Annales Mathematicae Silesianae 9. Katowice 1995, 123–135

Prace Naukowe Uniwersytetu Śląskiego nr 1523

# ON A SYSTEM OF SIMULTANEOUS ITERATIVE FUNCTIONAL EQUATIONS

JANUSZ MATKOWSKI

**Abstract.** A system of two simultaneous functional equations in a single variable, related to a generalized Gołąb–Schinzel functional equation, is considered.

**Introduction.** The Gołąb–Schinzel type functional equation

$$f(x+yf(x)^p)=f(x)f(y),$$

where $p$ is a fixed integer number, was studied in [3] (cf. also [4] where more general equation was considered). Suppose that a function $f:\mathbb{R}\to\mathbb{R}$ is a solution of this equation. Setting here $x=\alpha$ and next $x=\beta$ gives the system of two simultaneous Schröder functional equations

$$f(a^px+\alpha)=af(x),\qquad f(b^px+\beta)=bf(x),$$

which may be interpreted as a Gołąb–Schinzel type equation on a restricted domain. In the present note we examine a little more general system

$$(*)\qquad f(Ax+\alpha)=af(x),\qquad f(Bx+\beta)=bf(x).$$

We show that, in the case when $A\beta+\alpha\neq B\alpha+\beta$, under some modest regularity assumptions, there are only constant solutions. Therefore, the main results are concerned with the case $A\beta+\alpha=B\alpha+\beta$. It turns out that, in this case, if $\log A$ and $\log B$ are not commensurable, and system $(*)$ has a nontrivial continuous solution, then there exists a real $p\neq 0$, such that

$$A=a^p,\qquad B=b^p.$$

AMS (1991) Subject Classification: Primary 39B62.

The main results give the general form of solutions which are continuous at a point or Lebesgue measurable.

The Lebesgue measurable solutions of $(*)$ with $A = B = 1$ was considered by W.E. Clark and A. Mukherjea [2]. The continuous (at least at one point) solutions of the system of functional equations

$$f(x+a) = f(x) + \alpha, \qquad f(x+b) = f(x) + \beta,$$

was considered by the present author in [9] (cf. also M. Kuczma, B. Choczewski and R. Ger [7], §§ 9.5, 9.6.6 and 6.1).

**1. Some auxiliary results.** Denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, respectively, the set of positive integers, integers, and rational numbers.

LEMMA 1. *Let $\alpha$, $\beta$, $a$, $b$, $A$, $B$; $A \neq 0 \neq B$, be fixed real numbers. Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies the system of functional equations*

$$f(Ax+\alpha) = af(x), \qquad f(Bx+\beta) = bf(x), \qquad x \in \mathbb{R}. \tag{1}$$

$1^0$. *If $A\beta + \alpha \neq B\alpha + \beta$ then $f$ is periodic, and for every $n, m \in \mathbb{N}$,*

$$p_{n,m} := \beta(1+B+\ldots+B^{m-1})(A^n-1) + \alpha(1+A+\ldots+A^{n-1})(1-B^m) \tag{2}$$

*is a period of $f$.*

$2^0$. *If $A\beta + \alpha = B\alpha + \beta$ and $A \neq 1$ then*

$$f\left(A^n B^m \left(x - \frac{\alpha}{1-A}\right) + \frac{\alpha}{1-A}\right) = a^n b^m f(x), \qquad x \in \mathbb{R}, \ n, m \in \mathbb{Z}.$$

PROOF. From (1), by induction,

$$f\left(A^n x + \alpha(1+A+\ldots+A^{n-1})\right) = a^n f(x),$$

$$f\left(B^m x + \beta(1+B+\ldots+B^{m-1})\right) = b^m f(x),$$

for all $x \in \mathbb{R}$ and $n, m \in \mathbb{N}$. Hence, replacing $x$ by $B^m x + \beta(1+B+\ldots+B^{m-1})$ in the first of these equations, we get

$$f\left(A^n B^m x + \beta A^n(1+B+\ldots+B^{m-1}) + \alpha(1+A+\ldots+A^{n-1})\right) = a^n b^m f(x),$$

for all $x \in \mathbb{R}$ and $n, m \in \mathbb{N}$. In the same way, replacing $x$ by $A^n x + \alpha(1 + A + \ldots + A^{n-1})$ in the second equation gives

