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ON THE SYSTEM OF THE ABEL EQUATIONS ON THE PLANE

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Abstract. We find all of continuous, homeomorphic and C^k solutions of the system of the Abel equations

$$\begin{cases} \varphi(f(x)) = \varphi(x) + a \\ \varphi(g(x)) = \varphi(x) + b \end{cases} \quad \text{for} \quad x \in \mathbb{R}^2,$$

where a, b are linearly independent vectors and f, g are commutable orientation preserving homeomorphisms of the plane onto itself satisfying some condition which is equivalent to the fact that there exists a homeomorphic solution of the system above.

In the present paper we shall be concerned with the system of the Abel equations

$$(1) \qquad \begin{cases} \varphi(f(x)) = \varphi(x) + a \\ \varphi(g(x)) = \varphi(x) + b \end{cases} \quad \text{for} \quad x \in \mathbb{R}^2,$$

where a, b are linearly independent vectors. The Abel equation

$$\varphi(f(x)) = \varphi(x) + a \qquad \text{for} \quad x \in \mathbb{R}^2,$$

where $a \neq (0,0)$, has been considered in [5].

By a *line* we mean a homeomorphic image of a straight line which is a closed set. We assume that f, g are *free mappings* (i.e. orientation preserving homeomorphisms of the plane onto itself which have no fixed points - for the definition of an orientation preserving homeomorphism see e.g. [6], p. 198 or [2], p. 395) such that

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$$f \circ g = g \circ f$$

and satisfy the following condition:

(D) there exist lines K^0 and K_0 such that

(2) $$K^0 \cap f[K^0] = \emptyset,$$

(3) $$K_0 \cap g[K_0] = \emptyset,$$

(4) $$U^0 \cap f[U^0] = \emptyset,$$

(5) $$U_0 \cap g[U_0] = \emptyset,$$

(6) $$\bigcup_{n\in\mathbb{Z}} f^n[U^0] = \mathbb{R}^2,$$

(7) $$\bigcup_{m\in\mathbb{Z}} g^m[U_0] = \mathbb{R}^2,$$

(8) $$f[K_0] = K_0,$$

(9) $$g[K^0] = K^0,$$

(10) $$\mathrm{card}(K^0 \cap K_0) = 1,$$

where $U^0 := M^0 \cup f[K^0]$, $U_0 := M_0 \cup g[K_0]$, M^0 and M_0 are the strips bounded by K^0 and $f[K^0]$ and by K_0 and $g[K_0]$ (Fig. 1).

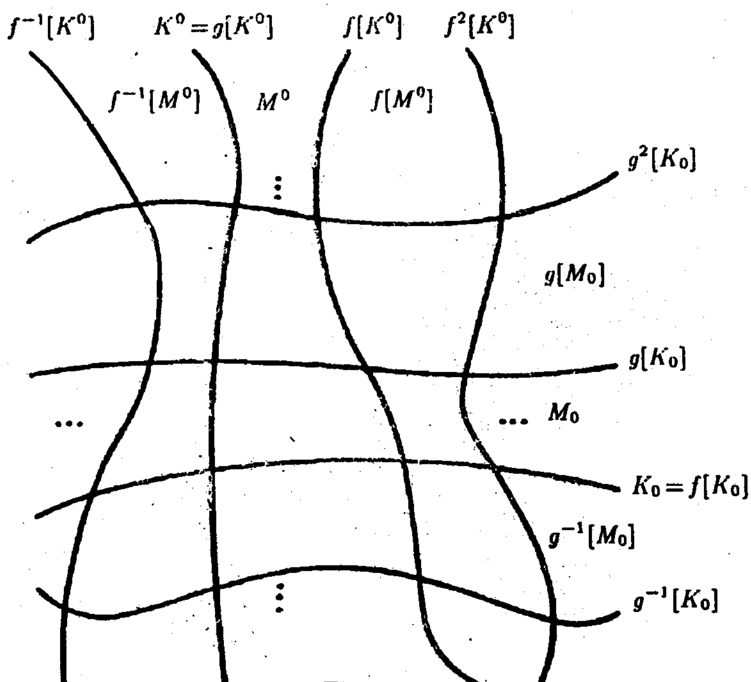

Fig. 1

Let us note that if f and g are orientation preserving homeomorphisms of the plane onto itself such that $f \circ g = g \circ f$ and satisfy condition (D), then they have no fixed points, and so they are free mappings.

1. First note the following

PROPOSITION 1. *Let $a = (a_1, a_2) \in \mathbf{R}^2$ and $b = (b_1, b_2) \in \mathbf{R}^2$ be linearly independent vectors (i.e. $a_1 b_2 - a_2 b_1 \neq 0$). Put $T_a(x) := x + a$ and $T_b(x) := x + b$ for $x \in \mathbf{R}^2$. Then there exists a homeomorphism ψ of the plane onto itself such that*

$$\begin{cases} T_1 = \psi^{-1} \circ T_a \circ \psi \\ T_2 = \psi^{-1} \circ T_b \circ \psi, \end{cases} \tag{11}$$

where

$$T_1(x_1, x_2) := (x_1, x_2) + (1, 0) \quad \text{for} \quad (x_1, x_2) \in \mathbf{R}^2, \tag{12}$$

and

$$T_2(x_1, x_2) := (x_1, x_2) + (0, 1) \quad \text{for} \quad (x_1, x_2) \in \mathbf{R}^2. \tag{13}$$

PROOF. It suffices to put

$$\psi(x_1, x_2) := (a_1 x_1 + b_1 x_2, a_2 x_1 + b_2 x_2) \quad \text{for} \quad (x_1, x_2) \in \mathbf{R}^2. \tag{14}$$

Then

$$\psi^{-1}(x_1, x_2) = \left(\frac{b_2}{a_1 b_2 - a_2 b_1} x_1 - \frac{b_1}{a_1 b_2 - a_2 b_1} x_2, -\frac{a_2}{a_1 b_2 - a_2 b_1} x_1 + \frac{a_1}{a_1 b_2 - a_2 b_1} x_2 \right)$$

for $(x_1, x_2) \in \mathbf{R}^2$. □

From now on we may assume that $a = (1, 0)$ and $b = (0, 1)$, since we have

PROPOSITION 2. *Let $a = (a_1, a_2) \in \mathbf{R}^2$ and $b = (b_1, b_2) \in \mathbf{R}^2$ be linearly independent vectors. Then φ is a solution of (1) if and only if it has the form*

$$\varphi = \psi \circ \varphi_0,$$

where φ_0 *is a solution of the system*

$$(15)\qquad \begin{cases} \varphi(f(x)) = \varphi(x) + (1,0) \\ \varphi(g(x)) = \varphi(x) + (0,1) \end{cases} \quad \text{for} \quad x \in \mathbb{R}^2,$$

and ψ *is given by (14).*

PROOF. If φ_0 is a solution of (15), then by (11)

$$\varphi_0(f(x)) = (\psi^{-1} \circ T_a \circ \psi)(\varphi_0(x)) \qquad \text{for} \quad x \in \mathbb{R}^2,$$

and

$$\varphi_0(g(x)) = (\psi^{-1} \circ T_b \circ \psi)(\varphi_0(x)) \qquad \text{for} \quad x \in \mathbb{R}^2,$$

where ψ is given by (14). Hence $\psi \circ \varphi_0$ is a solution of (1).

