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ON SOME CONDITIONAL FUNCTIONAL EQUATIONS OF GOŁĄB—SCHINZEL TYPE

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Abstract. We consider equation (6) in the class of continuous functions $f : I \to \mathbb{R}$ satisfying (7), where I is a non-trivial real interval and n, k are fixed positive integers. The obtained results are applied to get the solutions of the system of functional equations (3)–(5) in the class of pairs of functions $f, g : I \to \mathbb{R}$ such that f is continuous. Some connections between solutions of the equations and a class of subsemigroups of some Lie groups are established as well.

Let $\mathbf{N}, \mathbf{Z}, \mathbf{Q}$, and $\mathbf{R}$ stand for the sets of positive integers, integers, rationals, and reals, respectively. The system of functional equations

$$f(xf(y)^2 + yf(x)) = f(x)f(y), \tag{1}$$

$$g(xf(y)^2 + yf(x)) = f(x)g(y) + 3xyf(y) + g(x)f(y)^3, \tag{2}$$

where the unknown functions f and g map $\mathbf{R}$ into $\mathbf{R}$, has been introduced by S. Midura (see [7]) in connection with the problem of finding subsemigroups of the Lie group L_3^1. It is solved in [7] under the assumption that f is continuous. Namely we have the following

THEOREM M (see [7], p. 47–48). *A pair of functions $f, g : \mathbf{R} \to \mathbf{R}$ such that f is continuous satisfies the system of functional equations (1), (2) iff either*

$$f(x) \equiv 0 \quad \text{and} \quad g(0) = 0$$

or

$$f(x) \equiv 1 \quad \text{and} \quad g(x) = h(x) + \frac{3}{2}x^2 \quad \text{for} \quad x \in \mathbf{R}$$

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with some additive function $h : \mathbb{R} \to \mathbb{R}$.

In the paper we consider the following conditional generalization of the system (1), (2):

$$f(xf(y)^2 + yf(x)) = f(x)f(y) \tag{3}$$

whenever $f(x)f(y) \neq 0$,

$$g(xf(y)^2 + yf(x)) = f(x)g(y) + 3xyf(y) + g(x)f(y)^3 \tag{4}$$

where the unknown functions f and g map a non-trivial real interval I into $\mathbb{R}$ with the additional assumption that

$$xf(y)^2 + yf(x) \in I \quad \text{for every} \quad x, y \in I \ \text{ with } \ f(x)f(y) \neq 0. \tag{5}$$

Let us note that every pair of functions $f, g : \mathbb{R} \to \mathbb{R}$ satisfying (1), (2) is also a solution of (3), (4). The converse is not true. For instance, a pair of functions $f_0, g_0 : \mathbb{R} \to \mathbb{R}$, given by: $g_0(x) \equiv 0$ and

$$f_0(x) = \begin{cases} 1 & \text{if} \quad x = 0; \\ 0 & \text{if} \quad x \neq 0 \end{cases} \qquad \text{for} \quad x \in \mathbb{R},$$

is a solution of the system (3), (4) and does not satisfy (1), (2).

We as well will investigate the conditional equation

$$f(xf(y)^k + yf(x)^n) = f(x)f(y) \quad \text{whenever} \quad f(x)f(y) \neq 0 \tag{6}$$

in the class of continuous functions $f : I \to \mathbb{R}$ satisfying

$$xf(y)^k + yf(x)^n \in I \qquad \text{for every} \quad x, y \in I \ \text{ with } \ f(x)f(y) \neq 0, \tag{7}$$

where $k, n \in \mathbb{N}$ are fixed. Equation (6) is a conditional generalization of (1) and the functional equation

$$f(f(y)^k x + f(x)^n y) = f(x)f(y) \tag{8}$$

studied in different cases e.g. in [2]–[4] and [7]–[9]. For example it is known that, for $n, k \in \mathbb{N}$, the only continuous solutions of (8) in the class of functions mapping a real topological linear space into $\mathbb{R}$ are functions $f_1(x) \equiv 1$ and $f_2(x) \equiv 0$ (cf. [3] and [9]). In the class of functions $f : [0, +\infty) \to [0, +\infty)$ equation (8) has been considered in [8].

Now, we will give a justification for the study of (3)–(5) and (6)–(7). For this we need some definitions.