$$f\left(A^n B^m x + \alpha B^m (1 + A + \ldots + A^{n-1}) + \beta(1 + B + \ldots + B^{m-1})\right) = a^n b^m f(x),$$

for all $x \in \mathbb{R}$ and $n, m \in \mathbb{N}$. Comparing the left-hand sides of the above two formulas with $x$ replaced by $A^{-n}B^{-m}x$, we immediately get

$$f(x + p_{n,m}) = f(x), \qquad x \in \mathbb{R}, \ n, m \in \mathbb{N}.$$

Since $p_{1,1} = \beta(A-1) + \alpha(1-B) = (A\beta + \alpha) - (B\alpha + \beta) \neq 0$, the function $f$ is periodic. This proves $1^0$.

To prove $2^0$ note that

$$\beta = \alpha \frac{B-1}{A-1}.$$

Hence, applying the first formula of the previous part of the proof, we get

$$\begin{aligned}
&a^n b^m f(x) \\
&= f\left(A^n B^m x + \beta A^n (1 + B + \ldots + B^{m-1}) + \alpha(1 + A + \ldots + A^{n-1})\right) \\
&= f\left(A^n B^m x + \alpha \frac{B-1}{A-1} A^n (1 + B + \ldots + B^{m-1}) + \alpha(1 + A + \ldots + A^{n-1})\right) \\
&= f\left(A^n B^m x + \frac{\alpha}{A-1}(A^n B^m - 1)\right)
\end{aligned}$$

for all $x \in \mathbb{R}$ and $n, m \in \mathbb{N}$. It is easy to check that this formula is also true for all $n, m \geq 0$, $n, m \in \mathbb{Z}$. Taking $n = 0$ we obtain

$$f\left(B^m x + \frac{\alpha}{A-1}(B^m - 1)\right) = b^m f(x), \qquad x \in \mathbb{R}, \ m \in \mathbb{Z}, \ m \geq 0.$$

Replacing here $x$ by $B^{-m}[x - \frac{\alpha}{A-1}(B^m - 1)]$ gives

$$f\left(B^{-m} x + \frac{\alpha}{A-1}(B^{-m} - 1)\right) = b^{-m} f(x), \qquad x \in \mathbb{R}, \ m \in \mathbb{N}.$$

Thus we have shown that

$$f\left(B^m x + \frac{\alpha}{A-1}(B^m - 1)\right) = b^m f(x), \qquad x \in \mathbb{R}, \ m \in \mathbb{Z}.$$

In the same way we prove that

$$f\left(A^n x + \frac{\alpha}{A-1}(A^n - 1)\right) = a^n f(x), \qquad x \in \mathbb{R},\ n \in \mathbb{Z}.$$

Take now arbitrary $n, m \in \mathbb{Z}$. Applying the last two formulas we have

$$a^n b^m f(x) = a^n\left(b^m f(x)\right) = a^n f\left(B^m x + \frac{\alpha}{A-1}(B^m - 1)\right)$$
$$= f\left(A^n[B^m x + \frac{\alpha}{A-1}(B^m - 1)] + \frac{\alpha}{A-1}(A^n - 1)\right)$$
$$= f\left(A^n B^m x + \frac{\alpha}{A-1}(A^n B^m - 1)\right) = f\left(A^n B^m\left(x - \frac{\alpha}{1-A}\right) + \frac{\alpha}{1-A}\right)$$

for all $x \in \mathbb{R}$, which completes the proof. □

A function $f : \mathbb{R} \to \mathbb{R}$ is called microperiodic if it has arbitrarily small positive periods. In the sequel we need also the following result due to A. Lomnicki [8] (for short proofs cf. R. Ger, Z. Kominek and M. Sablik [5], and M. Kuczma [6]).