Conversely, if φ is a solution of (1), then $\varphi_0 := \psi^{-1} \circ \varphi$, where ψ is given by (14), satisfies (15). This completes the proof. □

Let us introduce the following condition:

(A) there exists a homeomorphism of the plane onto itself which satisfies system (15).

Now we shall show

PROPOSITION 3. *If* f *and* g *satisfy* (A), *then they are orientation preserving homeomorphisms of the plane onto itself such that* $f \circ g = g \circ f$ *and satisfy condition* (D).

PROOF. Let φ be a homeomorphism of the plane onto itself which is a solution of (15). Then

$$f = \varphi^{-1} \circ T_1 \circ \varphi$$

and

$$g = \varphi^{-1} \circ T_2 \circ \varphi,$$

where T_1 and T_2 are given by (12) and (13), resp. It is clear that f, g are homeomorphisms of the plane onto itself which preserve orientation. Since $T_1 \circ T_2 = T_2 \circ T_1$, we have $f \circ g = g \circ f$.

Let $L^0 := \{0\} \times \mathbb{R}$ and $L_0 := \mathbb{R} \times \{0\}$. Putting $K^0 := \varphi^{-1}[L^0]$ and $K_0 := \varphi^{-1}[L_0]$, we get condition (D). □

2. In this section we study continuous and homeomorphic solutions of system (15). By an *arc* we mean any continuous and one-to-one function γ defined on a compact segment of $\mathbb{R}$ taking its values from the plane. The set of values of the function is denoted by γ^*. Similarly, by a *Jordan curve* we mean any continuous and one-to-one function J of the unit circle into $\mathbb{R}^2$

and denote the set of its values by J^*. Let C be a homeomorphic image of a straight line. For all $a, b \in C$ denote by $[a,b]$ the set of values of an arc with endpoints a and b contained in C. Let $(a,b] := [a,b] \setminus \{a\}$.

L. E. J. Brouwer has proved the following

LEMMA 1. ([1]) *Let f be a free mapping. Let C be a homeomorphic image of a straight line such that $f[C] = C$. Then if $\gamma_1^* \cup \gamma_2^*$ is the set of values of a Jordan curve and for an $x_0 \in C$ the set $[x_0, f(x_0)] \subset C$ is a proper subset of γ_2^*, then $\gamma_1^* \cap f[\gamma_1^*] \neq \emptyset$.*

Using the lemma above we shall prove

LEMMA 2. *Let $F_{00}, G_{00} : [0,1] \to \mathbf{R}^2$ be arcs such that $F_{00}(0) = G_{00}(0)$, $F_{00}(1) = F_{00}(0) + (0,1)$ and $G_{00}(1) = G_{00}(0) + (1,0)$. Assume that $F_{00}^* \cup G_{00}^* \cup (F_{00}^* + (1,0)) \cup (G_{00}^* + (0,1))$ is the set of values of a Jordan curve J and $F_{00}^* \cap (F_{00}^* + (0,1)) = \{F_{00}(0) + (0,1)\}$ (or $G_{00}^* \cap (G_{00}^* + (1,0)) = \{F_{00}(0) + (1,0)\}$). Then*

(a) *$L^0 := \bigcup_{k \in \mathbf{Z}} (F_{00}^* + (0,k))$ is a line and $L^0 \cap (L^0 + (1,0)) = \emptyset$;*

(b) *$L_0 := \bigcup_{k \in \mathbf{Z}} (G_{00}^* + (k,0))$ is a line and $L_0 \cap (L_0 + (0,1)) = \emptyset$;*

(c) *$\{W_0^0 + (n,m) : n, m \in \mathbf{Z}\}$ is a family of pairwise disjoint sets such that*

$$\bigcup_{n,m \in \mathbf{Z}} W_0^0 + (n,m) = \mathbf{R}^2,$$

where $W_0^0 := B_0^0 \setminus (F_{00}^* \cup G_{00}^*)$ and B_0^0 is the sum of J^* and the inside of J^*.

PROOF. Put $p_{00} := F_{00}(0)$. Let $p_{kl} := p_{00} + (k,l)$, $F_{kl} := F_{00} + (k,l)$, $G_{kl} := G_{00} + (k,l)$ for all $k, l \in \mathbf{Z}$. Since $F_{00}^* \cap (F_{00}^* + (0,1)) = \{p_{01}\}$ and T_2 is a free mapping, the set $L^0 = \bigcup_{k \in \mathbf{Z}} F_{0k}^*$ is a homeomorphic image of a straight line (see [1]). It is easy to see that L^0 is a closed set. Thus L^0 is a line, and consequently so is $L^1 := L^0 + (1,0)$.

First we shall prove that $G_{00}^* \cap L^1 = \{p_{10}\}$. Let

$$s_1 := \min \{s \in [0,1] : G_{00}(s) \in L^1\}$$

(by the Weierstrass theorem the minimum exists). Put $\gamma_1 := G_{00}|_{[0,s_1]}$. Suppose, on the contrary, that $s_1 < 1$. Then $\gamma_1(s_1) \notin F_{10}^*$ (since $G_{00}^* \cap F_{10}^* = \{p_{10}\}$) and $\gamma_1(s_1) \notin F_{1,-1}^*$ (since $(G_{00}^* + (0,1)) \cap F_{10}^* = \{p_{11}\}$, and so $G_{00}^* \cap F_{1,-1}^* = \{p_{10}\}$). Thus either $\gamma_1(s_1) \in L^{1+}$, or $\gamma_1(s_1) \in L^{1-}$, where $L^{1+} := \bigcup_{k=1}^{+\infty} F_{1k}^*$ and $L^{1-} := \bigcup_{k=-2}^{-\infty} F_{1k}^*$.

In case $\gamma_1(s_1) \in L^{1+}$ we put $s_2 := \max\{s \in [s_1, 1] : G_{00}(s) \in L^{1+}\}$, $s_3 := \min\{s \in [s_2, 1] : G_{00}(s) \in L^{1-} \cup \{p_{10}\}\}$ and $\gamma_2 := G_{00}|_{[s_2,s_3]}$. Then the sum of γ_2^* and $[\gamma_2(s_2), \gamma_2(s_3)] \subset L^1$ is the set of values of a Jordan curve. Since $\gamma_2(s_2) \neq p_{11}$, the set $F_{10}^* = [p_{10}, p_{11}] \subset L^1$ is a proper subset of $[\gamma_2(s_2), \gamma_2(s_3)] \subset L^1$. From Lemma 1 (for T_2 in place of f) we get

$$\gamma_2^* \cap (\gamma_2^* + (0,1)) \neq \emptyset,$$

which contradicts the facts that $G_{00}^* \cap (G_{00}^* + (0,1)) = \emptyset$ and $\gamma_2^* \subset G_{00}^*$. The same arguments apply to the case where $\gamma_1(s_1) \in L^{1-}$. Thus $s_1 = 1$, whence $G_{00}^* \cap L^1 = \{p_{10}\}$.

