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A DIRECT PROOF OF A THEOREM OF K. BARON

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Abstract. In his work on the Gołąb–Schinzel equation, K. Baron shows a theorem concerning continuous complex-valued solutions, defined on the complex plane. In this note, we will give a direct proof of this theorem, which does not use the form of the general solution of the Gołąb–Schinzel equation.

Among other things, K. Baron shows in [2] the following theorem, concerning continuous solutions of the Gołąb–Schinzel equation

$$f(x + f(x)\,y) = f(x)f(y). \tag{1}$$

THEOREM. *A function $f : \mathbf{C} \to \mathbf{C}$ with $f(\mathbf{C}) \not\subseteq \mathbf{R}$ is a continuous solution of (1) if and only if f has the form*

$$f(x) = 1 + cx \qquad (x \in \mathbf{C}), \tag{2}$$

where c is a complex constant.

In his proof, K. Baron uses a theorem on the general solution of equation (1) (see for example [4]). We will give a direct proof that does not depend on this theorem.

Obviously, a function of form (2) is a continuous solution of (1). Therefore, let f be an arbitrary continuous solution of equation (1). The following relations hold:

(a) $P := f^{-1}(\{1\})$ is the group of periods of f.
(b) If $f(x) = f(y) \neq 0$, then $x - y \in P$.
(c) $G := f(\mathbf{C}) \setminus \{0\}$ is a multiplicative subgroup of $\mathbf{C} \setminus \{0\}$.
(d) $G \cdot P = P$.

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The first three properties are proved in [1], we will only show property (d). Let $p \in P$ and $f(y) \in G$. We get

$$f(y + f(y)\,p) = f(y)f(p) = f(y) \neq 0.$$

Property (b) implies $y + f(y)\,p - y = f(y)\,p \in P$, hence $G \cdot P \subseteq P$. The other inclusion is trivial.

Assume that P is not discrete. Then there is $z \in P$, $z \neq 0$, so that the straight line $\{\, sz \mid s \in \mathbb{R}\}$ is contained in P (see [3, VII F51.2, Prop. 3] [3]). As $f(\mathbb{C}) \not\subseteq \mathbb{R}$, there exists $x_0 \in \mathbb{C}$ with $z_0 = f(x_0) \in G \setminus \mathbb{R}$. Let z_1 be an arbitrary complex number. Considered as elements of $\mathbb{R}^2$, z and $z_0 z$ are linearly independent, hence we get

$$z_1 = sz + tz_0 z$$

with some $s, t \in \mathbb{R}$. Property (d) implies $z_0 z \in P$, therefore we get $z_1 \in P$. It follows $P = \mathbb{C}$, meaning $f(x) \equiv 1$, which is a contradiction.

Now we will show that the discrete group P only consists of the number 0. Suppose $P \neq \{0\}$. Then Property (d) implies that $G = f(\mathbb{C}) \setminus \{0\}$ is discrete as well. On the other hand, $f(\mathbb{C})$ is connected, because f is continuous. Again, we arrive at a contradiction and $P = \{0\}$ is proved.

Now, we choose two arbitrary elements x, y of the set $A := \{\, x \in \mathbb{C} \mid f(x) \neq 0\,\}$. We get

$$f(x + f(x)\,y) = f(x)f(y) = f(y + f(y)\,x) \neq 0$$

and with (b)

$$x + f(x)\,y - (y + f(y)\,x) \in P.$$

$P = \{0\}$ implies

$$x + f(x)\,y = y + f(y)\,x,$$

therefore

$$\frac{f(x) - 1}{x} = \frac{f(y) - 1}{y} \quad (x, y \in A \setminus \{0\}).$$

With $c := \frac{f(x)-1}{x}$ $(x \in A \setminus \{0\})$, f has the form

$$f(x) = \begin{cases} 1 + cx, & 1 + cx \in G \\ 0, & \text{otherwise} \end{cases}$$

Because of $f(\mathbb{C}) \not\subseteq \mathbb{R}$, $c \neq 0$ holds. Further, we argue as K. Baron did in his paper [2]. We have

$$f(x) = 0 \iff 1 + cx \notin G \iff x = \frac{g - 1}{c} \qquad \text{with} \quad g \notin G.$$

or

$$M := f^{-1}(\{0\}) = \left\{ \frac{1}{c}(g-1) \middle| g \notin G \right\} = \frac{1}{c}\left(G^C - 1\right).$$

The set M is a closed subset of $\mathbb{C}$, therefore the sets cM and $1 + cM = G^C$ are closed as well. It follows that G is an open set. Observing that G is a multiplicative subgroup of $\mathbb{C} \setminus \{0\}$, we get $G = \mathbb{C} \setminus \{0\}$. That means, f has the form (2).

References

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