} \quad r_1 - 1/n < x < r_1 \\ g_1(r_1,0) - \mu(r_1 - 1/n - x) & \text{for} \quad x \le r_1 - 1/n \end{cases} \tag{2.5}$$

$$\tilde g_{2n}(x,y) = \begin{cases} g_2(x,R_2) & \text{for} \quad y > R_2 \\ g_2(x,y) & \text{for} \quad R_1 \le y \le R_2 \\ g_2(x,R_1) & \text{for} \quad y < R_1, \end{cases} \tag{2.6}$$

and, for fixed $n \in \mathbb{N}$, $n > 1$, study auxiliary problems

$$x'' = \lambda f(t,x,x'), \quad \lambda \in [0,1], \tag{2.7}$$

$$\lambda \tilde g_{1n}(x(a),x'(a)) = 0, \quad \lambda \tilde g_{2n}(x(b),x'(b)) = -\nu x'(b)/n. \tag*{$(2.8_n)_\lambda$}$$

If we set $\mathbf{X} = \mathbf{C}^1(\mathbf{J})$, $\mathbf{Y} = \mathbf{L}(\mathbf{J}) \times \mathbb{R}^2$, dom $L = \mathbf{A}\ \mathbf{C}^1(\mathbf{J})$,

$$L : \mathrm{dom} L \longrightarrow \mathbf{Y}, \quad x \longmapsto (x'', 0, -\nu x'(b)/n), \quad N : \mathbf{X} \longrightarrow \mathbf{Y},$$

$$x \longmapsto (f(\cdot, x(\cdot), x'(\cdot)), \quad \tilde{g}_{1n}(x(a), x'(a)), \quad \tilde{g}_{2n}(x(b), x'(b)))$$

we can write (2.7), $(2.8_n)_\lambda$ in the form

$$(2.9)_\lambda \qquad Lx = \lambda N x.$$

We can see that L is a Fredholm map of index zero, because Ker $L = \{x \in X :\ x(t) \equiv c,\ c \in \mathbb{R}\}$, Im $L = \mathbf{L}(\mathbf{J}) \times \{0\} \times \mathbb{R}$ is closed in $\mathbf{Y}$ and dim Ker $L =$ codim Im $L = 1$.

Further, the maps

$$P : \mathbf{X} \longrightarrow \mathbf{X}, \quad x \longrightarrow x(a), \quad Q : \mathbf{Y} \longrightarrow \mathbf{Y}, \quad (y, \alpha, \beta) \longmapsto (0, \alpha, 0)$$

are continuous projectors and we can find the generalized inverse (to L) operator $K_P : \mathrm{Im}\, L \longrightarrow \mathrm{Ker}\, P \cap \mathrm{dom}\, L$
in the form

$$K_P : (y, 0, \beta) \longmapsto -n\beta(t-a)/\nu - \int_a^t \int_\tau^b y(s)\, ds\, d\tau.$$

Then $QN :\ X \longrightarrow \mathbf{X},\ x \longrightarrow (0, \tilde{g}_{1n}(x(a), x'(a)), 0)$,

$$K_P(I-Q)N :\ X \to X,$$

$$x \longmapsto -n\tilde{g}_{2n}(x(b), x'(b))(t-a)/\nu - \int_a^t \int_\tau^b f(s, x(s), x'(s))\, ds\, d\tau.$$

Using the Arzelà–Ascoli Theorem we can show that for any open bounded set $\Omega \subset \mathbf{X}$, the sets $QN(\overline{\Omega}) \subset \mathbf{Y}$ and $K_P(I-Q)N(\overline{\Omega}) \subset \mathbf{X}$ are relatively compact. This means that $N : \overline{\Omega} \longrightarrow \mathbf{Y}$ is L–compact. Therefore we can use the Continuation Theorem for problems $(2.9)_\lambda$.

2. *A priori estimates.*

Before using the Continuation Theorem we need to find a priori estimates for solutions of $(2.9)_\lambda$. Thus, suppose that problem (1.7), $(1.8_n)_\lambda$ has a solution u for some $\lambda \in (0,1]$ and some $n \in \mathbb{N}$. Let $u'(b) > R_2$. Then $\nu\lambda\tilde{g}_{2n}(u(b), u'(b)) = -u'(b)/n < 0$. But, by (2.2), $\nu\lambda\tilde{g}_{2n}(u(b), u'(b)) = \nu\lambda g_2(u(b), R_2) \geq 0$, a contradiction. Similarly, for $u'(b) < R_1$, we get