Now we shall show that $G_{00}^* \cap L^0 = \{p_{00}\}$. Since J^* is the set of values of a Jordan curve, we have $G_{00}^* \cap ((F_{0,-1}^* \cup F_{00}^*) \setminus \{p_{00}\}) = \emptyset$. Suppose $G_{00}^* \cap (L^{0+} \cup L^{0-}) \neq \emptyset$, where $L^{0+} := \bigcup_{k=1}^{+\infty} F_{0k}^*$ and $L^{0-} := \bigcup_{k=-2}^{-\infty} F_{0k}^*$. Let $s_4 := \min\{s \in [0,1] : G_{00}(s) \in L^{0+} \cup L^{0-}\}$ (the minimum exists, since $L^{0+} \cup L^{0-}$ is a closed set). Put $\gamma_3 := G_{00}|_{[0,s_4]}$. Then by Lemma 1

$$\gamma_3^* \cap (\gamma_3^* + (0,1)) \neq \emptyset,$$

contrary to the fact that $G_{00}^* \cap (G_{00}^* + (0,1)) = \emptyset$. Thus $G_{00}^* \cap L^0 = \{p_{00}\}$. Since $L^0 + (0,1) = L^0$ and $L^1 + (0,1) = L^1$, we have $G_{01}^* \cap L^0 = \{p_{01}\}$ and $G_{01}^* \cap L^1 = \{p_{11}\}$.

Next we shall prove that $L^0 \cap L^1 = \emptyset$. Suppose, on the contrary, that $L^0 \cap L^1 \neq \emptyset$. Then $F_{00}^* \cap L^1 \neq \emptyset$. Let

$$t_1 := \min\{t \in [0,1] : F_{00}(t) \in L^1\}$$

(on account of the Weierstrass theorem the minimum exists). Since $G_{00}^* \cap L^1 = \{p_{10}\}$, we have $p_{00} \notin L^1$, whence $t_1 > 0$. Put $\gamma_4 := F_{00}|_{[0,t_1]}$. Obviously $\gamma_4(t_1) \notin F_{10}^*$. Thus one of the following three cases holds: $\gamma_4(t_1) \in L^{1+} \setminus \{p_{11}\}$, $\gamma_4(t_1) \in L^{1-} \setminus \{p_{1,-1}\}$, $\gamma_4(t_1) \in F_{1,-1}^* \setminus \{p_{10}\}$.

First suppose $\gamma_4(t_1) \in L^{1+} \setminus \{p_{11}\}$. Then the sum of γ_4^*, G_{00}^* and $[\gamma_4(t_1), p_{10}] \subset L^1$ is the set of values of a Jordan curve such that $F_{10}^* = [p_{11}, p_{10}]$ is a proper subset of $[\gamma_4(t_1), p_{10}] \subset L^1$. Hence by Lemma 1

$$(\gamma_4^* \cup G_{00}^*) \cap ((\gamma_4^* \cup G_{00}^*) + (0,1)) \neq \emptyset.$$

From the fact that $(G_{00}^* + (0,1)) \cap L^1 = \{p_{11}\}$ we get $p_{01} \notin L^1$, whence $\gamma_4(t_1) \neq p_{01}$. Hence $\gamma_4^* \cap (\gamma_4^* + (0,1)) = \emptyset$, since $F_{00}^* \cap (F_{00}^* + (0,1)) = \{p_{01}\}$. Moreover $\gamma_4^* \cap (G_{00}^* + (0,1)) = \emptyset$ (since $F_{00}^* \cap (G_{00}^* + (0,1)) = \{p_{01}\}$ and $p_{01} \notin \gamma_4^*$) and by assumptions $G_{00}^* \cap (G_{00}^* + (0,1)) = \emptyset$. Consequently $G_{00}^* \cap (\gamma_4^* + (0,1)) \neq \emptyset$ (Fig. 2). Let $s_5 := \min\{s \in [0,1] : G_{00}(s) \in$

$\gamma_4^* + (0,1)\}$ and $\gamma_5 := G_{00}|_{[0,s_5]}$. Then the sum of γ_5^* and $[p_{00}, \gamma_5(s_5)] \subset L^0$ is the set of values of a Jordan curve. Since $\gamma_5(s_5) \in F_{01}^* \setminus \{p_{01}\}$, the set $F_{00}^* = [p_{00}, p_{01}] \subset L^0$ is a proper subset of $[p_{00}, \gamma_5(s_5)]$. Hence by Lemma 1

$$\gamma_5^* \cap (\gamma_5^* + (0,1)) \neq \emptyset,$$

which is impossible, since $G_{00}^* \cap (G_{00}^* + (0,1)) = \emptyset$. Thus $\gamma_4(t_1) \notin L^{1+} \setminus \{p_{11}\}$. In the similar manner we can show that $\gamma_4(t_1) \notin L^{1-} \setminus \{p_{1,-1}\}$.

Consider the case where $\gamma_4(t_1) \in F_{1,-1}^* \setminus \{p_{10}\}$. We shall show that $F_{00}(t_2) \in L^{1-} \cup F_{1,-1}^*$, where $t_2 := \max\{t \in [0,1] : F_{00}(t) \in L^1\}$. Let $t_3 := \max\{t \in [0,1] : F_{00}(t) \in L^{1-} \cup F_{1,-1}^*\}$. Suppose, on the contrary, that $t_3 < t_2$. Put $t_4 := \min\{t \in [t_3, 1] : F_{00}(t) \in L^{1+}\}$ and $\gamma_6 := F_{00}|_{[t_3,t_4]}$.

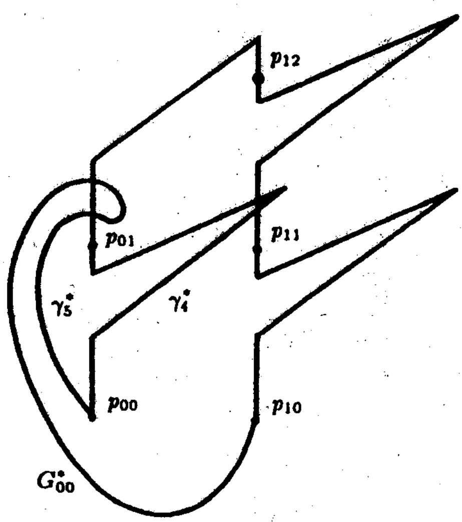

Fig. 2

Then by Lemma 1 $\gamma_6^* \cap (\gamma_6^* + (0,1)) \neq \emptyset$, which contradicts the fact that $(F_{00}^* \setminus \{p_{00}\}) \cap ((F_{00}^* \setminus \{p_{00}\}) + (0,1)) = \emptyset$, since $\gamma_6^* \subset F_{00}^* \setminus \{p_{00}\}$. Thus $t_3 = t_2$.

