Annales Mathematicae Silesianae 9. Katowice 1995, 33–41

Prace Naukowe Uniwersytetu Śląskiego nr 1523

SOME REMARKS ON THE PROPERTY (N) OF LUZIN

A. B. KHARAZISHVILI

Abstract. We consider the classical property (N) of Luzin for various mappings in connection with a measure extension problem. We give some examples of Borel measurable mappings and of Lebesgue measurable mappings which transform all compact sets with measure zero into sets with measure zero but do not have the property (N) of Luzin.

Let E_1 and E_2 be two spaces equipped with σ-finite measures μ_1 and μ_2, respectively, and let f be a mapping from E_1 into E_2. According to a well known definition of Luzin (see [1]) this mapping has the property (N) if for every set $X \subseteq E_1$ with outer μ_1-measure zero the image $f(X)$ has outer μ_2-measure zero. Notice that Luzin himself considered this property only for continuous functions f acting from (R, λ) into (R, λ), where R is the real line and λ is the standard Lebesgue measure on R. There are some interesting and important results in the theory of real functions concerning continuous functions having the property (N). Recall, for instance, the Banach–Zarecki theorem stating that a continuous real function f defined on a segment of R is absolutely continuous if and only if it has the property (N) and is of finite variation. Luzin proved in [1] that a sufficient condition for a continuous real function f defined on a segment of R to have the property (N) is the following one: for any compact subset X of R with $\lambda(X) = 0$ the set $f(X)$ also satisfies the equality $\lambda(f(X)) = 0$. From the point of view of modern mathematical analysis and measure theory this classical result of Luzin can be obtained as an easy consequence of Choquet's theorem on capacities or of the Kuratowski–Ryll–Nardzewski theorem on measurable selectors. In the present paper we shall consider the property (N) of Luzin for those mappings f which are not necessarily continuous. In particular, we shall investigate

AMS (1991) subject classification: 28A05, 28A20.

a question connected with a possibility of a generalization of Luzin's result mentioned above for mappings f which are measurable in various senses, for instance, Borel measurable or Lebesgue measurable. We shall see that this question is closely connected with the typical problem of measure theory: the problem on the existence of an extension of a measure, given on a subalgebra of a Borel σ-algebra, to a Borel measure. This problem plays the important rôle in several questions of analysis and probability theory. In the sequel we need an auxiliary result due to M.Ershov (see [2]) concerning that problem.

LEMMA 1. *Let E be a Polish topological space, let $B(E)$ be a Borel σ-algebra of E, let S be a countably generated σ-algebra contained in $B(E)$ and let μ be a probability measure defined on S. Then there exists a Borel measure $\overline{\mu}$ on E extending the original measure μ.*

Let us make some remarks concerning the lemma just formulated. The proof of this lemma is based on the well known properties of the so called Marczewski characteristic function of a countable family of sets and on a theorem about the existence of measurable selectors. In fact, it is sufficient to apply that version of the theorem about the existence of measurable selectors which is implied by Choquet's theorem on capacities (see, for instance, [3]). Notice also that simple examples show that if the given σ-algebra S is not countably generated then the conclusion of lemma is not true in general. On the other hand, the lemma is valid not only for a Polish topological space E and for a countably generated σ-algebra $S \subseteq B(E)$ but also for every analytic space E and every countably generated σ-algebra $S \subseteq B(E)$, since it is easy to see that the property of an extendability of a probability measures from a countably generated σ-subalgebras of a Borel σ-algebra to the whole Borel σ-algebra is preserved under surjective Borel mappings of topological spaces. So, in the further consideration we shall use the above lemma for analytic spaces. Finally, let us remark that for a coanalytic space E and for a countably generated σ-algebra $S \subseteq B(E)$ the mentioned lemma fails in general. More exactly, if we assume Gödel's Constructibility Axiom then there exist a coanalytic space E, a countably generated σ-algebra $S \subseteq B(E)$ and a probability measure μ defined on S such that μ cannot be extended to a Borel measure $\overline{\mu}$ on the space E (see Example 1 below). Notice that this fact is of some independent interest because each coanalytic space E is a Radon topological space, and from the point of view of topological measure theory has rather good properties.

Now, let us formulate the first proposition concerning the property (N) of Luzin.