Put $D_1 = (\mathbb{R}\setminus\{0\})\times\mathbb{R}$ and $D_2 = (\mathbb{R}\setminus\{0\})\times\mathbb{R}\times\mathbb{R}$ and fix $n, k \in \mathbb{N}$. We define binary operations $\cdot : D_1 \times D_1 \to D_1$ and $\cdot : D_2 \times D_2 \to D_2$ by the formulas:

$$(x_1, x_2)\cdot(y_1, y_2) = (x_1y_1, x_1^n y_2 + y_1^k x_2) \quad \text{for} \quad (x_1, x_2), (y_1, y_2) \in D_1,$$
$$(x_1, x_2, x_3)\cdot(y_1, y_2, y_3) = (x_1y_1, x_1y_2 + y_1^2 x_2, x_1y_3 + 3x_2y_2y_1 + x_3y_1^3)$$
$$\text{for} \quad (x_1, x_2, x_3), (y_1, y_2, y_3) \in D_2.$$

It is easy to check that $(D_1, \cdot)$ (with any $k, n \in \mathbb{N}$) and $(D_2, \cdot)$ are groups. Furthermore (see e.g. [3], p. 261, [6], p. 6, and [4], p. 60):

– if $n = 0$ and $k = 1$, D_1 is isomorphic to the one–dimensional affine group,
– if $n = 1$ and $k = 2$, D_1 is isomorphic to the Lie group L_2^1,
– if $n = 1$, D_1 is isomorphic to a subgroup of the Lie group L_k^1,
– if $n = k = 1$, D_1 is isomorphic to the Clifford group of the field $\mathbb{R}$,
– D_2 is isomorphic to the Lie group L_3^1.

Let us introduce the following.

DEFINITION 1. Let $S \neq \emptyset$ be a set. We say a pair (D, F) is a parametrization of S provided D is a non–empty set and F is a function mapping D onto S.

Of course every non–empty set S has a parametrization; it suffices to take $D = S$ and $F(a) = a$ for every $a \in D$.

DEFINITION 2. A subset S of D_1 (D_2 respectively) is of type j, with $j \in \{1, 2\}$ ($j \in \{1, 2, 3\}$, resp.), provided there is a parametrization (D, F) of S such that the function F_j is one–to–one, where $F = (F_1, F_2)$ ($F = (F_1, F_2, F_3)$, resp.).

Now, we are in a position to formulate the subsequent

PROPOSITION 1. *Let $n, k \in \mathbb{N}$. The following two conditions are valid.*

(i) *S is a subsemigroup of type 2 of the group D_1 iff there is a function $f : \mathbb{R} \to \mathbb{R}$ satisfying the conditional functional equation (6) such that $f(\mathbb{R}) \neq \{0\}$ and*
$$S = \{(f(x), x) : x \in \mathbb{R}, f(x) \neq 0\}.$$

(ii) *S is a subsemigroup of type 2 of the group D_2 iff there is a pair of functions $f, g : \mathbb{R} \to \mathbb{R}$ satisfying the system (3), (4) such that $f(\mathbb{R}) \neq \{0\}$ and*
$$S = \{(f(x), x, g(x)) : x \in \mathbb{R}, f(x) \neq 0\}.$$

PROOF. We prove only (ii). The proof of (i) is analogous.

Let S be a subsemigroup of type 2 of D_2 and (D, F), $F = (F_1, F_2, F_3)$, be a parametrization of S with F_2 one–to–one. Put

$$f_0 = F_1 \circ F_2^{-1} \qquad \text{and} \quad g_0 = F_3 \circ F_2^{-1}.$$

Then

$$S = \{(f_0(x), x, g_0(x)) : \quad x \in F_2(D)\}.$$

Further, since for every $(x_1, x_2, x_3), (y_1, y_2, y_3) \in S$

$$(x_1y_1, x_1y_2 + y_1^2x_2, x_1y_3 + 3x_2y_2y_1 + x_3y_1^3) = (x_1, x_2, x_3) \cdot (y_1, y_2, y_3) \in S,$$

we have

(9) $(f_0(x)f_0(y), f_0(x)y + f_0(y)^2x, f_0(x)g_0(y) + 3xyf_0(y) + g_0(x)f_0(y)^3) \in S$

for every $x, y \in F_2(D)$. Thus, for every $x, y \in F_2(D)$,

$$f_0(f_0(x)y + f_0(y)^2x) = f_0(x)f_0(y) \tag{10}$$

$$g_0(f_0(x)y + f_0(y)^2x) = f_0(x)g_0(y) + 3xyf_0(y) + g_0(x)f_0(y)^3. \tag{11}$$

This implies that the functions $f, g : \mathbb{R} \to \mathbb{R}$, defined by:

$$f(x) = \begin{cases} f_0(x) & \text{if } x \in F_2(D); \\ 0 & \text{otherwise,} \end{cases}$$

$$g(x) = \begin{cases} g_0(x) & \text{if } x \in F_2(D); \\ 0 & \text{otherwise,} \end{cases}$$

for every $x \in \mathbb{R}$, satisfy the system (3), (4).