Put $\gamma_7 := F_{00}|_{[t_2,1]}$. Then the sum of γ_7^*, $G_{00}^* + (0,1)$ and $[\gamma_7(t_2), p_{11}] \subset L^1$ is the set of values of a Jordan curve (Fig. 3). On account of Lemma 1 we have

$$(\gamma_7^* \cup (G_{00}^* + (0,1))) \cap ((\gamma_7^* - (0,1)) \cup G_{00}^*) \neq \emptyset.$$

In the same manner as before we can show that this is impossible. Consequently none of the above three cases can hold. Thus $L^0 \cap L^1 = \emptyset$.

Denote by N^0 the strip bounded by L^0 and L^1. Let $W^0 := N^0 \cup L^1$. Put $W^k := W^0 + (k,0)$ for $k \in \mathbf{Z}$. Then, by the definition of L^0, $W^0 \cap W^k = \emptyset$ for $k \in \mathbf{Z} \setminus \{0\}$ and $\bigcup_{k\in\mathbf{Z}} W^k = \mathbb{R}^2$. Because $G^*_{00} \cap L^0 = \{p_{00}\}$, $G^*_{00} \cap (L^0 + (1,0)) = \{p_{10}\}$ and $W^0 \cap W^1 = \emptyset$, we have $G^*_{00} \cap (G^*_{00} + (1,0)) = \{p_{10}\}$. Hence $L_0 = \bigcup_{k\in\mathbf{Z}} G^*_{k0}$ is a line (the arguments are the same as that for L^0).

Let $L_1 := L_0 + (0,1)$. Note that $L_0 \cap L_1 = \emptyset$. Indeed, from the fact that $G^*_{00} \cap (G^*_{00} + (0,1)) = \emptyset$, we get $(G^*_{00} + (k,0)) \cap (G^*_{00} + (k,1)) = \emptyset$ for $k \in \mathbf{Z}$. Moreover $(G^*_{00} + (k,0)) \cap (G^*_{00} + (l,1)) = \emptyset$ for $k,l \in \mathbf{Z}$, $k \neq l$, since $W^k \cap W^l = \emptyset$, $p_{k0} \neq p_{l-1,1}$ and $p_{k-1,0} \neq p_{l1}$.

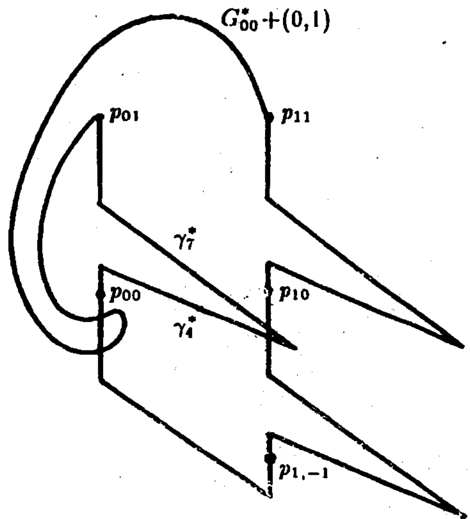

Fig. 3

Denote by N_0 the strip bounded by L_0 and L_1. Let $W_0 := N_0 \cup L_1$ and $W_k := W_0 + (0,k)$ for $k \in \mathbf{Z}$. Then $W_0 \cap W_k = \emptyset$ for $k \in \mathbf{Z} \setminus \{0\}$ and $\bigcup_{k\in\mathbf{Z}} W_k = \mathbb{R}^2$. Let $W^n_m := W^n \cap W_m$ for $n,m \in \mathbf{Z}$. Then

$$W^n = \bigcup_{m\in\mathbf{Z}} W^n_m \qquad \text{for} \quad n \in \mathbf{Z},$$

and

$$W_m = \bigcup_{n\in\mathbf{Z}} W^n_m \qquad \text{for} \quad m \in \mathbf{Z}.$$

Thus

$$\bigcup_{n,m\in\mathbf{Z}} W^n_m = \bigcup_{n\in\mathbf{Z}} W^n = \mathbb{R}^2.$$

Take any $k \in \mathbf{Z}$. Then $W_k^l \cap W_k^n = \emptyset$ for $l \neq n$, since $W_k^l \subset W^l$, $W_k^n \subset W^n$ and $W^l \cap W^n = \emptyset$. Hence by the fact that $W_k \cap W_m = \emptyset$ for $k \neq m$ we get $W_k^l \cap W_m^n = \emptyset$ for $(l,k) \neq (n,m)$. This completes the proof. □

From now on, let $U_0^0 := M_0^0 \cup F_0^1 \cup G_1^0$, $M_0^0 := M^0 \cap M_0$, $F_0^0 := (x_B, g(x_B)] \subset K^0$, $F_0^1 := (f(x_B), (g \circ f)(x_B)] \subset f[K^0]$, $G_0^0 := (x_B, f(x_B)] \subset K_0$, $G_1^0 := (g(x_B), (g \circ f)(x_B)] \subset g[K_0]$ and $x_B \in K^0 \cap K_0$ in the case where f and g are orientation preserving homeomorphisms of the plane onto itself such that $f \circ g = g \circ f$ and satisfy (D).

All of continuous and homeomorphic solutions of system (15) we get from

THEOREM 1. *Let f and g be orientation preserving homeomorphisms of the plane onto itself such that $f \circ g = g \circ f$ and satisfying (D). Let φ_0 be a continuous mapping defined on $U_0^0 \cup F_0^0 \cup G_0^0 \cup \{x_B\}$ such that*

$$\varphi_0(f(x)) = \varphi_0(x) + (1,0) \qquad \text{for} \quad x \in F_0^0 \cup \{x_B\}, \tag{16}$$

$$\varphi_0(g(x)) = \varphi_0(x) + (0,1) \qquad \text{for} \quad x \in G_0^0 \cup \{x_B\}. \tag{17}$$

Then

(a) *there exists exactly one function φ satisfying system (15) such that*

$$\varphi(x) = \varphi_0(x) \qquad \text{for} \quad U_0^0 \cup F_0^0 \cup G_0^0 \cup \{x_B\}. \tag{18}$$

This φ is continuous.

Moreover

(b) *if φ_0 is one-to-one and*

$$\varphi_0[\{x_B\} \cup F_0^0] \cap (\varphi_0[\{x_B\} \cup F_0^0] + (0,1)) = \{\varphi_0(x_B) + (0,1)\}$$

or

$$\varphi_0[\{x_B\} \cup G_0^0] \cap (\varphi_0[\{x_B\} \cup G_0^0] + (1,0)) = \{\varphi_0(x_B) + (1,0)\},$$

then φ is a homeomorphism of the plane onto itself (Fig. 4).