PROPOSITION 1. *Let E_1 be an analytic space equipped with a σ-finite Borel measure μ_1, let E_2 be a metric space equipped with a σ-finite Borel* *measure μ_2 and let f be a Borel mapping from E_1 into E_2. If for each compact set $K \subseteq E_1$ with $\mu_1(K) = 0$ the image $f(K)$ is of outer μ_2-measure zero, then the mapping f has the property (N) of Luzin.*

PROOF. It is obvious that, without loss of generality, we may assume that both given measures μ_1 and μ_2 are probability measures. We also may suppose that the space E_2 is analytic (because the image $f(E_1)$ is a separable metric space and therefore $f(E_1)$ is an analytic subset of the completion of $f(E_1)$). Now, let us suppose that there exists a set $E \subseteq E_1$ with μ_1-measure zero such that the set $f(E)$ has a nonzero outer μ_2-measure. Of course, we can assume that E is the G_δ-subset of the space E_1, and hence E is the analytic space, too. Denote by the symbol g the restriction of f to the set E, and consider the following family of sets

$$S = \{g^{-1}(Z) : \ Z \in B(g(E))\}.$$

It is evident that S is a countably generated σ-algebra of subsets of E contained in the Borel σ-algebra $B(E)$. Let us put

$$\mu(g^{-1}(Z)) = \mu_2^*(Z) \qquad (Z \in B(g(E))),$$

where μ_2^*, as usual, denotes the outer measure associated with μ_2. Hence we have a finite measure μ on the σ-algebra S. According to Lemma 1, there exists a Borel measure $\overline{\mu}$ on the space E extending μ. As

$$\overline{\mu}(E) = \mu_2^*(g(E)) > 0,$$

the measure $\overline{\mu}$ is not identically equal to zero. But for every compact set $K \subseteq E$ there exists a set $Z \in B(g(E))$ such that the inclusion $g(K) \subseteq Z$ holds and

$$\mu_2^*(Z) = \mu_2^*(g(K)) = 0.$$

Hence we have

$$\overline{\mu}(K) \leq (\overline{\mu})^*(g^{-1}(g(K))) \leq (\overline{\mu})^*(g^{-1}(Z)) = \mu(g^{-1}(Z)) = \mu_2^*(Z) = 0.$$

So, we see that the measure $\overline{\mu}$ vanishes on all compact subsets of E. But we know that every analytic space is a Radon space. This implies that our measure $\overline{\mu}$ must be identically equal to zero. The obtained contradiction completes the proof. □

Notice that another proof of Proposition 1 can be made directly using Choquet's theorem on capacities. But here we prefer the proof based on measure extension theorem. The reason of this preference will be discussed later. The following example shows that Proposition 1 is not true in general for a coanalytic space E_1 and even for a continuous mapping f from E_1 into the unit segment $[0, 1]$.

EXAMPLE 1. Assume Gödel's Constructibility Axiom. Then, as it is well known, there exist a coanalytic set $X \subseteq [0, 1]$ and a continuous mapping φ from X into $[0, 1]$ such that the image $\varphi(X)$ is a Vitali type subset of $[0, 1]$. In particular, $\varphi(X)$ is λ–nonmeasurable subset of $[0, 1]$, where λ denotes the classical Lebesgue measure on the real line R. Let us take the segment $[2, 3]$ on R and let ψ be the identity transformation of this segment. Afterwards consider the coanalytic space $E_1 = X \cup [2, 3]$ and equip this space with a diffused probability Borel measure μ_1 concentrated on the segment $[2, 3]$ and coinciding on this segment with the restriction of λ to $[2, 3]$. Let f be a common extension of the mappings φ and ψ. Of course, we may consider f as a continuous (hence, also as a Borel) mapping from the measure space (E_1, μ_1) into the measure space (R, λ). Obviously, we have

$$\mu_1(X) = 0, \qquad \lambda^*(f(X)) > 0.$$

Now, let K be any compact subset of E_1 with $\mu_1(K) = 0$. Then it is clear that $K \cap X$ is a compact subset of X and $K \cap [2, 3]$ is a compact subset of $[2, 3]$. We can write

$$\lambda(f(K)) = \lambda(f(K \cap X)) + \lambda(f(K \cap [2, 3])).$$

But it is evident that

$$\lambda(f(K \cap [2, 3])) = \lambda(K \cap [2, 3]) = \mu_1(K \cap [2, 3]) = 0.$$

On the other hand, the set $f(K \cap X)$ is a compact subset of a Vitali set $\varphi(X)$. We know that any Lebesgue measurable subset of a Vitali set has measure zero. This immediately follows from the Steinhaus property of Lebesgue measurable sets with a strictly positive measure. Hence we also have $\lambda(f(K \cap X)) = 0$ and finally $\lambda(f(K)) = 0$. So we see that a continuous mapping f transforms all compact subsets of E_1 with μ_1–measure zero into sets of λ–measure zero. But there exists a set X with μ_1–measure zero whose image is of strictly positive outer λ–measure.