Now, suppose that we are given functions $f, g : \mathbb{R} \to \mathbb{R}$, $f(\mathbb{R}) \neq \{0\}$, satisfying the system (3), (4). Put $S(f) = \mathbb{R} \setminus f^{-1}(\{0\})$, $f_0 = f|_{S(f)}$ and $g_0 = g|_{S(f)}$. It is easily seen that f_0 and g_0 are solutions of (10) and (11). Thus (9) holds for every $x, y \in S(f)$, because

$$S = \{(f_0(x), x, g_0(x)) : \quad x \in S(f)\}.$$

Hence S is a subsemigroup of D_2. Next $(S(f), F)$, where $F(x) = (f_0(x), x, g_0(x))$ for $x \in S(f)$, is a parametrizaton of S. This completes the proof. $\square$

Consideration of subsemigroups of types 1 and 3 of D_2 leads to some other systems of functional equations. Results concerning them will be published separately.

We must mention yet that similar methods, as those presented in Proposition 1, of determining algebraic substructures were already used e.g. in [2], [3], [6], [7], and [4].

REMARK 1. Observe that if a function $f : I \to \mathbb{R}$, where I is a real interval, satisfies (6) and (7), then the function $\overline{f} : \mathbb{R} \to \mathbb{R}$ given by:

$$\overline{f}(x) = \begin{cases} f(x) & \text{if } x \in I; \\ 0 & \text{otherwise}, \end{cases} \qquad \text{for} \quad x \in \mathbb{R},$$

is a solution of equation (6). Thus, according to Proposition 1, f (if $f(I) \neq \{0\}$) determines a subsemigroup of type 2 of D_1.

REMARK 2. For every $k, n \in \mathbb{N} \cup \{0\}$ equation (6) can be written in the following equivalent form

$$f(x)f(y)(f(xf(y)^k + yf(x)^n) - f(x)f(y)) = 0$$

(Of course, if we confine ourselves e.g. to a class of functions mapping a real interval I into $\mathbb{R}$ and satisfying (7).)

Now, we will prove some lemmas useful in the sequel.

LEMMA 1. *Let I be a non-trivial real interval, $n, k \in \mathbb{N}$, and $f : I \to \mathbb{R}$ be a function satisfying (6) and (7). Then, for every $x \in S(f) := \{x \in I : f(x) \neq 0\}$, the functions*

$$S(f) \ni z \to xf(z)^n + zf(x)^k \in \mathbb{R},$$
$$S(f) \ni z \to zf(x)^n + xf(z)^k \in \mathbb{R}$$

are one-to-one.

PROOF. We prove only that the first function is one-to-one. The proof for the second function is analogous.

Fix $x \in S(f)$ and suppose that

$$xf(z)^n + zf(x)^k = xf(w)^n + wf(x)^k$$

for some $z, w \in S(f)$. Then, by (6), $f(x)f(z) = f(x)f(w)$ and consequently $f(z) = f(w)$. Thus

$$0 = (xf(z)^n + zf(x)^k) - (xf(w)^n + wf(x)^k) = (z - w)f(x)^k,$$

which means that $z = w$. This ends the proof. $\square$

LEMMA 2. *Let I, n, k, and f be the same as in Lemma 1. Assume that f is continuous and put $I^+ = \{x \in I : x \geq 0\}$ and $I^- = \{x \in I : x \leq 0\}$. Then*

$$f^{-1}(\{0\}) \in \{\emptyset, \{0\}, I^+, I^-, I\}.$$

PROOF. We will show that

$$\text{either } \ f(I^+) = \{0\} \quad \text{or} \quad 0 \notin f(I^+ \setminus \{0\})$$

and

$$\text{either } \ f(I^-) = \{0\} \quad \text{or} \quad 0 \notin f(I^- \setminus \{0\})$$

It is easily seen that those conditions yield the statement.

Let $I_0^+ = I^+ \setminus \{0\}$ and $I_0^- = I^- \setminus \{0\}$. For the proof by contradiction suppose that there are $a, b \in I_0^+$ ($a, b \in I_0^-$, respectively) with

$$f(b) \neq 0 = f(a).$$

Then, by (6) and (7), $f(b)^2 \in f(I)$ and the continuity of f implies

$$[0, f(b)^2] \subset f(I). \tag{12}$$

Since

$$L_1 := f^{-1}(\{0\}) \cap I_0^+ \neq \emptyset \quad (L_1 := f^{-1}(\{0\}) \cap I_0^- \neq \emptyset, \text{ resp.})$$

and

$$L_2 := I_0^+ \setminus L_1 \neq \emptyset \quad (L_2 := I_0^- \setminus L_1 \neq \emptyset, \text{ resp.})$$

is open in I_0^+ (I_0^-, resp.), there are $e \in L_1$ and a sequence $\{y_m\}_{m \in \mathrm{N}} \subset L_2$ with

$$e = \lim_{m \to \infty} y_m.$$

Note that, in view of (7),

$$z_m = y_m f(y_m)^k + y_m f(y_m)^n \in I \quad \text{for every} \quad m \in \mathrm{N}$$

and consequently

$$0 = \lim_{m \to \infty} z_m = \inf I^+ \quad (0 = \lim_{m \to \infty} z_m = \sup I^-, \quad \text{resp.}). \tag{13}$$