LEMMA 2. *Every Lebesgue measurable microperiodic function $f : \mathbb{R} \to \mathbb{R}$ is constant almost everywhere.*

**2. Main results.** We begin this section with the following

PROPOSITION 1. *Let $\alpha$, $\beta$, $a$, $b$, $A$, $B$; $A \neq 0 \neq B$, be fixed real numbers such that*

$$A\beta + \alpha \neq B\alpha + \beta,$$

*and*

$$\text{(3)}\quad \inf\{up_{k,l} + vp_{n,m} :\ up_{k,l} + vp_{n,m} > 0;\ k, l, m, n \in \mathbb{N};\ u, v \in \mathbb{Z}\} = 0,$$

*where the numbers $p_{n,m}$ are defined by (2). Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies the system of functional equations*

$$f(Ax + \alpha) = af(x), \qquad f(Bx + \beta) = bf(x), \qquad x \in \mathbb{R}.$$

$1^0$. *If $f$ is continuous at least at one point then $f$ is constant. Moreover, if $a \neq 1$ or $b \neq 1$ then $f \equiv 0$.*

$2^0$. *If $f$ is Lebesgue measurable then $f$ is constant almost everywhere in $\mathbb{R}$. Moreover, if $a \neq 1$ or $b \neq 1$ then $f = 0$ almost everywhere.*

PROOF. Put

$$\mathbf{D} := \{up_{k,l} + vp_{n,m} : \ k,l,m,n \in \mathbf{N}; \ u,v \in \mathbf{Z}\}.$$

According to Lemma 1.1[0] we have

$$f(x + p_{n,m}) = f(x), \qquad x \in \mathbf{R}.$$

It follows that $f(x+p) = f(x)$ for all $p \in \mathbf{D}$ and $x \in \mathbf{R}$. By (3) the set $\mathbf{D}$ is dense in $\mathbf{R}$, and consequently $f$ is microperiodic. The continuity of $f$ at least at one point implies that $f$ is continuous everywhere and, of course, $f$ must be constant. The part $2^0$ is a consequence of Lemma 2. □

REMARK 1. Note that the condition (3) is satisfied if for some $k,l,m,n \in \mathbf{N}$ the numbers $p_{k,l}$ and $p_{n,m}$ are not commensurable.

The above proposition shows that the case $A\beta + \alpha \neq B\alpha + \beta$ is not very interesting. Therefore in the sequel we assume that

$$A\beta + \alpha = B\alpha + \beta.$$

REMARK 2. Suppose that $A \neq 1 \neq B$. Then the numbers $\alpha/(1-A)$ and $\beta/(1-B)$ are, respectively, the unique fixed points of the functions $g_1, g_2 : \mathbf{R} \to \mathbf{R}$, $g_1(x) := Ax + \alpha$ and $g_2(x) = Bx + \beta$. Since the condition $A\beta + \alpha = B\alpha + \beta$ can be written in the form

$$\frac{\alpha}{1-A} = \frac{\beta}{1-B},$$

it means that $\xi := \alpha/(1-A)$ is a common fixed point of these functions. If moreover $A$ and $B$ are positive then

$$g_i\left((\xi,\infty)\right) = (\xi,\infty) \quad and \quad g_i\left((-\infty,\xi)\right) = (-\infty,\xi), \quad i = 1,2.$$

It follows that for every function $f : (\xi,\infty) \to \mathbf{R}$ satisfying $(*)$ for all $x \in (\xi,\infty)$, the counterpart of Lemma 1.2[0] remains true.

REMARK 3. To obtain another interpretation of the condition $A\beta + \alpha = B\alpha + \beta$ suppose that there exists a bijective solution of $(*)$. Then the inverse function $f^{-1}$ satisfies the functional equations

$$Af^{-1}(x) + \alpha = f^{-1}(ax), \qquad Bf^{-1}(x) + \beta = f^{-1}(bx), \qquad x \in \mathbf{R}.$$

Setting here $x = 0$ we get $Af^{-1}(0) + \alpha = f^{-1}(0)$ and $Bf^{-1}(0) + \beta = f^{-1}(0)$ which implies that $\alpha/(1-A) = f^{-1}(0) = \beta/(1-B)$ must be a common fixed point of the linear functions mentioned in Remark 2.