$\nu\lambda\tilde{g}_{2n}(u(b), u'(b)) = -u'(b)/n > 0$, and according to (2.2), $\nu\lambda\tilde{g}_{2n}(u(b), u'(b)) = \nu\lambda g_2(u(b), R_1) \leq 0$, a contradiction. Thus

$$R_1 \leq u'(b) \leq R_2. \tag{2.10}$$

Further, let $u(a) > r_2 + 1/n$. Then, by (2.2), (2.5), $\mu\lambda\tilde{g}_{1n}(u(a), u'(a)) = \lambda\mu(g_1(r_2, 0) + \mu(u(a) - r_2 - 1/n)) > 0$, which contradicts $(2.8_n)_\lambda$. Similarly, if $u(a) < r_1 - 1/n$, we get $\mu\lambda\tilde{g}_{1n}(u(a), u'(a)) = \lambda\mu(g_1(r_1, 0) - \mu(r_1 - 1/n - u(a)) < 0$, which also contradicts $(2.8_n)_\lambda$. Thus

$$r_1 - 1/n \leq u(a) \leq r_2 + 1/n. \tag{2.11}$$

Now, integrating (2.3) from b to $t \in \mathbf{J}$, we get

$$\varrho_1 \leq u'(t) \leq \varrho_2 \qquad \text{for each} \quad t \in \mathbf{J}, \tag{2.12}$$

where $\varrho_1 = R_1 - \int_a^b \varphi(t)dt$, $\varrho_2 = R_2 + \int_a^b \varphi(t)dt$. Finally, integrating (2.12) from a to $t \in \mathbf{J}$, we have

$$c_1 - 1/n \leq u(t) \leq c_2 + 1/n, \tag{2.13}$$

where $c_i = r_i + \varrho_i(b-a)$, $i = 1, 2$.

3. *Application of the Continuation Theorem to problem $(2.9)_\lambda$.*

Let us put $\Omega = \{x \in X : \ c_1 - 1 < x(t) < c_2 + 1, \ \varrho_1 - 1 < x'(t) < \varrho_2 + 1$ for each $t \in \mathbf{J}\}$. Then $\operatorname{Ker} L \cap \delta\Omega = \{x \in \mathbf{X}\colon \ x(t) \equiv c, \ c = c_1 - 1$ or $c = c_2 + 1\}$. Estimates (2.12) and (2.13) ensure that condition (a) of the Continuation Theorem is valid.

Let us check condition (b). The equation $QNx = 0$ for $x \in \partial\Omega \cap \operatorname{Ker} L$ has the form $\tilde{g}_{1n}(c_1 - 1, 0) = 0$ or $\tilde{g}_{1n}(c_2 + 1, 0) = 0$. But $\mu\tilde{g}_{1n}(c_1 - 1, 0) = \mu g_1(r_1, 0) - (r_1 - 1/n - c_1 + 1) < 0$ and $\mu\tilde{g}_{1n}(c_2 + 1, 0) = \mu g_1(r_2, 0) + (c_2 + 1 - r_2 - 1/n) > 0$. Now put $\mathcal{J} : \operatorname{Im} Q \longrightarrow \operatorname{Ker} L$, $(0, \alpha, 0) \longmapsto \alpha$. Then $N_0 = \mathcal{J}QN :$ $\operatorname{Ker} L \longrightarrow \operatorname{Ker} L$ has the form $N_0(c) = \tilde{g}_{1n}(c, 0)$, $c \in (c_1 - 1, c_2 + 1)$. Since $\operatorname{sgn} \tilde{g}_{1n}(c_1 - 1, 0) \neq \operatorname{sgn} \tilde{g}_{1n}(c_2 + 1, 0)$, we have $\mathrm{d}[N_0, (c_1 - 1, c_2 + 1), 0] \neq 0$ and (c) is valid. Therefore problem $(2.9)_1$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$.

4. *Limitting process*

In order to complete the proof observe that for any $n \in \mathbb{N}$ $n > 1$, problem (2.7), $(2.8_n)_1$ has a solution u_n satisfying (2.10), (2.11), (2.12), (2.13). By the Arzelà–Ascoli Theorem and the integrated form of the equation one gets the existence of a uniformly converging subsequence of $(u_n)_1^\infty$ whose limit is a solution of (1.1), (1.2) satisfying (2.4). The proof is complete. □

Conditions (2.1) and (2.2) can be modified in the following way:

THEOREM 2.2. *Let $r_1, r_2 \in \mathbb{R}$, $r_1 \le r_2$, $-R_1, R_2 \in [0,\infty)$, $\mu, \nu \in \{-1,1\}$ and $\varphi \in \mathbf{L}(\mathbf{J})$ be such that for a.e. $t \in \mathbf{J}$ and each $x, y \in \mathbb{R}$ the conditions (2.3),*

$$\mu g_2(r_1, 0) \le 0, \quad \mu g_2(r_2, 0) \ge 0, \tag{2.14}$$

$$\nu g_1(x, R_1) \le 0, \quad \nu g_1(x, R_2) \ge 0 \tag{2.15}$$

are satisfied.

The problem (1.1), (1.2) has a solution u with

$$r_1 \le u(b) \le r_2, \quad R_1 \le u'(a) \le R_2. \tag{2.16}$$

3. Unbounded nonlinearity

Here, we replace the boundedness (1.5) in Theorems 2.1 and 2.2 by appropriate sign conditions.

THEOREM 3.1. *Let there exist $r_1, r_2 \in \mathbb{R}$, $r_1 \le r_2$, $-R_1, R_2 \in [0,\infty)$, $\mu, \nu \in \{-1,1\}$ such that for a.e. $t \in \mathbf{J}$ and each $x \in [d_1, d_2]$ with $d_i = r_i + R_i(b-a)$, $i = 1,2$, the conditions (2.1), (2.2) and*

$$f(t, x, R_2) \ge 0, \quad f(t, x, R_1) \le 0 \tag{3.1}$$

are fulfilled.