Further, by (12), there is $d > 0$ with $(0,d) \subset \{ef(y)^k : y \in I\}$ ($(-d,0) \subset \{ef(y)^k : y \in I\}$, resp.) and, for every $y \in I$ with $ef(y)^k \in I \setminus \{0\}$,

$$\begin{aligned} f(ef(y)^k) &= \lim_{m\to\infty} f(y_m f(y)^k + y f(y_m)^n) \\ &= \lim_{m\to\infty} f(y)f(y_m) = f(y)f(e) = 0. \end{aligned}$$

Thus, according to (13), there exists $c > 0$ such that

$$(0,c) \subset L_1 \quad ((-c,0) \subset L_1, \text{ resp.}).$$

On account of (12), there is $b_0 \in I$ with

$$0 < f(b_0)^k < ce^{-1} \quad (0 < f(b_0)^k < -ce^{-1}, \text{ resp.}).$$

Put

$$w_m := b_0 f(y_m)^n + y_m f(b_0)^k \qquad \text{for every} \quad m \in \mathbf{N}.$$

Then (6) gives

$$(14) \qquad f(w_m) = f(b_0)f(y_m) \neq 0 \qquad \text{for every} \quad m \in \mathbf{N}.$$

We have as well

$$0 < \lim_{m\to\infty} w_m = ef(b_0)^k < c \quad (0 > \lim_{m\to\infty} w_m = ef(b_0)^k > -c, \text{ resp.}).$$

Whence there is $m \in \mathbf{N}$ such that

$$0 < w_m < c \quad (0 > w_m > -c, \text{ resp.}),$$

which means that $f(w_m) = 0$. This brings a contradiction to (14). □

For the next lemma we need a theorem of J. Aczél [1] (see also [5]). Let us remind it.

THEOREM A (see [5], p. 307). *Let L be a real non-trivial interval and let $. : L \times L \to L$ be a continuous cancellative associative operation. Then there exists a continuous bijection $h : L \to J$ such that*

$$(15) \qquad x.y = h^{-1}(h(x) + h(y)) \qquad \textit{for every} \quad x, y \in L,$$

where J is a (necessarily unbounded) real interval.

LEMMA 3. *Let n, k, I, and f be just the same as in Lemma 2. Let $L \subset I$ be a non-trivial interval with*

$$xf(y)^n + yf(x)^k \in L \qquad \textit{for every} \quad x, y \in L$$

and $0 \notin f(L)$. Then either $f(x) = 1$ for every $x \in L$ or $f|_L$ is one–to–one, $f(L) \subset (0, +\infty)$ and there is $s \in \mathbb{R} \setminus \{0\}$ such that,

1^o in the case $n \neq k$,

$$f(s(y^n - y^k)) = y \qquad \text{for every} \quad y \in f(L);$$

2^o in the case $n = k$,

$$f(sy^n \ln(y)) = y \qquad \text{for every} \quad y \in f(L).$$

PROOF. The case $f|_L = \text{const} = c$ is trivial, because, according to the hypothesis, for $x \in L$

$$c = f(xf(x)^n + xf(x)^k) = f(x)^2 = c^2.$$

So suppose that $f|_L$ is not constant.

Setting $x = y \in L$ in (6) we get

$$f(L) \cap (0, +\infty) \neq \emptyset.$$

Since $0 \notin f(L)$ and f is continuous, this implies $f(L) \subset (0, +\infty)$.

Define a binary operation $.\ :\ L \times L \to L$ by the formula:

$$x.y = xf(y)^n + yf(x)^k \qquad \text{for every} \quad x, y \in L.$$

Then, according to (6), for every $x, y, z \in L$ we have

$$\begin{aligned} x.(y.z) =& xf(y.z)^n + (y.z)f(x)^k = xf(y)^n f(z)^n + yf(z)^n f(x)^k \\ &+ zf(y)^k f(x)^k = (x.y)f(z)^n + zf(x.y)^k = (x.y).z. \end{aligned}$$

Thus the operation is associative. Further, it is easy to see that it is continuous and, by Lemma 1, cancellative. Hence, on account of Theorem A, there exists a homeomorphism $h : L \to J$ (where J is an unbounded real interval) such that (15) holds and, whence,

$$\begin{aligned} f(h^{-1}(h(x)))f(h^{-1}(h(y))) =& f(x)f(y) = f(x.y) \\ =& f(h^{-1}(h(x) + h(y))) \quad \text{for every} \quad x, y \in L. \end{aligned}$$

Hence there is $d \in (0, +\infty) \setminus \{1\}$ such that

$$f \circ h^{-1}(y) = d^y \qquad \text{for every} \quad y \in J,$$

which means that $f|_L$ is one–to–one.