Note also that if system $(*)$ has a nontrivial solution satisfying a modest regularity condition, then the numbers $A$, $B$, $a$, and $b$ are dependent. In fact, we have the following

THEOREM 1. *Let $\alpha, \beta \in \mathbb{R}$ and $a, b, A, B \in (0, \infty)$, $A \neq 1 \neq B$, be such that*

$$\frac{\log A}{\log B} \notin \mathbb{Q}, \qquad \frac{\alpha}{1-A} = \frac{\beta}{1-B}.$$

*Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies the system of equations*

$$f(Ax + \alpha) = af(x), \qquad f(Bx + \beta) = bf(x), \qquad x \in \mathbb{R}.$$

*If $\log \circ |f|$ is bounded on a neighbourhood of a point then there exists $p \in \mathbb{R}$, $p \neq 0$, such that*

$$A = a^p, \qquad B = b^p.$$

PROOF. By assumption there exist $x_0 \in \mathbb{R}$, $\delta > 0$, and $M > 0$ such that

$$-M \leq \log | f(x) | \leq M, \qquad x \in (x_0 - \delta, x_0 + \delta).$$

Since $\log A$ and $\log B$ are not commensurable, in view of Kronecker theorem, the set

$$\{n \cdot \log A + m \cdot \log B : \ n, m \in \mathbb{Z}\}$$

is dense in $\mathbb{R}$. It follows that there exist sequences $n_k, m_k \in \mathbb{Z} \setminus \{0\}$, $k \in \mathbb{N}$, such that

$$\lim_{k\to\infty} (n_k \log A + m_k \log B) = 0.$$

Consequently,

$$\lim_{k\to\infty} \frac{m_k}{n_k} = -\frac{\log A}{\log B},$$

and

$$\lim_{k\to\infty} A^{n_k} B^{m_k} = 1.$$

From Lemma 1.2$^0$ we have

$$\left| f\left(A^{n_k} B^{m_k} (x_0 - \frac{\alpha}{1-A}) + \frac{\alpha}{1-A}\right) \right| = a^{n_k} b^{m_k} \ | f(x_0) |, \qquad k \in \mathbb{N}.$$

Since

$$\lim_{k\to\infty}\left(A^{n_k}B^{m_k}\left(x_0-\frac{\alpha}{1-A}\right)+\frac{\alpha}{1-A}\right)=x_0,$$

we infer that there is a $k_0 \in \mathbf{N}$ such that

$$-M \le \log\left(a^{n_k}b^{m_k} \mid f(x_0) \mid\right) \le M, \qquad k \ge k_0,$$

what can be written in the form

$$-M-\log \mid f(x_0) \mid \le n_k \log a + m_k \log b \le M - \log \mid f(x_0) \mid, \qquad k \ge k_0.$$

Note that

$$\lim_{k\to\infty} \mid n_k \mid = \lim_{k\to\infty} \mid m_k \mid = +\infty$$

(in the opposite case $\log A$ and $\log B$ would be commensurable). Dividing the last inequalities by $n_k$, and then letting $k \to \infty$ implies

$$\lim_{k\to\infty}\frac{m_k}{n_k} = -\frac{\log a}{\log b}.$$

It follows that

$$\frac{\log A}{\log B}=\frac{\log a}{\log b},$$

which may be written in the following equivalent form

$$\frac{\log A}{\log a}=\frac{\log B}{\log b}.$$

Hence, putting

$$p := \frac{\log A}{\log a},$$

we get $A = a^p$ and $B = b^p$ what was to be shown. □

Justified by Theorem 1 we examine system (*) assuming that there is a $p \in \mathbf{R}$, $p \neq 0$, such that $A = a^p$, $B = b^p$.

THEOREM 2. *Let $\alpha, \beta, p \in \mathbf{R}$, $p \neq 0$, and $a, b \in (0,\infty)$, $a \neq 1 \neq b$, be such that*

$$\frac{\log a}{\log b} \notin \mathbf{Q}, \qquad \frac{\alpha}{1-a^p}=\frac{\beta}{1-b^p},$$

*and put*

$$\xi := \frac{\alpha}{1-a^p}.$$

$1^o$. *If $f : (\xi, \infty) \to \mathbb{R}$ satisfies the system of functional equations*

$$f(a^p x + \alpha) = a f(x), \qquad f(b^p x + \beta) = b f(x), \qquad x > \xi,$$

*and it is continuous at least at one point, then there is a constant $c \in \mathbb{R}$ such that*

$$f(x) = c(x - \xi)^{1/p}, \qquad x > \xi.$$

$2^o$. *If $f : (-\infty, \xi) \to \mathbb{R}$ satisfies the system of functional equations*

$$f(a^p x + \alpha) = a f(x), \qquad f(b^p x + \beta) = b f(x), \qquad x < \xi,$$