In particular, from this example immediately follows that the coanalytic space E_1 considered above does not have the measure extension property, i.e. there exist a countably generated σ–algebra $S \subseteq B(E_1)$ and a probability measure μ on S such that μ can not be extended to a Borel measure on E_1.

The next example shows that Proposition 1 is not true in general even for Polish topological spaces if we take Lebesgue measurable mappings instead of Borel mappings.

EXAMPLE 2. Let E be any dense G_δ-subset of the segment $[0,1]$ with the Lebesgue measure equal to zero. Assume that the Continuum Hypothesis holds. Let $\{K_\xi : \xi < \omega_1\}$ be the family of all compact subsets of $[0,1]$ with the Lebesgue measure zero. Notice that each set K_ξ is nowhere dense in $[0,1]$, so the union of any countable family of these sets does not cover the residual set $E \subseteq [0,1]$. From this remark it follows that with the aid of transfinite recursion it can be constructed a family $\{x_\xi : \xi < \omega_1\}$ of points of E in such a way that

(1) all points x_ξ are distinct;
(2) if $\xi < \zeta < \omega_1$ then the point x_ζ does not belong to K_ξ.

As soon as the family $\{x_\xi : \xi < \omega_1\}$ of points of E has been constructed let us put

$$X = \{x_\xi : \xi < \omega_1\}.$$

By construction, the set X is uncountable and (since $X \subseteq E$) it has the Lebesgue measure zero. Now, let Z be a subset of $[0,1]$, either with a strictly positive Lebesgue measure or nonmeasurable in the Lebesgue sense, and let f be a bijective mapping from X onto Z. Let us extend the mapping f to a mapping defined on the whole segment $[0,1]$ putting

$$f([0,1] \setminus X) = \{0\}.$$

It is obvious that the extended mapping f coincides almost everywhere (in the sense of the Lebesgue measure λ) with a constant mapping. Hence f is a Lebesgue measurable mapping. It is also easy to see, taking into account the construction of X, that for any compact set $K \subseteq [0,1]$ with $\lambda(K) = 0$ the set $X \cap K$ is at most countable, so the set $f(K)$ is also at most countable and therefore has λ–measure zero. On the other hand, the image $f(X)$ of the set X of λ–measure zero is not of λ–measure zero. Moreover, $f(X)$ may be a λ–nonmeasurable subset of the segment $[0,1]$.

It is not difficult to see that a similar example can be constructed if we assume only Martin's Axiom (much weaker than the Continuum Hypothesis).

It is worth also to notice here that the property (N) of Luzin is closely connected with the class of those mappings which preserve measurable sets (i.e. the image of a measurable set is also a measurable set). From the theory of real functions it is well known that every continuous function from R into

R with the property (N) of Luzin preserves the class of Lebesgue measurable sets. This fact can be easily extended to a more general situation. Namely, we have the following result.

PROPOSITION 2. *Let E_1 be a Hausdorff topological space equipped with a σ–finite Radon measure μ_1, let E_2 be a metric space equipped with a σ–finite Borel measure μ_2 and let f be a mapping from E_1 into E_2 measurable with respect to the standard completion of the measure μ_1. Finally, let f have the property (N) of Luzin. Then for each set $X \subseteq E_1$ measurable with respect to the completion of μ_1 the set $f(X)$ is measurable with respect to the completion of μ_2.*

The proof of Proposition 2 is not difficult. Indeed, without loss of generality, we may assume that the space E_2 is separable. Moreover, we may assume that E_2 is a subspace of the unit segment $[0, 1]$. So, the given measurable mapping f has the classical property (C) of Luzin, i.e. for each Borel set $X \subseteq E_1$ and for each real number $r > 0$ there exists a compact set $K \subseteq X$ such that $\mu_1(X \setminus K) < r$ and the restriction of f to K is continuous. From these facts we easily obtain the required conclusion.