First consider the case $k \neq n$. Then (15) yields

$$xf(y)^n + yf(x)^k = x.y = y.x = yf(x)^n + xf(y)^k \qquad \text{for every} \quad x, y \in L.$$

Thus setting

$$s = (f(y_0)^n - f(y_0)^k)^{-1} y_0$$

for some fixed $y_0 \in L \setminus \{0\}$, $f(y_0) \neq 1$, we obtain 1^o.

Next, let $k = n$. Note that $g = (f|_L)^{-1}$ satisfies

$$g(xy) = g(x)y^n + g(y)x^n \qquad \text{for every} \quad x, y \in f(L).$$

Consequently the function $t : f(L) \to \mathbb{R}$, given by:

$$t(z) = g(z)z^{-n} \qquad \text{for every} \quad z \in f(L),$$

is a solution of the Cauchy equation:

$$t(xy) = t(x) + t(y).$$

Whence there exists $s \in \mathbb{R} \setminus \{0\}$ such that

$$t(x) = s \ln(x) \qquad \text{for every} \quad x \in f(L).$$

This implies 2^o and ends the proof. □

LEMMA 4. *Let n, k, I, and f be just the same as in Lemma 2. Then $f(x) \equiv 0$ or 1, or there exists $s \in \mathbb{R} \setminus \{0\}$ such for every $x \in S(f) := \{x \in I : \ f(x) \neq 0\}$,*

1^o in the case $n \neq k$, $x = s(f(x)^n - f(x)^k)$;

2^o in the case $n = k$, $x = sf(x)^n \ln |f(x)|$.

Consequently $f|_{S(f)}$ is one–to–one or f is constant.

PROOF. It is easy to see that if f = const, then $f(x) \equiv 1, 0$. So, it remains to study the situation where f is not constant. Then, according to Lemma 2,

$$f^{-1}(\{0\}) \in \{\emptyset, \{0\}, I^+, I^-\}.$$

The case $f^{-1}(\{0\}) \neq \{0\}$ results from Lemma 3 (with $L \in \{I_0^+, I_0^-, I\}$). Therefore suppose that $f^{-1}(\{0\}) = \{0\}$.

Observe that then, by (6), we get

$$xf(y)^n + yf(x)^k \neq 0 \qquad \text{for every} \quad x, y \in I \setminus \{0\}. \tag{16}$$

Next, since in view of the right hand side of equation (6)

$$f(I) \cap (0, +\infty) \neq \emptyset,$$

we have

$$f(I_0^+) \subset (0, +\infty) \quad \text{or} \quad f(I_0^-) \subset (0, +\infty).$$

First suppose that $f(I_0^+) \subset (0, +\infty)$ and $f(I_0^-) \subset (-\infty, 0)$ ($f(I_0^+) \subset (-\infty, 0)$ and $f(I_0^-) \subset (0, +\infty)$, respecively). Then, by (6), for every $x, y \in I_0^+$ and $x, y \in I_0^-$

$$f(xf(y)^n + yf(x)^k) = f(x)f(y) > 0,$$

which means that

$$\text{(17)} \qquad \begin{aligned} &xf(y)^n + yf(x)^k \in I_0^+ \quad (xf(y)^n + yf(x)^k \in I_0^-, \text{ resp.}) \\ &\qquad\qquad \text{for every} \quad x, y \in I_0^+ \text{ and } x, y \in I_0^-. \end{aligned}$$

Thus, on account of Lemma 3 with $L = I_0^+$ ($L = I_0^-$, resp.), there is $s \in \mathbb{R} \setminus \{0\}$ such that for every $x \in I_0^+$ ($x \in I_0^-$, resp.)

$$\text{(18)} \qquad x = \begin{cases} s(f(x)^n - f(x)^k) & \text{if } n \neq k; \\ sf(x)^n \ln |f(x)| & \text{if } n = k. \end{cases}$$

Moreover, in view of (17), for every $x \in I_0^-$ ($x \in I_0^+$, resp.)

$$xf(x)^n + xf(x)^k \in I_0^+ \quad (xf(x)^n + xf(x)^k \in I_0^-, \text{ resp.})$$

and consequently, by (6) and (18), in the case $n \neq k$,

$$xf(x)^n + xf(x)^k = s(f(x)^{2n} - f(x)^{2k}) \qquad \text{for} \quad x \in I_0^- \ (x \in I_0^+, \text{ resp.}),$$

and, in the case $n = k$,

$$2xf(x)^n = sf(x)^{2n} \ln(f(x)^2) \qquad \text{for} \quad x \in I_0^- \ (x \in I_0^+, \text{ resp.}).$$

Since, according to (16),

$$f(x)^n + f(x)^k \neq 0 \qquad \text{for every} \quad x \in I \setminus \{0\},$$

this implies that (18) holds also for every $x \in I_0^-$ ($x \in I_0^+$, resp.).