*and it is continuous at least at one point, then there is a constant $c \in \mathbb{R}$ such that*

$$f(x) = c(\xi - x)^{1/p}, \qquad x < \xi.$$

$3^o$. *If $f : \mathbb{R} \to \mathbb{R}$ satisfies the system of functional equations*

$$f(a^p x + \alpha) = a f(x), \qquad f(b^p x + \beta) = b f(x), \qquad x \in \mathbb{R},$$

*and in each of the intervals $(\xi, \infty)$ and $(-\infty, \xi)$ there is at least one point of continuity of $f$, then there are $c_1,\ c_2 \in \mathbb{R}$ such that*

$$f(x) = \begin{cases} c_1(x - \xi)^{1/p}, & x > \xi \\ 0, & x = \xi. \\ c_2(\xi - x)^{1/p}, & x < \xi \end{cases}$$

PROOF. $1^o$. Since $\frac{\log a}{\log b} \notin \mathbf{Q}$, by the Kronecker theorem, the set

$$\mathbf{D} = \{a^n b^m : \ n, m \in \mathbf{Z}\}$$

is dense in $(0, \infty)$. Applying Lemma $1.2^0$ with $A = a^p$ and $B = b^p$ (cf. also Remark 2) we obtain

$$f\left((x - \xi)t^p + \xi\right) = t f(x), \qquad x > \xi, \ t \in \mathbf{D}. \tag{4}$$

Let $x_0 > \xi$ be a point of the continuity of $f$, and $x > \xi$ arbitrary. Since $(x_0 - \xi)/(x - \xi) > 0$, there exists a sequence $t_k \in \mathbf{D}$, $k \in \mathbf{N}$, such that

$$\lim_{k \to \infty} t_k = \left(\frac{x_0 - \xi}{x - \xi}\right)^{1/p}.$$

Note that

$$\lim_{k\to\infty}((x-\xi)t_k^p+\xi)=x_0.$$

Taking $t=t_k$ in (4) gives

$$f((x-\xi)t_k^p+\xi)=t_kf(x), \qquad k\in\mathbf{N}.$$

Letting $k\to\infty$, and making use of the continuity of $f$ at the point $x_0$, in this relation yields

$$f(x_0)=\left(\frac{x_0-\xi}{x-\xi}\right)^{1/p}f(x).$$

Hence, putting

$$c:=f(x_0)(x_0-\xi)^{-1/p},$$

we obtain

$$f(x)=c(x-\xi)^{1/p},$$

which completes the proof of $1^0$.

To prove $2^0$ suppose that $f$ is continuous at a point $x_0<\xi$, and take an arbitrary $x<\xi$. Then $(x_0-\xi)/(x-\xi)$ is positive, and we can repeat the same argument as in the part $1^o$.

Suppose that $f:\mathbf{R}\to\mathbf{R}$ satisfies the considered system of functional equations. Setting $x=\xi$ in the first of these equations gives $f(\xi)=af(\xi)$. Since $a\neq 1$, we get $f(\xi)=0$. Now $3^0$ is a consequence of $1^o$ and $2^o$. This completes the proof. □

EXAMPLE. Consider the system of functional equations

$$f(4x+3)=2f(x), \qquad f(9x+8)=3f(x), \qquad x\in\mathbf{R},$$

where $f:\mathbf{R}\to\mathbf{R}$. Thus we have $a=2$, $\alpha=3$, $b=3$, $\beta=8$, and $p=2$. Because $\log 2/\log 3$ is irrational, and $\alpha/(1-a^p)=\beta/(1-b^p)=-1$, the numbers $\alpha$, $\beta$, $a$, $b$, and $p$ satisfy the assumptions of Theorem 2. If $f$ is continuous at two points $x_1$ and $x_2$ such that $x_1<-1<x_2$ then, by Theorem 2, there exist $c_1$, $c_2\in\mathbf{R}$ such that

$$f(x)=\begin{cases} c_1\sqrt{x+1}, & x>-1\\ 0, & x=-1.\\ c_2\sqrt{-1-x}, & x<-1\end{cases}$$

THEOREM 3. *Let $\alpha$, $\beta$, $p\in\mathbf{R}$, $p\neq 0$, and $a$, $b\in(0,\infty)$, $a\neq 1\neq b$, be such that*

$$\frac{\log a}{\log b}\notin\mathbf{Q}, \qquad \frac{\alpha}{1-a^p}=\frac{\beta}{1-b^p},$$