Of course, in Proposition 2 the measurability of f is essential. The following example shows that if f has the property (N) but is not measurable then pathological situations can happen.

EXAMPLE 3. Let us recall that an uncónuntable set $X \subseteq [0, 1]$ is a Sierpiński set on $[0, 1]$ if for every Lebesgue measure zero set $Y \subseteq [0, 1]$ the intersection $X \cap Y$ is at most countable. It is well known that if the Continuum Hypothesis holds then there exist Sierpiński subsets of $[0, 1]$. Moreover, using the method of tranfinite recursion it is not difficult to construct (also under the Continuum Hypothesis) a Sierpiński set $X \subseteq [0, 1]$ with the following additional property: for every Lebesgue measurable set $Y \subseteq [0, 1]$ with strictly positive λ–measure the intersection $X \cap Y$ is of the cardinality continuum. Let us take such Sierpiński set X on $[0, 1]$. Let f be the identity transformation of X. We can extend f to a mapping defined on the whole segment $[0, 1]$ putting

$$f([0, 1] \setminus X) = \{0\}.$$

It is easy to see that the extended mapping f from $[0, 1]$ into $[0, 1]$ satisfies the following two conditions:

(1) for each Lebesgue measure zero set $Y \subseteq [0, 1]$ the image $f(Y)$ is at most countable and hence is of Lebesgue measure zero;
(2) for each Lebesgue measurable set $Y \subseteq [0, 1]$ with a strictly positive measure the image $f(Y)$ is a Lebesgue nonmeasurable subset of $[0, 1]$.

Notice that condition (2) follows from the fact that any uncountable subset of a Sierpiński set is also a Sierpiński set, and hence it is Lebesgue nonmeasurable. So, we see that the mapping f has the property (N) of Luzin in a very strong form. But at the same time f transforms all Lebesgue measurable sets with a strictly positive measure into Lebesgue nonmeasurable sets.

Of course, a similar example can be constructed using Martin's Axiom and a generalized Sierpiński subset of $[0, 1]$.

In connection with Example 3 notice that if a mapping f from a measure space (E_1, μ_1) into a measure space (E_2, μ_2) transforms measurable sets into measurable sets (with respect to the completions of μ_1 and μ_2) then, as a rule, f has the property (N). In particular, this fact will be true in the case when every μ_2–measurable set with strictly positive μ_2–measure contains a nonmeasurable subset with respect to the completion of μ_2. But, of course, this fact is not true in general for all spaces (E_2, μ_2). For instance, if μ_2 is a universal measure on E_2, i.e. the domain of μ_2 coincides with the family of all subsets of E_2 then, obviously, every mapping f from E_1 into E_2 transforms μ_1–measurable sets into μ_2–measurable sets but f may be such that it does not have the property (N) of Luzin. Notice also that if μ_2 is a σ–finite measure on E_2 invariant (or, more generally, quasiinvariant) with respect to an uncountable group of transformations of the space E_2 acting freely on E_2, then each μ_2–measurable set with a strictly positive measure contains a nonmeasurable subset, so in this situation the property (N) follows from the property of preserving measurability (see [4]).

Now let us consider the case where the given measure spaces (E_1, μ_1) and (E_2, μ_2) are such that E_1 and E_2 are projective spaces, i.e. E_1 and E_2 are homeomorphic to some projective subsets of the real line R, and μ_1 and μ_2 are probability (or σ–finite) Borel measures on E_1 and E_2, respectively. Example 1 shows that, if we want to preserve for such spaces the assertion of Proposition 1, we need some additional set–theoretic axioms. It turns out that the standard Axiom of Projective Determinacy is sufficient for our purposes. In fact, we need the following three properties of projective sets:

(a) every projective set is a Radon space,

(b) every projective subset of the plane R^2 admits a uniformization by a projective set,

(c) every uncountable projective set contains an uncountable compact subset.

These three properties are implied by the Axiom of Projective Determinacy (see [5]). Remark here that relation (a) is equivalent to the following one: every projective subset of R is universally measurable with respect to the class of all σ–finite Borel measures on R. Similarly, if relation (a) holds then relation (c) is equivalent to the following one: every uncountable projective subset of R is not a universally measure zero set with respect to the class of all σ–finite diffused Borel measures on R.

Taking these facts into account we can formulate an appropriate analogue of Lemma 1.