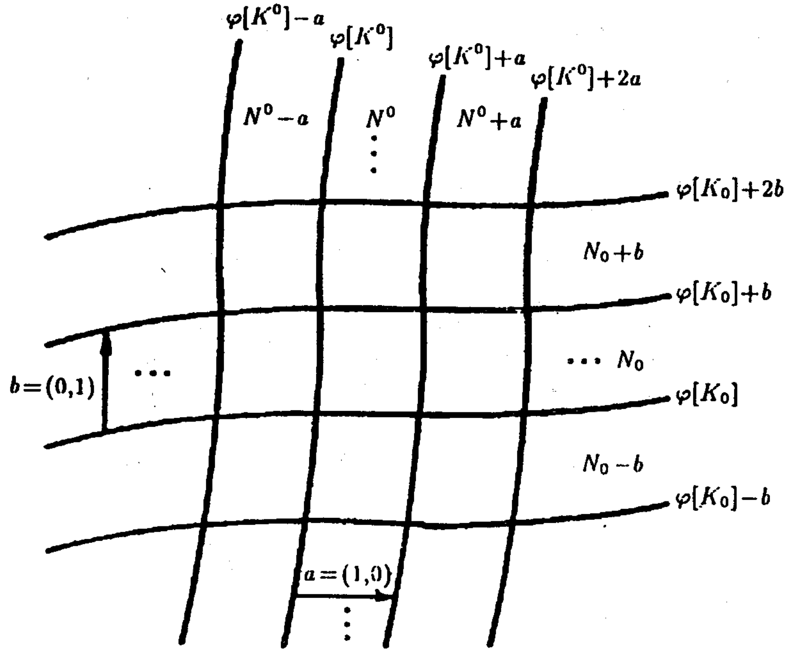

Fig. 4

Proof. From (2) and (3) it follows that

$$f^n[K^0] \cap f^{n+1}[K^0] = \emptyset \qquad \text{for} \quad n \in \mathbb{Z},$$

and

$$g^m[K_0] \cap g^{m+1}[K_0] = \emptyset \qquad \text{for} \quad m \in \mathbb{Z}.$$

Put $K^n := f^n[K^0]$, $K_n := g^n[K_0]$. Denote by M^n and M_n the strips bounded by K^n and K^{n+1}, and by K_n and K_{n+1}, resp. Let $U^n := M^n \cup K^{n+1}$ and $U_n := M_n \cup K_{n+1}$ for $n \in \mathbb{Z}$. Then $U^n = f^n[U^0]$ and $U_n = g^n[U_0]$ for $n \in \mathbb{Z}$, since f and g are homeomorphisms of the plane onto itself. From (4) and (5) we get $U^n \cap U^m = \emptyset$ and $U_n \cap U_m = \emptyset$ for $n, m \in \mathbb{Z}$, $n \neq m$. Since f and g are homeomorphisms of the plane onto itself, we have by (8) and (9)

$$(19) \qquad g[U^n] = U^n \qquad \text{for} \quad n \in \mathbb{Z},$$

and

$$f[U_n] = U_n \qquad \text{for} \quad n \in \mathbf{Z}. \tag{20}$$

Let

$$F_i^n := (g^i(f^n(x_B)), g^{i+1}(f^n(x_B))] \subset K^n \qquad \text{for} \quad n \in \mathbf{Z},\ i \in \mathbf{Z},$$

and

$$G_n^i := (f^i(g^n(x_B)), f^{i+1}(g^n(x_B))] \subset K_n \qquad \text{for} \quad n \in \mathbf{Z},\ i \in \mathbf{Z}.$$

Then

$$F_i^n = f^n[F_i^0], \qquad F_i^n = g^i[F_0^n]$$

and

$$G_n^i = g^n[G_0^i], \qquad G_n^i = f^i[G_n^0].$$

Moreover

$$K^n = \bigcup_{i \in \mathbf{Z}} F_i^n$$

and

$$K_n = \bigcup_{i \in \mathbf{Z}} G_n^i$$

for $n \in \mathbf{Z}$.

Let

$$M_m^n := M^n \cap M_m \qquad \text{for} \quad n, m \in \mathbf{Z}.$$

Put

$$U_m^n := M_m^n \cup F_m^{n+1} \cup G_{m+1}^n \qquad \text{for} \quad n, m \in \mathbf{Z}.$$

Then

$$(f^{n+1} \circ g^{m+1})(x_B) \in U_m^n \qquad \text{for} \quad n, m \in \mathbf{Z}.$$

It is easy to see that

$$U^n = \bigcup_{m \in \mathbf{Z}} U_m^n \qquad \text{for} \quad n \in \mathbf{Z}$$

and

$$U_m = \bigcup_{n \in \mathbf{Z}} U_m^n \qquad \text{for} \quad m \in \mathbf{Z}.$$

Hence

$$\bigcup_{n,m \in \mathbf{Z}} U_m^n = \mathbf{R}^2$$

and the sets U_m^n, for $n, m \in \mathbf{Z}$, are pairwise disjoint.

Define the function φ by the formula

(21) $$\varphi(x) := \varphi_0((f^{-n} \circ g^{-m})(x)) + (n, m), \qquad x \in U^n_m,\ n, m \in \mathbf{Z}.$$

It is clear that φ is a unique solution of system (15) satisfying (18) and that φ is continuous in $\bigcup_{n,m\in\mathbf{Z}} M^n_m$.

Now we shall show that φ is continuous in $G^0_0 \setminus \{f(x_B)\}$. Take any $x_0 \in G^0_0 \setminus \{f(x_B)\}$. Let R be an open disc with centre at x_0 such that

$$R \subset (G^0_0 \setminus \{f(x_B)\}) \cup M^0_0 \cup M^0_{-1}.$$

Put

$$R_1 := R \cap M^0_0, \qquad R_2 := R \cap M^0_{-1}, \qquad R_0 := R \cap G^0_0.$$

Then

$$\varphi(x) = \begin{cases} \varphi_0(x) & \text{for } x \in R_1, \\ \varphi_0(g(x)) - (0,1) & \text{for } x \in R_2 \cup R_0. \end{cases}$$

Hence by (21) $\varphi(x) = \varphi_0(x)$ for $x \in R_0$.

Let $x_k \to x_0$ as $k \to +\infty$, where $x_k \in R$. If $x_k \in R_1 \cup R_0$, then

$$\lim_{k\to+\infty} \varphi(x_k) = \lim_{k\to+\infty} \varphi_0(x_k) = \varphi_0(x_0) = \varphi(x_0),$$

since φ_0 is continuous in $R_1 \cup R_0 \subset G^0_0 \cup M^0_0$. If $x_k \in R_2$, then $g(x_k) \in M^0_0$ and $g(x_k) \to g(x_0) \in G^0_1$ as $k \to +\infty$. Then

$$\lim_{k\to+\infty} \varphi(x_k) = \lim_{k\to+\infty} (\varphi_0(g(x_k)) - (0,1)) = \varphi_0(g(x_0)) - (0,1) = \varphi(x_0),$$

since φ_0 is continuous and $x_0 \in G^0_0$. Consequently φ is continuous at $x_0 \in G^0_0 \setminus \{f(x_B)\}$. In the similar way we can show that φ is continuous in $F^0_0 \setminus \{g(x_B)\}$.