*and put*

$$\xi := \frac{\alpha}{1-a^p}.$$

**1°.** *If $f : (\xi, \infty) \to \mathbb{R}$ satisfies the system of functional equations*

$$f(a^p x + \alpha) = af(x), \qquad f(b^p x + \beta) = bf(x), \qquad x > \xi,$$

*and for a nonempty open interval $I \subset (\xi, \infty)$ the restriction $f|_I$ is Lebesgue measurable, then there is a constant $c \in \mathbb{R}$ such that*

$$f(x) = c(x-\xi)^{1/p} \qquad \text{a. e. in } (\xi, \infty).$$

**2°.** *If $f : (-\infty, \xi) \to \mathbb{R}$ satisfies the system of functional equations*

$$f(a^p x + \alpha) = af(x), \qquad f(b^p x + \beta) = bf(x), \qquad x < \xi,$$

*and for a nonempty open interval $I \subset (-\infty, \xi)$ the restriction $f|_I$ is Lebesgue measurable, then there is a constant $c \in \mathbb{R}$ such that*

$$f(x) = c(\xi - x)^{1/p} \qquad \text{a. e. in } (-\infty, \xi).$$

**3°.** *If $f : \mathbb{R} \to \mathbb{R}$ satisfies the system of functional equations*

$$f(a^p x + \alpha) = af(x), \qquad f(b^p x + \beta) = bf(x), \qquad x \in \mathbb{R},$$

*and each of the intervals $(\xi, \infty)$ and $(-\infty, \xi)$ contains a nonempty open interval $I$ such that $f|_I$ is Lebesgue measurable, then there are $c_1$, $c_2 \in \mathbb{R}$ such that*

$$f(x) = \begin{cases} c_1(x-\xi)^{1/p}, & x > \xi \\ 0, & x = \xi \\ c_2(\xi - x)^{1/p}, & x < \xi \end{cases} \qquad \text{a.e. in } \mathbb{R}$$

PROOF. **1°.** Let $f : (\xi, \infty) \to \mathbb{R}$ be a solution of the considered system of functional equations which is Lebesgue measurable on a nonempty open interval $I \in (\xi, \infty)$. Define $f_0 : (\xi, \infty) \to \mathbb{R}$ by

$$f_0(x) = (x - \xi)^{1/p}, \qquad x > \xi.$$

It is easy to verify that the function $\phi : (\xi, \infty) \to \mathbb{R}$,

$$\phi(x) := \frac{f(x)}{f_0(x)}, \qquad x > \xi,$$

satisfies the simultaneous system of functional equations

$$\phi(a^p x + \alpha) = \phi(x), \qquad \phi(b^p x + \beta) = \phi(x), \qquad x > \xi. \tag{5}$$

Note that the family of functions $(h^t : t \in \mathbf{R})$, $h^t : (\xi, \infty) \to \mathbf{R}$, defined by

$$h^t(x) = a^{pt}(x - \xi) + \xi, \qquad x > \xi, \ t \in \mathbf{R}, \tag{6}$$

is a continuous iteration group. Thus there exists a homeomorphism $\gamma : \mathbf{R} \to (\xi, \infty)$ (cf. J. Aczél [1], Chapter 6) such that

$$h^t(x) = \gamma\left(\gamma^{-1}(x) + t\right), \qquad x > \xi, \ t \in \mathbf{R}. \tag{7}$$

Hence

$$h^1(x) = a^p x + \alpha = \gamma\left(\gamma^{-1}(x) + 1\right), \qquad x > \xi.$$

Put

$$r := \frac{\log b}{\log a}.$$

Taking $t = r$ in (6), and making use of the assumption $a^p\beta + \alpha = b^p\alpha + \beta$, gives

$$h^r(x) = b^p x + \beta = \gamma\left(\gamma^{-1}(x) + r\right), \qquad x > \xi.$$

Therefore we can write (5) in the form

$$\phi\left[\gamma\left(\gamma^{-1}(x) + 1\right)\right] = \phi(x), \qquad \phi\left[\gamma\left(\gamma^{-1}(x) + r\right)\right] = \phi(x), \qquad x > \xi.$$

