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CATEGORIES OF UNIVERSAL ALGEBRAS IN WHICH DIRECT PRODUCTS ARE TENSOR PRODUCTS

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Abstract. In categories of commutative universal algebras of given types we discover full subcategories in which direct products coincide with tensor products.

A concrete category K of structured sets and structure-compatible maps, i.e. a category K with a faithful (=forgetful) functor $|\ |: K \to \mathrm{Set}$, will be called a *construct*. Given two objects A and B of a construct K (we write $A, B \in K$), by $\mathrm{Hom}(A,B)$ we denote the set of all morphisms from A into B in K. The usual symbols $\times$ and $\otimes$ will be used for denotation of the cartesian product and the tensor product, respectively, in a construct. We shall need the following known result (see e.g. [1]): Let K be a semifinally complete construct with a unit object. If for arbitrary objects $A, B \in K$ there exists a subobject $[A, B]$ of the cartesian product $B^{|A|}$ such that $|[A,B]| = \mathrm{Hom}(A,B)$, then for any object $A \in K$ the functor $A \otimes - : K \to K$ is a left adjoint to the functor $[A, -]$ (and vice versa).

By a *type* we mean a family $\tau = (n_\lambda;\ \lambda \in \Omega)$ where Ω is a set and n_λ is a cardinal for each $\lambda \in \Omega$. A *universal algebra* (briefly an *algebra*) of type $\tau = (n_\lambda;\ \lambda \in \Omega)$ is a pair $A = \langle X, (p_\lambda;\ \lambda \in \Omega)\rangle$ where X is a set – the so called *underlying set* of A – and p_λ is an n_λ-ary operation on X, i.e. $p_\lambda : X^{n_\lambda} \to X$, for each $\lambda \in \Omega$. An algebra $\langle X, (p_\lambda;\ \lambda \in \Omega)\rangle$ of type $\tau = (n_\lambda;\ \lambda \in \Omega)$ is called

idempotent if for any $\lambda \in \Omega$ and any $x \in X$ from $x_i = x$ for each $i \in n_\lambda$ it follows that $p_\lambda(x_i;\ i \in n_\lambda) = x$,

diagonal (c.f. [9]) if there holds

$$p_\lambda(p_\lambda(x_{ij};\ j \in n_\lambda);\ i \in n_\lambda) = p_\lambda(x_{ii};\ i \in n_\lambda)$$

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whenever $\lambda \in \Omega$ and $x_{ij} \in X$ for all $i, j \in n_\lambda$,

commutative (see [7]) if the identity

$$p_\lambda(p_\mu(x_{ij};\ j \in n_\mu);\ i \in n_\lambda) = p_\mu(p_\lambda(x_{ij};\ i \in n_\lambda);\ j \in n_\mu)$$

holds for all $\lambda, \mu \in \Omega$ and all $x_{ij} \in X$, $i \in n_\lambda$, $j \in n_\mu$.

Given a type τ, by Alg_τ we denote the category of all idempotent, diagonal and commutative algebras of type τ with homomorphisms as morphisms. Clearly, for arbitrary type τ, Alg_τ is a semifinally complete construct where $|A|$ is the underlying set of A for any algebra $A \in \mathrm{Alg}_\tau$. It is also clear that Alg_τ is closed with respect to cartesian (i.e. direct) products of objects. The idempotency implies that every algebra $A \in \mathrm{Alg}_\tau$ with $\mathrm{card}|A| = 1$ is a unit object of Alg_τ. By [7], from the commutativity it follows that for any two algebras $A, B \in \mathrm{Alg}_\tau$ there exists a subalgebra $[A, B]$ of the direct product $B^{|A|}$ such that $|[A, B]| = \mathrm{Hom}\ (A, B)$. Thus, $A\otimes -$ is a left adjoint to the functor $[A, -] : \mathrm{Alg}_\tau \to \mathrm{Alg}_\tau$ for any type τ and any algebra $A \in \mathrm{Alg}_\tau$ (see also [2], [8]). Using this fact we prove

THEOREM. *For any type τ and any algebras $A, B \in Alg_\tau$ there holds $A \times B = A \otimes B$.*

PROOF. Let $\tau = (n_\lambda;\ \lambda \in \Omega)$ be an arbitrary type. According to the previous considerations it is sufficient to show that $A\times -$ is a left adjoint to $[A, -]$ for each $A \in \mathrm{Alg}_\tau$. In other words, we are to prove that for any two algebras $A, B \in \mathrm{Alg}_\tau$ there is a homomorphism $f \in \mathrm{Hom}(B, [A, A \times B])$ such that for any algebra $C \in \mathrm{Alg}_\tau$ and any homomorphism $g \in \mathrm{Hom}(B, [A, C])$ there exists a unique homomorphism $g^* \in \mathrm{Hom}(A \times B, C)$ with the property that $g(y) = g^* \circ f(y)$ for each $y \in |B|$.