Then problem (1.1), (1.2) has a solution u satisfying

$$r_1 \le u(a) \le r_2, \quad R_1 \le u'(t) \le R_2 \quad \text{for each} \quad t \in \mathbf{J}. \tag{3.2}$$

PROOF. Let us set

$$\sigma(x) = \begin{cases} d_2 & \text{for} \quad x > d_2 \\ x & \text{for} \quad d_1 \le x \le d_2 \,, \\ d_1 & \text{for} \quad x < d_1 \end{cases}$$

$$\varrho(y) = \begin{cases} R_2 & \text{for} \quad y > R_2 \\ y & \text{for} \quad R_1 \le y \le R_2 \,, \\ R_1 & \text{for} \quad y < R_1 \end{cases}$$

$$\tilde{f}(t,x,y) = f(t,\sigma(x),\varrho(y)) + \frac{y-\varrho(y)}{|y-\varrho(y)|+1},$$

$$\tilde{g}_1(x,y) = g_1(\sigma(x),y), \quad i = 1,2,$$

and consider the problem

$$x'' = \tilde{f}(t,x,x') \tag{3.3}$$

$$\tilde{g}_1(x(a),x'(a)) = 0, \quad \tilde{g}_2(x(b),x'(b)) = 0. \tag{3.4}$$

The function $\tilde{f}$ fulfils (2.3) of Theorem 2.1 with $\varphi(t) = \sup\{|f(t,x,y)| : x \in [d_1,d_2],\ y \in [R_1,R_2]\} + 1$ and $\tilde{g}_1, \tilde{g}_2$ fulfil (2.1), (2.2) respectively for each $x \in \mathbb{R}$. So, problem (3.3), (3.4) has a solution u satisfying (2.4). Suppose $\max\{u'(t) : t \in \mathbf{J}\} = u'(t_0) > R_2$. Then $t_0 \in [a,b)$ and we can find $\delta > 0$ such that $R_2 < u'(t) \leq u'(t_0)$ for each $t \in (t_0, t_0+\delta]$. On the other hand, by (3.1),

$$\int_{t_0}^{t_0+\delta} u''(\tau)d\tau = \int_{t_0}^{t_0+\delta} \left[f(\tau,\sigma(u(\tau)),R_2) + \frac{u'(\tau)-R_2}{u'(\tau)-R_2+1}\right] d\tau > 0,$$

a contradiction. The inequality $R_1 \leq u'(t)$ for each $t \in \mathbf{J}$ can be proved by similar arguments. Thus $R_1 \leq u'(t)' \leq R_2$ for each $t \in \mathbf{J}$. Integrating the latter from a to $t \in \mathbf{J}$, we get $d_1 \leq u(t) \leq d_2$ and we can see that u is a solution of (1.1), (1.2) as well. □

THEOREM 3.2. *Let there exist $r_1, r_2 \in \mathbb{R}$, $r_1 \leq r_2$, $-R_1, R_2 \in [0,\infty)$, $\mu, \nu \in \{-1,1\}$ such that for a.e. $t \in \mathbf{J}$ and each $x \in [d_1,d_2]$ with $d_i = r_i + R_i(b-a)$, $i = 1,2$, the conditions (2.14), (2.15) and*

$$f(t,x,R_2) \leq 0, \qquad f(t,x,R_1) \geq 0$$

are fulfilled.

Then problem (1.1), (1.2) has a solution u satisfying

$$r_1 \leq u(b) \leq r_2, \quad R_1 \leq u'(t) \leq R_2 \quad \text{for each} \quad t \in \mathbf{J}.$$

PROOF. Using Theorem 2.2 instead of Theorem 2.1 and setting $\tilde{f}(t,x,y) = f(t,\sigma(x),\varrho(y)) + (\varrho(y)-y)/(|y-\varrho(y)|+1)$, we can prove Theorem 3.2 similarly as Theorem 3.1. □

EXAMPLE. Let us show f, g_1, g_2 satisfying the conditions of Theorem 3.1. Suppose $\varphi_1, \varphi_2 \in L(\mathbf{J})$, $\varphi_3 \in L^\infty(\mathbf{J})$, $\varphi_1 \geq 0$, $n \in \mathbb{N}$. We can choose both quickly growing functions, e.g.

$$f(t,x,y) = \varphi_1(t) \cdot e^x \cdot y^{2n-1} + \varphi_3(t),$$
$$g_1(x,y) = x^2 \cdot e^y + e^x \cdot y, \quad g_2(x,y) = y^{2n} \cdot e^x - 1,$$

or oscillating functions, e.g.

$$f(t,x,y) = \sin^2(x + \varphi_2(t)) \cdot \cos y,$$
$$g_1(x,y) = \sin(x+y) + y + 1, \quad i = 1,2.$$

Let us note that the theorems of [6] cannot be used in this case.

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