To complete the proof it remains to study the case where $f(I_0^+) \cup f(I_0^-) \subset (0,+\infty)$. Using Lemma 3, first with $L = I_0^+$ and next with $L = I_0^-$, we obtain then that there are $s, s_0 \in \mathbf{R} \setminus \{0\}$ such that (18) holds for every $x \in I_0^+$ and

$$x = \begin{cases} s_0(f(x)^n - f(x)^k) & \text{if} \quad n \neq k; \\ s_0 f(x)^n \ln(f(x)) & \text{if} \quad n = k, \end{cases} \qquad \text{for} \quad x \in I_0^-. \tag{19}$$

Fix $y \in I_0^-$ with $f(y) \neq 1$. Note that $0f(y)^n + yf(0)^k = 0$ and

$$yf(y)^n + yf(y)^k < 0.$$

Thus Lemma 1 yields

$$xf(y)^n + yf(x)^k > 0 \qquad \text{for every} \quad x \in I_0^+.$$

Hence, according to (6), (18) (with $x \in I_0^+$), and (19), for every $x \in I_0^+$, in the case $n \neq k$,

$$\begin{aligned} &s(f(x)^n - f(x)^k)f(y)^n + s_0(f(y)^n - f(y)^k)f(x)^k \\ &\quad = xf(y)^n + yf(x)^k = s(f(x)^n f(y)^n - f(x)^k f(y)^k) \end{aligned}$$

and, in the case $n = k$,

$$\begin{aligned} &(sf(x)^n \ln(f(x)))f(y)^n + (s_0 f(y)^n \ln(f(y)))f(x)^n \\ &\quad = xf(y)^n + yf(x)^n = sf(x)^n f(y)^n \ln(f(x)f(y)). \end{aligned}$$

Whence, for every $x \in I_0^+$, in the case $n \neq k$,

$$(s_0 f(x)^k - sf(x)^k)(f(y)^n - f(y)^k) = 0$$

and, in the case $n = k$,

$$s_0 \ln(f(y)) = s \ln(f(y)),$$

which means that $s_0 = s$, because $f(I_0^+) \neq \{1\}$ and $f(y) \neq 1$. This implies the statement. $\square$

Now, we have all tools to prove the following

THEOREM 1. *Let $n, k \in \mathbf{N}$, I be a non-trivial real interval, and f be a function mapping I into $\mathbf{R}$. Then f is a non-constant continuous solution of*

(6), (7) *iff there are* $s \in \mathbb{R} \setminus \{0\}$ *and a real interval* K *such that the function* $t : K \to \mathbb{R}$, *defined by:*

$$(20)\quad t(y) = \begin{cases} s(y^n - y^k) & \text{if } n \neq k; \\ sy^n \ln |y| & \text{if } n = k \text{ and } y \neq 0; \\ 0 & \text{if } 0 \in K \text{ and } y = 0 \end{cases} \qquad \text{for } y \in K,$$

is one–to–one and one of the following two conditions is valid:

(i) $xy \in K$ *for every* $x, y \in K$, $t(K) = I$, *and* $f = t^{-1}$;

(ii) $0 \in K$, $K \subset [0,1)$, $t(K) \in \{I_0^+, I_0^-\}$, *and*

$$(21)\qquad f(x) = \begin{cases} t^{-1}(x) & \text{if } x \in t(K); \\ 0 & \text{otherwise} \end{cases} \qquad \text{for every } x \in I.$$

Furthermore, f *is a constant solution of* (6), (7) *iff* $f(x) \equiv 0$ *or, only in the case where* $x + y \in I$ *for every* $x, y \in I$, $f(x) \equiv 1$.

PROOF. The case where f is constant is trivial (see e.g. Lemma 4). Therefore assume that f is not constant.

First we will show that if f has the form described in the statement, then it is a continuous solution of (6), (7). Since the cases $n \neq k$ and $n = k$ are analogous, we consider only the first one.

Fix $x, y \in I$ with $f(x)f(y) \neq 0$. Then according to the definition of f, $f(x) = t^{-1}(x)$ and $f(y) = t^{-1}(y)$. Next $f(x), f(y) \in K$ which implies $f(x)f(y) \in K$ and $t(f(x)f(y)) = t(f(x))f(y)^n + t(f(y))f(x)^k$. Thus $xf(y)^n + yf(x)^k \in I$ and

$$f(xf(y)^n + yf(x)^k) = f(x)f(y).$$

To complete the first part of the proof it suffices to observe that f is continuous, because t is continuous.