On that account, let $A = \langle X, (p_\lambda;\ \lambda \in \Omega)\rangle$, $B = \langle Y, (q_\lambda;\ \lambda \in \Omega)\rangle$ and let $f : Y \to (X \times Y)^X$ be the map given by $f(y)(x) = (x, y)$. Denote $A\times B = \langle X\times Y,\ (r_\lambda;\ \lambda \in \Omega)\rangle$, $[A, A\times B] = \langle \mathrm{Hom}(A, A\times B),\ (s_\lambda;\ \lambda \in \Omega)\rangle$, and let $\lambda \in \Omega$ and $y_i \in Y$ for each $i \in n_\lambda$. Then we have

$$\begin{aligned} f(q_\lambda(y_i;\ i \in n_\lambda))(x) &= (x, q_\lambda(y_i;\ i \in n_\lambda)) = (p_\lambda(x;\ i \in n_\lambda),\ q_\lambda(y_i;\ i \in n_\lambda)) \\ &= r_\lambda((x, y_i);\ i \in n_\lambda) = r_\lambda(f(y_i)(x);\ i \in n_\lambda) \\ &= s_\lambda(f(y_i);\ i \in n_\lambda)(x) \end{aligned}$$

for any $x \in X$. Hence $f \in \mathrm{Hom}(B, [A, A \times B]$.

Next, let $C = \langle Z, (t_\lambda;\ \lambda \in \Omega)\rangle \in \mathrm{Alg}_\tau$ be an algebra and let $g \in \mathrm{Hom}(B, [A, C])$ be a homomorphism. For any $(x, y) \in X \times Y$ put $g^*(x, y) = g(y)(x)$. Then we have defined a map $g^* : X \times Y \to Z$. Let $(x_i, y_i) \in X \times Y$

for each $i \in n_\lambda$ and denote $[A, C] = \langle \operatorname{Hom}(A, C), (u_\lambda;\ \lambda \in \Omega)\rangle$. There holds

$$\begin{aligned} g^*(r_\lambda((x_i, y_i);\ i \in n_\lambda)) =& g^*(p_\lambda(x_i;\ i \in n_\lambda),\ q_\lambda(y_i;\ i \in n_\lambda)) \\ =& g(q_\lambda(y_i;\ i \in n_\lambda))(p_\lambda(x_i;\ i \in n_\lambda)) \\ =& u_\lambda(g(y_i);\ i \in n_\lambda)(p_\lambda(x_i;\ i \in n_\lambda)) \\ =& t_\lambda(g(y_i)(p_\lambda(x_i;\ i \in n_\lambda));\ j \in n_\lambda) \\ =& t_\lambda(t_\lambda(g(y_j)(x_i);\ i \in n_\lambda); j \in n_\lambda) \\ =& t_\lambda(g(y_i)(x_i);\ i \in n_\lambda) \\ =& t_\lambda(g^*(x_i, y_i);\ i \in n_\lambda). \end{aligned}$$

We have shown that $g^* \in \operatorname{Hom}(A \times B, C)$.

As the equality $g(y) = g^* \circ f(y)$ for any $y \in Y$ and the uniqueness of g^* are obvious, the proof is complete. □

REMARK. (a) A finitely productive construct K having the property that for any its object A the functor $A \times - :\ K \to K$ has a right adjoint is called cartesian closed – see [4]. Thus, we have proved that Alg_τ is cartesian closed for any type τ.

(b) Given a natural number n, by an n–ary algebra we understand a set with one n–ary operation on it. For n–ary algebras the commutativity (more often called the mediality) results from the diagonality. The diagonal and idempotent n–ary algebras are studied in [9]. The medial groupoids are fully dealt with in [6]. One can easily prove the following criterion for the diagonality of n–ary algebras: An n–ary algebra $\langle X, p\rangle$ is diagonal iff

$$\begin{aligned} p(x_{11}, x_{22}, \dots, x_{nn}) =& p(p(x_{11}, x_{12}, \dots, x_{1n}), x_{22}, \dots, x_{nn}) \\ =& p(x_{11}, p(x_{21}, x_{22}, \dots, x_{2n}), x_{33}, \dots, x_{nn}) = \dots \\ =& p(x_{11}, x_{22}, \dots, x_{n-1,n-1}, p(x_{n1}, x_{n2}, \dots, x_{nn})) \end{aligned}$$

holds for any $x_{ij} \in X,\ i, j = 1, 2, \dots, n$. Consequently, a groupoid $\langle X, \cdot\rangle$ is diagonal iff it is a semigroup with $xyz = xz$ for each $x, y, z \in X$. For example, any rectangular band (see [3]) is a diagonal (and idempotent) groupoid and hence an object of Alg_τ for $\tau = (2)$.

A criterion for the diagonality of algebras of arbitrary types is given in [10].

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