Next we shall prove that φ is continuous at x_B. Let R be an open disc with centre at x_B such that

$$\begin{aligned} R \subset & M^0_0 \cup M^0_{-1} \cup M^{-1}_0 \cup M^{-1}_{-1} \cup \\ & (G^0_0 \setminus \{f(x_B)\}) \cup (F^0_0 \setminus \{g(x_B)\}) \cup G^{-1}_0 \cup F^0_{-1}. \end{aligned}$$

Put $R_1 := R \cap M^0_0$, $R_2 := R \cap M^0_{-1}$, $R_3 := R \cap M^{-1}_0$, $R_4 := R \cap M^{-1}_{-1}$, $R_5 := R \cap G^0_0$, $R_6 := R \cap F^0_0$, $R_7 := R \cap G^{-1}_0$, $R_8 := R \cap F^0_{-1}$. Then

$$\varphi(x) = \begin{cases} \varphi_0(x) & \text{for } x \in R_1, \\ \varphi_0(g(x)) - (0,1) & \text{for } x \in R_2 \cup R_5, \\ \varphi_0(f(x)) - (1,0) & \text{for } x \in R_3 \cup R_6, \\ \varphi_0((f \circ g)(x)) - (1,1) & \text{for } x \in R_4 \cup R_7 \cup R_8, \end{cases}$$

since $R_1 \subset U_0^0$, $R_2 \cup R_5 \subset U_{-1}^0$, $R_3 \cup R_6 \subset U_0^{-1}$, $R_4 \cup R_7 \cup R_8 \subset U_{-1}^{-1}$. Hence by (21) $\varphi(x) = \varphi_0(x)$ for $x \in R_5 \cup R_6$.

Let $x_k \to x_B$ as $k \to +\infty$, where $x_k \in R$. Considering the four cases: $x_k \in R_1 \cup R_5 \cup R_6$, $x_k \in R_3$, $x_k \in R_2$, $x_k \in R_4 \cup R_7 \cup R_8$, we see that in each of these cases

$$\lim_{k\to+\infty} \varphi(x_k) = \varphi(x_B).$$

Thus φ is continuous at x_B.

Fix an arbitrary $x_0 \in \mathbb{R}^2 \setminus \bigcup_{n,m\in\mathbb{Z}} M_m^n$. Then there exist $n, m \in \mathbb{Z}$ such that $x_0 \in U_m^n \setminus M_m^n$. Hence, by the definition of U_m^n, one of the following three cases holds: $x_0 \in F_m^{n+1} \setminus \{(f^{n+1} \circ g^{m+1})(x_B)\}$, $x_0 \in G_{m+1}^n \setminus \{(f^{n+1} \circ g^{m+1})(x_B)\}$, $x_0 = (f^{n+1} \circ g^{m+1})(x_B)$.

Let $x_0 \in F_m^{n+1} \setminus \{(f^{n+1} \circ g^{m+1})(x_B)\}$ and let P be an open disc with centre at x_0 such that

$$P \subset (F_m^{n+1} \setminus \{(f^{n+1} \circ g^{m+1})(x_B)\}) \cup M_m^n \cup M_m^{n+1}.$$

Then

$$(f^{-n-1} \circ g^{-m})[P] \subset (F_0^0 \setminus \{g(x_B)\}) \cup M_0^{-1} \cup M_0^0$$

and $(f^{-n-1} \circ g^{-m})[P]$ is a neighbourhood of the point $(f^{-n-1} \circ g^{-m})(x_0) \in F_0^0 \setminus \{g(x_B)\}$. Since φ is continuous in $F_0^0 \setminus \{g(x_B)\}$, it is continuous at x_0. Similar arguments apply to the cases $x_0 \in G_{m+1}^n \setminus \{(f^{n+1} \circ g^{m+1})(x_B)\}$ and $x_0 = (f^{n+1} \circ g^{m+1})(x_B)$. Consequently φ is continuous on the whole plane.

Assume, in addition, that φ_0 is one-to-one and

$$\varphi_0[\{x_B\} \cup F_0^0] \cap (\varphi_0[\{x_B\} \cup F_0^0] + (0,1)) = \{\varphi_0(x_B) + (0,1)\}.$$

Then φ_0 is a homeomorphism, since $U_0^0 \cup F_0^0 \cup G_0^0 \cup \{x_B\}$ is compact. Thus $J^* := \varphi_0[F_0^0 \cup G_0^0 \cup F_0^1 \cup G_1^0 \cup \{x_B\}]$ is the set of values of a Jordan curve.

From assertions (a) and (b) of Lemma 2 we obtain that $L^0 := \bigcup_{k\in\mathbb{Z}}(\varphi_0[\{x_B\} \cup F_0^0] + (0,k))$, $L_0 := \bigcup_{k\in\mathbb{Z}}(\varphi_0[\{x_B\} \cup G_0^0] + (k,0))$ are lines and $L^0 \cap (L^0 + (1,0)) = \emptyset$, $L_0 \cap (L_0 + (0,1)) = \emptyset$.

Denote by N^0 the strip bounded by L^0 and $L^0 + (1,0)$. Then $\varphi_0[U^0] \cap (\varphi_0[U^0] + (k,0)) = \emptyset$ for every $k \in \mathbb{Z} \setminus \{0\}$, since $\varphi_0[U^0] \subset W^0$, where $W^0 := N^0 \cup (L^0 + (1,0))$ (see [5], the proof of Theorem 2, part (c)). Likewise, we get that $\varphi_0[U_0] \cap (\varphi_0[U_0] + (0,k)) = \emptyset$ for every $k \in \mathbb{Z} \setminus \{0\}$, since $\varphi_0[U_0] \subset W_0$, where $W_0 := N_0 \cup (L_0 + (0,1))$ and N_0 denotes the strip bounded by L_0 and $L_0 + (0,1)$.

We shall show that φ is one-to-one. Let $x, y \in \mathbb{R}^2$ and $\varphi(x) = \varphi(y)$. Then there exist $k, l, m, n \in \mathbb{Z}$ such that $x \in U_k^l$, $y \in U_m^n$. Hence $x \in U^l$, $y \in U^n$. From the fact that $\varphi_0[U^0] \cap (\varphi_0[U^0] + (l,0)) = \emptyset$ for every $l \in \mathbb{Z} \setminus \{0\}$ it follows that $l = n$ (see [5], the proof of Theorem 2, part (b)). On the other hand $x \in U_k$, $y \in U_m$. Applying the same method as above we get $k = m$ (now we use the fact that $\varphi_0[U_0] \cap (\varphi_0[U_0] + (0,k)) = \emptyset$ for every $k \in \mathbf{Z} \setminus \{0\}$). Therefore $x = y$, since φ_0 is one-to-one. Note that φ, being a continuous one-to-one mapping of the plane into itself, is a homeomorphism (see e.g. [3], p. 186).