Now, assume that f is a continuous and non–constant solution of (6), (7). In view of Lemma 4, $f|_{S(f)}$ is one–to–one and there is $s \in \mathbb{R} \setminus \{0\}$ such that conditions 1°, 2° of that lemma are valid for every $x \in S(f)$. Hence, if card $f^{-1}(\{0\}) \leq 1$, f is one–to–one and it suffices to put $K = f(I)$ and $t = f^{-1}$. So it remains to study the case card $f^{-1}(\{0\}) > 1$.

According to Lemma 2 we have then

$$f^{-1}(\{0\}) \in \{I^+, I^-\}.$$

Consequently $S(f) \in \{I_0^+, I_0^-\}$ and, by Lemma 3 (with $L = I_0^+$ or $L = I_0^-$, respectively), $f(I) \subset [0, +\infty)$.

Let $K = f(I)$ and

$$t(y) = \begin{cases} f^{-1}(y) & \text{if} \quad y \in K \setminus \{0\}; \\ 0 & \text{if} \quad y = 0. \end{cases}$$

Then (21) holds, $0 \in K$,

$$t(K) = S(f) \cup \{0\} \in \{I^-, I^+\},$$

and, since $0 \notin S(f)$ and $f|_{S(f)}$ is one–to–one, t is one–to–one. Consequently $K \subset [0, 1)$. This ends the proof. □

Using Theorem 1 one can easily determine the continuous solutions $f : I \to \mathbb{R}$ of (6), (7) for any given k, n, and I. In particular we have the following

COROLLARY 1. Let $n, k \in \mathbb{N}$. A function $f : \mathbb{R} \to \mathbb{R}$ is a continuous solution of (6) iff $f(x) \equiv 0$ or $f(x) \equiv 1$.

PROOF. Let f be a continuous solution of (6). Then, according to Theorem 1, f is constant or there are $s \in \mathbb{R} \setminus \{0\}$ and a real interval K such that the function $t : K \to \mathbb{R}$, defined by (20), is one–to–one and one of conditions (i), (ii) of that theorem is valid. It is easily seen that, for every real interval K, t is not one–to–one or $t(K) \neq \mathbb{R}$ and, for every interval K described by (ii), $t(K)$ is a bounded set. Hence f is constant, which ends the proof. □

Finally we have the given below theorem.

THEOREM 2. *Let I be a non–trivial real interval and $f, g : I \to \mathbb{R}$. Assume that f is continuous. Then f, g fulfil the system of functional equations (3), (4) and condition (5) iff one of the following three conditions holds:*

1° $f(x) \equiv 0$,

2° $f(x) \equiv 1$, $x + y \in I$ for every $x, y \in I$, and there is an additive function $h : I \to \mathbb{R}$ such that $g(x) = h(x) + \frac{3}{2}x^2$ for every $x \in I$,

3° f has the form described in Theorem 1, by (i) or (ii), and there is $C \in \mathbb{R}$ such that

$$g(x) = Cf(x)^3 - 3s^2 f(x)^2 + (3s^2 - C)f(x) \qquad \textit{for} \quad x \in I \quad \textit{with} \quad f(x) \neq 0.$$

PROOF. First assume that f is constant. Then $f(x) \equiv 0$ or $f(x) \equiv 1$. In the case $f(x) \equiv 0$ we get 1°. So suppose that $f(x) \equiv 1$ and g satisfy (3)–(5). Then, by (4) and (5), $x + y \in I$ for every $x, y \in I$ and

$$g(x + y) = g(x) + 3xy + g(y) \qquad \text{for every} \quad x, y \in I.$$

Thus the function $h : I \to \mathbb{R}$, defined by:

$$h(x) = g(x) - \frac{3}{2}x^2 \qquad \text{for every} \quad x, y \in I,$$

satisfies

$$h(x+y) = h(x) + h(y) \qquad \text{for every} \quad x, y \in I.$$

Consequently 2^o holds. Since it is easy to check that functions f, g, described by 2^o, are solutions of (3)–(5), this ends the proof in the case where f is constant.

Now suppose that f is non–constant and continuous and f, g satisfy (3)–(5). Then f has the form described in Theorem 1, which means that $f|_{S(f)}$ is one–to–one and

$$x = s(f(x)^2 - f(x)) \qquad \text{for every} \quad x \in S(f) \tag{22}$$

with some $s \in \mathbb{R} \setminus \{0\}$. Hence, in view of symmetry of the right hand side of (3) with respect to x and y, for every $x, y \in \mathbb{R}(f)$ we have

$$xf(y)^2 + yf(x) = yf(x)^2 + xf(y)$$

and consequently, by (4),

$$f(x)g(y) + 3xyf(y) + f(y)^3 g(x) = f(y)g(x) + 3yxf(x) + f(x)^3 g(y).$$

From this and (22), for every $x, y \in S(f)$ with $f(y) \notin \{-1, 1\}$, we get

$$\begin{aligned} g(x) =&(f(y)^3 - f(y))^{-1}(g(y)(f(x)^3 - f(x)) + 3xyf(x) - 3xyf(y)) \\ =&(f(y)^3 - f(y))^{-1}(g(y) + 3s^2(f(y)^2 - f(y)))f(x)^3 - 3s^2 f(x)^2 \\ &+ (f(y)^3 - f(y))^{-1}(3s^2(f(y)^3 - f(y)^2) - g(y))f(x). \end{aligned}$$