It follows that the function $\phi \circ \gamma : \mathbf{R} \to \mathbf{R}$ satisfies the system of equations

$$\phi \circ \gamma(s + 1) = \phi \circ \gamma(s), \qquad \phi \circ \gamma(s + r) = \phi \circ \gamma(s), \qquad s \in \mathbf{R},$$

which means that $\phi \circ \gamma$ is periodic of periods 1 and $r$. Hence, by an obvious induction,

$$\phi \circ \gamma(s + n + mr) = \phi \circ \gamma(s), \qquad s \in \mathbf{R}, \ n, m \in \mathbf{Z}.$$

Since $r$ is irrational, $\{n + mr : \ n, m \in \mathbf{Z}\}$ is a dense set in $\mathbf{R}$, and consequently $\phi \circ \gamma$ is microperiodic.

From (6) and (7) we get

$$\gamma\left(\gamma^{-1}(x) + t\right) = a^{pt}(x - \xi) + \xi, \qquad x > \xi, \ t \in \mathbf{R}.$$

Setting $x = \gamma(0)$ gives

$$\gamma(t) = a^{pt}\left(\gamma(0) - \xi\right) + \xi, \qquad t \in \mathbf{R},$$

so $\gamma$ is a diffeomorphism. By assumption $\phi = f/f_0$ is measurable on a nonempty open interval $I \subset (\xi, \infty)$. It follows that the function $\phi \circ \gamma$ is measurable on the open interval $\gamma^{-1}(I)$. The microperiodicity of $\phi \circ \gamma$ implies that it is Lebesgue measurable on $\mathbb{R}$. By Lemma 2 there is a $c \in \mathbb{R}$ such that $\phi \circ \gamma = c$ almost everywhere in $\mathbb{R}$. Hence $\phi = c$ a.e. in $\mathbb{R}$, and from the definition of $\phi$ we obtain $f = cf_0$ a.e. in $\mathbb{R}$. This completes the proof of $1^o$. The proof of $2^o$ is analogous. Part $3^o$ is an obvious consequence of $1^o$ and $2^o$. □

REMARK 4. In Theorem 1 (and consequently in Theorems 2 and 3) we have assumed that $A, B \in (0, \infty)$ and $A \neq 1 \neq B$. It is easy to verify that if $A \neq 1$, $B = 1$, or $A = 1$, $B \neq 1$, the condition (3) is fulfilled and we can apply the Proposition. The case $A = 1 = B$, as we have already mentioned, was considered in [2].

Note also that if $f : \mathbb{R} \to \mathbb{R}$ is a solution of system (*), then

$$f\left(A^2x + \alpha(A+1)\right) = a^2 f(x)\,, \quad f\left(B^2x + \beta(B+1)\right) = b^2 f(x)\,, \quad x \in \mathbb{R}\,.$$

Thus, without any loss of generality we could assume that the numbers $A$, $B$, $a$, and $b$ are positive.

**Acknowledgment.** The author would like to thank Professor K. Baron for his valuable comments.

## REFERENCES

[1] J. Aczél, *Lectures on Functional Equations and their Applications*, New York, London, 1966.

[2] W. E. Clark, A. Mukherjea, *Comments on a Functional Equation*, Real Analysis Exchange **6** (1980–81), 192–199.

[3] N. Brillouët–Bellout, *On some functional equations of Gołąb–Schinzel type*, Aequationes Math. **42** (1991), 239–270.

[4] J. Brzdęk, *Subgroups of the group $Z_n$ and a generalization of the Gołąb–Schinzel functional equation*, Aequationes Math. **43** (1992), 59–71.

[5] R. Ger, Z. Kominek, M. Sablik, *Generalized Smital's lemma and a theorem of Steinhaus*, Radovi Mat. **1** (1985), 101–119.

[6] M. Kuczma, *Note on Microperiodic Functions*, Radovi Matematički **5** (1989), 127–140.

[7] M. Kuczma, B. Choczewski, R.Ger, *Iterative Functional Equations*, Cambridge University Press 1990.

[8] A. Lomnicki, *O wielookresowych funkcjach jednoznacznych zmiennej rzeczywistej*, Sprawozdania Tow. Nauk. Warsz. **6** (1918), 808–846.

[9] J. Matkowski, *Cauchy functional equation on a restricted domain and commuting functions*, Lecture Notes in Mathematics **1163** (1985), 101–106.

Department of Mathematics
Technical University
Willowa 2
43–309 Bielsko–Biala
Poland