It remains to prove that $\varphi[\mathbf{R}^2] = \mathbf{R}^2$. Let

$$A := \bigcup_{n \in \mathbf{Z}} L^0 + (n, 0) \cup \bigcup_{n \in \mathbf{Z}} L_0 + (0, n).$$

Then $A \subset \varphi[\mathbf{R}^2]$. Take any $x_0 \in \mathbf{R}^2 \setminus A$. Put $N_0^0 := N^0 \cap N_0$ and $W_0^0 := W^0 \cap W_0$. Then $W_0^0 = N_0^0 \cup (\varphi_0[F_0^0] + (1,0)) \cup (\varphi_0[G_0^0] + (0,1))$. Let $N_m^n := N_0^0 + (n, m)$ and $W_m^n := W_0^0 + (n, m)$ for $n, m \in \mathbf{Z}$. Then by assertion (c) of Lemma 2

$$W_k^l \cap W_m^n = \emptyset \qquad \text{for} \quad (n, m) \neq (l, k),$$

and

$$\bigcup_{n,m \in \mathbf{Z}} W_m^n = \mathbf{R}^2.$$

Therefore there exist $n, m \in \mathbf{Z}$ such that $x_0 \in N_m^n$, since $x_0 \notin A$. Note that $(J^* + (n, m))$ is the set of values of a Jordan curve and $N_m^n = \operatorname{ins}(J^* + (n, m))$. Hence $x_0 \in \varphi[\mathbf{R}^2]$, since $J^* + (n, m) \subset \varphi[\mathbf{R}^2]$ and $\varphi[\mathbf{R}^2]$ is a simply connected region (i.e. for every Jordan curve γ such that $\gamma^* \subset \varphi[\mathbf{R}^2]$ we have $\operatorname{ins} \gamma^* \subset \varphi[\mathbf{R}^2]$). This completes the proof. □

From Theorem 1 (b) - by the Schönflies theorem (see e.g. [2], p. 370) - we obtain

COROLLARY 1. *If f and g are orientation preserving homeomorphisms of the plane onto itself such that $f \circ g = g \circ f$, then (D) implies (A).*

As a consequence of Proposition 3 and Corollary 1 we get

COROLLARY 2. *Let f and g be orientation preserving homeomorphisms of the plane onto itself such that $f \circ g = g \circ f$. Then conditions (A) and (D) are equivalent.*

3. Now we proceed to the construction of solutions of system (15) which are of class C^p $(p > 0)$. First we quote the following

LEMMA 3. ([4]) *If the functions h and ψ are of class C^p $(p > 0)$ in a region $V \subset \mathbf{R}^2$ such that $h[V] \subset V$, then for $x \in V$*

$$\frac{\partial^q}{\partial x_{i_1} \ldots \partial x_{i_q}} \psi[h(x)] = \sum_{k=1}^{q} \sum_{j_1, \ldots, j_k = 1}^{2} b_{i_1 \ldots i_q}^{j_1 \ldots j_k}(x)\, \psi_{j_1 \ldots j_k}[h(x)],$$

$q = 1, \ldots, p$, where

$$\psi_{i_1 \ldots i_k}(x) = \frac{\partial^k}{\partial x_{i_1} \ldots \partial x_{i_k}} \psi(x), \tag{22}$$

$b^{j_1 \ldots j_k}_{i_1 \ldots i_q}(x)$ may be expressed by means of sums and products of $a^j_i(x), \ldots,$ $a^j_{i_1 \ldots i_{q-k+1}}(x), a^j_{i_1 \ldots i_k}(x) = \frac{\partial^k}{\partial x_{i_1} \ldots \partial x_{i_k}} h_j(x)$, $k = 1, \ldots, p$ and $h = (h_1, h_2)$. Consequently $b^{j_1 \ldots j_k}_{i_1 \ldots i_q}$ are of class $C^{p-q+k-1}$. In particular,

$$b^{j_1 \ldots j_q}_{i_1 \ldots i_q}(x) = a^{j_1}_{i_1}(x) \cdot \ldots \cdot a^{j_q}_{i_q}(x).$$

Let f and g be orientation preserving homeomorphisms of the plane onto itself such that $f \circ g = g \circ f$ and satisfying (D). Let ψ be a continuous function defined on $U^0_0 \cup F^0_0 \cup G^0_0 \cup \{x_B\}$, p times continuously differentiable in M^0_0. We write

$$\psi_{i_1 \ldots i_k}(x_0) := \lim_{\substack{x \to x_0 \\ x \in M^0_0}} \frac{\partial^k}{\partial x_{i_1} \ldots \partial x_{i_k}} \psi(x), \quad k = 1, \ldots, p \tag{23}$$

for $x_0 \in F^0_0 \cup G^0_0 \cup F^1_0 \cup G^0_1 \cup \{x_B\}$ (provided this limit exists), and for $x \in M^0_0$ the function $\psi_{i_1 \ldots i_k}$ is given by (22). The function ψ is said to be of *class* C^p in $U^0_0 \cup F^0_0 \cup G^0_0 \cup \{x_B\}$, if all the functions $\psi, \psi_i, \ldots, \psi_{i_1 \ldots i_p}$ are continuous in this set.

We have the following

THEOREM 2. *Let f and g be orientation preserving C^p diffeomorphisms mapping $\mathbb{R}^2$ onto itself such that $f \circ g = g \circ f$ and satisfying (D). Assume that for every $x_1 \in F^0_0 \setminus \{f(x_B)\}$, for every $x_2 \in G^0_0 \setminus \{g(x_B)\}$ and for the point $x_B \in K^0 \cap K_0$ there exist three pairs of linearly independent vectors $u^1_{x_1}$ and $u^2_{x_1}$, $u^1_{x_2}$ and $u^2_{x_2}$, $u^1_{x_B}$ and $u^2_{x_B}$ and there exist constants $t^0_{x_1}, t^0_{x_2}, t^0_{x_B} > 0$ such that each of the sets $I^1_{x_1} \cap (F^0_0 \cup \{x_B\})$, $I^2_{x_1} \cap (F^0_0 \cup \{x_B\})$, $I^1_{x_2} \cap (G^0_0 \cup \{x_B\})$, $I^2_{x_2} \cap (G^0_0 \cup \{x_B\})$, $I^1_{x_B} \cap (F^0_0 \cup G^0_0 \cup \{x_B\})$, $I^2_{x_B} \cap (F^0_0 \cup G^0_0 \cup \{x_B\})$ is at most denumerable, where $I^1_x := \{x + tu^1_x : |t| \le t^0_x\}$, $I^2_x := \{x + tu^2_x : |t| \le t^0_x\}$. Let ψ be a C^p function from $U^0_0 \cup F^0_0 \cup G^0_0 \cup \{x_B\}$ into $\mathbb{R}^2$ which satisfies*

$$\psi[f(x)] = \psi(x) + (1, 0) \qquad \text{for} \quad x \in F^0_0 \cup \{x_B\},$$