Now, setting

$$C = (f(y)^3 - f(y))^{-1}(g(y) + 3s^2(f(y)^2 - f(y))),$$

with some fixed $y \in S(f)$, $f(y) \notin \{-1, 1\}$, we obtain

$$3s^2 - C = (f(y)^3 - f(y))^{-1}(3s^2(f(y)^3 - f(y)^2) - g(y))$$

and

$$g(x) = Cf(x)^3 - 3s^2 f(x)^2 + (3s^2 - C)f(x) \qquad \text{for every} \quad x \in S(f).$$

To complete the proof suppose that f and g are described by 3^o. Then, on account of Theorem 1, for every $x, y \in I$ with $f(x)f(y) \neq 0$, (3) and (5) hold and consequently

$$\begin{aligned} g(xf(y)^2 + yf(x)) =& Cf(x)^3 f(y)^3 - 3s^2 f(x)^2 f(y)^2 + (3s^2 - C)f(x)f(y) \\ =& f(x)(Cf(y)^3 - 3s^2 f(y)^2 + (3s^2 - C)f(y)) \\ &+ 3s^2(f(y)^2 - f(y))(f(x)^2 - f(x))f(y) \\ &+ (Cf(x)^3 - 3s^2 f(x)^2 + (3s^2 - C)f(x))f(y)^3 \\ =& f(x)g(y) + 3xyf(y) + g(x)f(y)^3. \end{aligned}$$

This ends the proof. □

REMARK 3. Observe that, in point 3^o of Theorem 2, by virtue of (22), g can be written in the following equivalent forms:

$$g(x) = C(f(x)^3 - f(x)) - 3sx = \frac{C}{s} f(x)x + \left(\frac{C}{s} - 3s\right)x \quad \text{for } x \in I,\ f(x) \neq 0.$$

REMARK 4. We may consider equation (6) also for every $n, k \in \mathbf{Z}$ or even for $n = n_1 n_2^{-1}$ and $k = k_1 k_2^{-1}$ with some odd $n_2, k_2 \in \mathbf{Z} \setminus \{0\}$ and $n_1, k_1 \in Z$, $n_1^2 + k_1^2 \neq 0$. The statements and proofs of Lemmas 1, 3, and 4 remain true then. This is also valid if $f : I \to [0, +\infty)$ and $n, k \in \mathbb{R}$, $n^2 + k^2 \neq 0$. However, since it is not the case for Lemma 2 and a suitable modification of it demands some additional long and complicated reasoning, we have confined our consideration to $n, k \in \mathbf{N}$. Some results concerning the mentioned situations will be published separately.

REFERENCES

[1] J. Aczél, *Sur les opération définies pour nombres réel*, Bull. Soc. Math. France **76** (1949), 59–64.

[2] J. Aczél and S. Gołąb, *Remark on one-parameter subsemigroups of the affine group and their homo- and isomorphisms*, Aequationes Math. **4** (1970), 1–10.

[3] N. Brillouët et J. Dhombres, *Equation fonctionnelles et recherche de sous-groupes*, Aequationes Math. **31** (1986), 253–293.

[4] J. Brzdęk, *Subgroups of the group Z_n and a generalization of the Gołąb-Schinzel functional equation*, Aequationes Math. **43** (1992), 59–71.

[5] R. Craigen and Z. Páles, *The associativity equation revisited*, Aequationes Math. **37** (1989), 306–312.

[6] S. Midura, *Sur la détermination de certains sous-groupes du groupe L_s^1 à l'aide d'équations functionnelles*, Dissertationes Math. No. **105**, IMPAN, Warsaw, 1973.

[7] S. Midura, *Sur certains sous-demigroupes à un parametre des groupes L_2^l et L_3^l déterminés à l'aide d'équations fonctionnelles*, Rocznik Nauk.-Dydakt. Wyż. Szkoly Ped. w Rzeszowie **7/62** (1985), 37–50.

[8] P. Urban, *Równania funkcyjne typu Gołąba-Schinzla.* [Doctor thesis], Uniwersytet Śląski, Instytut Matematyki, Katowice, 1987.

[9] P. Urban, *Continuous solutions of the functional equation* $f(xf(y)^k + yf(x)^l) = f(x)f(y)$, Demonstratio Math. **16** (1983), 1019–1025.

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