LEMMA 2. *Let the conditions (a) and (b) be satisfied, let E be a projective space, let S be a countably generated σ–subalgebra of the Borel σ–algebra $B(E)$ and let μ be a probability measure defined on S. Then there exists a Borel measure $\overline{\mu}$ on E extending μ.*

PROOF. The argument is quite similar to the proof of Lemma 1 (cf. [2]). Let $(X_i)_{i \in \omega}$ be a countable family of Borel subsets of E generating the σ–algebra S. Let φ be the characteristic function of this family. Obviously, φ is a Borel mapping from E into the Cantor space 2^ω and

$$S = \{\varphi^{-1}(Z) : \quad Z \in B(2^\omega)\}.$$

Let us define a Borel measure ν on 2^ω by the formula

$$\nu(Z) = \mu(\varphi^{-1}(Z)) \quad (Z \in B(2^\omega)).$$

Let us denote by the same symbol ν the completion of this Borel measure on 2^ω. Let us consider the graph

$$G = \{(x, \varphi(x)) : \quad x \in E\}$$

of the mapping φ. It is not difficult to check that this graph is a projective subset of the product space $E \times 2^\omega \subseteq R \times 2^\omega$. Using the uniformization property (b), we can define a mapping h from $pr_2(G)$ into E such that

(1) the graph of h is a projective subset of $2^\omega \times R$;
(2) $(\varphi \circ h)(y) = y$ for each point $y \in pr_2(G)$.

Using property (a) we see that the mapping h is ν–measurable. Now, let us put

$$\overline{\mu}(X) = \nu(h^{-1}(X)) \quad (X \in B(E)).$$

Then it is not difficult to check that the measure $\overline{\mu}$ is the required extension of the original measure μ. □

We need also the following auxiliary proposition.

LEMMA 3. *Let E_1 be a projective space, let E_2 be a metric space and let f be a Borel mapping from E_1 into E_2. If condition (c) holds then the image $f(E_1)$ is separable.*

The proof of Lemma 3 is not difficult. Notice in connection with this lemma that if the inequality $2^{\omega} < 2^{\omega_1}$ holds, then a metrizable Borel image of a separable metric space is also separable.

As soon as Lemmas 2 and 3 have been proved we can deduce from them the following proposition.

PROPOSITION 3. *Let (E_1, μ_1) be a projective space equipped with a σ-finite Borel measure μ_1, let (E_2, μ_2) be a metric space equipped with a σ-finite Borel measure μ_2 and let f be a Borel mapping from E_1 into E_2. Suppose that the conditions (a), (b) and (c) hold and let for every compact set $K \subseteq E_1$ with μ_1-measure zero the image $f(K)$ be of outer μ_2-measure zero. Then the mapping f has the property (N) of Luzin.*

Notice that the proof of Proposition 3 is quite similar to the proof of Proposition 1.

Finally, let us remark that in the case of projective spaces we cannot apply directly Choquet's theorem, so the approach to the property (N) of Luzin with the help of the measure extension theorem is more preferable. Of course, in concrete situations when we know of what projective class is the given space E_1, we do not need the whole power of the properties (a), (b) and (c). For instance, if our space E_1 is of the class Σ_2^1 then from an old result of Kondo we have the uniformization property for this class. Hence we do not need here the whole Axiom of Projective Determinacy but it is sufficiently, for instance, to assume the existence of a measurable cardinal (see [5]). From this additional set-theoretic assertion Lemmas 2 and 3 and Proposition 3 follow for all projective spaces of the class Σ_2^1.

REFERENCES

[1] N. Luzin, *Integral and Trigonometric series*, Moscow, 1956 (in Russian).

[2] M. Ershov, *Measure Extensions and Stochastic Equations*, Probability Theory and its Applications, 19 (3), 1974 (in Russian).

[3] C. Dellacherie, *Capacities et Processus Stochastiques*, Springer–Verlag, Berlin, 1972.

[4] A. Kharazishvili, *Some questions of set theory and measure theory*, Tbil. Gos. Univ., Tbilisi, 1978 (in Russian).

[5] Handbook of Mathematical Logic, (Ed. J. Barwise), North–Holland Publ. Comp., Amsterdam, 1977.

INSTITUTE OF APPLIED MATHEMATICS
TBILISI STATE UNIVERSITY
UNIVERSITY STREET 2
380043 TBILISI 43