$$\psi[g(x)] = \psi(x) + (0, 1) \qquad \text{for} \quad x \in G^0_0 \cup \{x_B\},$$

and for $q = 1, \dots, p$, $i_1, \dots, i_q = 1, 2$

$$\sum_{k=1}^{q} \sum_{j_1 \dots j_k = 1}^{2} b_{i_1 \dots i_q}^{j_1 \dots j_k}(x) \psi_{j_1 \dots j_k}[f(x)] = \psi_{i_1 \dots i_q}(x) \qquad \text{for} \quad x \in F_0^0 \cup \{x_B\},$$

$$\sum_{k=1}^{q} \sum_{j_1 \dots j_k = 1}^{2} \bar{b}_{i_1 \dots i_q}^{j_1 \dots j_k}(x) \psi_{j_1 \dots j_k}[g(x)] = \psi_{i_1 \dots i_q}(x) \qquad \text{for} \quad x \in G_0^0 \cup \{x_B\},$$

where the functions $b_{i_1 \dots i_q}^{j_1 \dots j_k}$, $\bar{b}_{i_1 \dots i_q}^{j_1 \dots j_k}$ *are those occuring in Lemma 3 for* $h = f$, $h = g$, *resp. Then there exists a unique solution* φ *of system (15) such that*

$$\varphi(x) = \psi(x) \qquad \text{for} \quad x \in U_0^0 \cup F_0^0 \cup G_0^0 \cup \{x_B\}.$$

This solution is of class C^p *in the plane.*

PROOF. Define φ by setting

$$\varphi(x) := \psi((f^{-n} \circ g^{-m})(x)) + (n, m) \qquad \text{for} \quad x \in U_m^n, \ n, m \in \mathbf{Z}.$$

From Theorem 1 we get that φ is continuous in $\mathbf{R}^2$. The fact that φ is of class C^p in the plane can be obtained in the same way as that of Theorem 3.1 in [4], part 2 (with the partial derivatives replaced by directional derivatives in the directions of the linearly independent vectors which occurs in our assumptions). This completes the proof. □

In particular, from Theorem 2 we obtain the existence of a C^p solution of system (15) provided the desired pairs of vectors exist (the main theorem of [7] yields the existence of ψ satisfying the assumptions of Theorem 2).

4. The last section deals with families of homeomorphic images of a straight line which fill the plane. Let us introduce the following conditions:

(E) there exist lines K^0, K_0 and families of homeomorphic images of a straight line $\{C_\alpha : \alpha \in I_1\}$, $\{C^\alpha : \alpha \in I_2\}$ satisfying (8), (9), (10) and

(24) $$f[C_\alpha] = C_\alpha \qquad \text{for} \quad \alpha \in I_1,$$

(25) $$C_\alpha \cap C_\beta = \emptyset \qquad \text{for} \quad \alpha, \beta \in I_1, \alpha \neq \beta,$$

(26) $$\text{card}(K^0 \cap C_\alpha) = 1 \qquad \text{for} \quad \alpha \in I_1,$$

(27) $$\bigcup_{\alpha \in I_1} C_\alpha = \mathbb{R}^2;$$

(28) $$g[C^\alpha] = C^\alpha \qquad \text{for} \quad \alpha \in I_2,$$

(29) $$C^\alpha \cap C^\beta = \emptyset \qquad \text{for} \quad \alpha, \beta \in I_2, \alpha \neq \beta,$$

(30) $$\text{card}\,(K_0 \cap C^\alpha) = 1 \qquad \text{for} \quad \alpha \in I_2,$$

(31) $$\bigcup_{\alpha \in I_2} C^\alpha = \mathbb{R}^2;$$

(E') there exist families of lines $\{C_\alpha : \alpha \in I_1\}$, $\{C^\alpha : \alpha \in I_2\}$ satisfying (24), (25), (27), (28), (29), (31) and

(32) $$\text{card}(C_\alpha \cap C^\beta) = 1 \qquad \text{for} \quad \alpha \in I_1, \beta \in I_2.$$

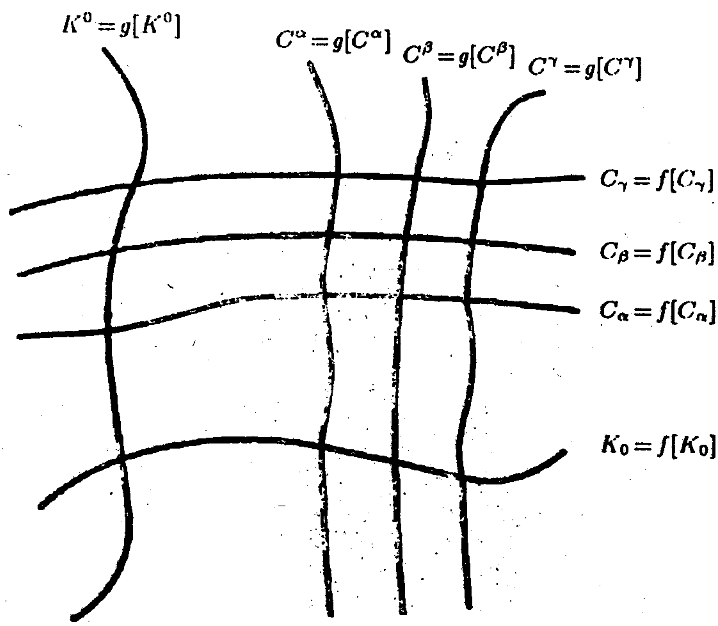

Fig. 5

The situation described in (E) is presented in Fig. 5.

PROPOSITION 4. *If f and g satisfy (A), then they satisfy (E').*

PROOF. Let φ be a homeomorphism of the plane onto itself such that $\varphi \circ f = T_1 \circ \varphi$ and $\varphi \circ g = T_2 \circ \varphi$. Put $D_\alpha := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = \alpha\}$ and $D^\alpha := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = \alpha\}$ for every $\alpha \in \mathbb{R}$. Let $C_\alpha := \varphi^{-1}[D_\alpha]$ and $C^\alpha := \varphi^{-1}[D^\alpha]$. It is easy to see that condition (E') is satisfied. □

We also have the following

PROPOSITION 5. *Let f and g be homeomorphisms of the plane onto itself without fixed points such that $f \circ g = g \circ f$. Then (E) implies (D).*

PROOF. Let K^0, K_0 be lines and let $\{C_\alpha : \alpha \in I_1\}$, $\{C^\alpha : \alpha \in I_2\}$ be families of homeomorphic images of a straight line such that condition (E) holds. Then K^0 satisfies (2), (4) and (6) (see [5], Theorem 3 and Corollary 3). In the same manner we can see that K_0 satisfies (3), (5) and (7). This completes the proof. □

From Propositions 4 and 5, and Corollary 2 we get

PROPOSITION 6. *Let f and g be free mappings such that $f \circ g = g \circ f$. Then conditions (A), (D), (E) and (E') are equivalent.*

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