

## ON TWO GEOMETRIC INEQUALITIES

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**Abstract.** In inner product spaces the Ptolemaic inequality (1) and the quadrilateral inequality (2) are well known. By using the identity (3), we derive here (2) from (1). The rest of the paper is devoted to some comments on (1), (2), (3).

**1. Main result.** Let  $E$  be a real or complex inner product space, e.g. a Euclidean  $\mathbb{R}^N$ . The main purpose of this note is the use of the Ptolemaic inequality

$$(1) \quad \|a - b\| \cdot \|c - d\| \leq \|a - c\| \cdot \|b - d\| + \|a - d\| \cdot \|b - c\|$$

( $a, b, c, d \in E$ ) for proving the quadrilateral inequality

$$(2) \quad \|a + b\| + \|a + c\| + \|b + c\| \leq \|a\| + \|b\| + \|c\| + \|a + b + c\|.$$

A further ingredient of the proof will be the identity

$$(3) \quad \|a + b\|^2 + \|a + c\|^2 + \|b + c\|^2 = \|a\|^2 + \|b\|^2 + \|c\|^2 + \|a + b + c\|^2,$$

which is easily verified. Observe that (1) is equivalent to

$$(4) \quad \|x + y\| \cdot \|x + z\| \leq \|y\| \cdot \|z\| + \|x\| \cdot \|x + y + z\| \quad (x, y, z \in E)$$

(use the transformation  $x = c - b$ ,  $y = a - c$ ,  $z = b - d$ ). To prove (2), square both sides of this inequality, simplify by means of (3), and divide by two. It remains

$$\begin{aligned} & \|a + b\| \cdot \|a + c\| + \|a + b\| \cdot \|b + c\| + \|a + c\| \cdot \|b + c\| \\ & \leq [\|b\| \cdot \|c\| + \|a\| \cdot \|a + b + c\|] + [\|a\| \cdot \|c\| + \|b\| \cdot \|a + b + c\|] \\ & \quad + [\|a\| \cdot \|b\| + \|c\| \cdot \|a + b + c\|], \end{aligned}$$

which is an obvious consequence of (4).

The foregoing procedure has been inspired by a question of Dennis C. Russell during the 1990 General Inequalities Conference at Oberwolfach, where he asked for "easy" proofs of (2). In the following paragraphs we shall add some comments on (1), (2), (3).

**2. Comments on the Ptolemaic inequality.** For sake of completeness let us start with a proof of (1). We shall repeat the proof from Alsina and Garcia-Roig [1]: Inequality (1) is equivalent to

$$(5) \quad \|x - y\| \cdot \|z\| \leq \|x - z\| \cdot \|y\| + \|y - z\| \cdot \|x\|$$

(use the transformation  $x = a - d$ ,  $y = b - d$ ,  $z = c - d$ ). Now

$$\frac{\|x - y\|}{\|x\| \cdot \|y\|} = \left\| \frac{x}{\|x\|^2} - \frac{y}{\|y\|^2} \right\| \quad (x, y \in E \setminus \{0\})$$

(together with the triangle inequality applied to the right-hand side) yields

$$\frac{\|x - y\|}{\|x\| \cdot \|y\|} \leq \frac{\|x - z\|}{\|x\| \cdot \|z\|} + \frac{\|z - y\|}{\|z\| \cdot \|y\|} \quad (x, y, z \in E \setminus \{0\}),$$

from which (5) follows immediately.

By Schoenberg's results [8] the Ptolemaic inequality characterizes inner product spaces: In any normed space, where (1) holds true, the norm may be generated by an inner product. For further going discussion of this subject cf. Day [2].

**3. Comments on the quadrilateral inequality.** Inequality (2) is included in a general class of inequalities given by Hornich [3]. There (2) also is proved by the following procedure due to Hlawka: Using nothing but (3), one can show that

$$(6) \quad \begin{aligned} & (\|a\| + \|b\| + \|c\| + \|a + b + c\| - \|a + b\| - \|a + c\| - \|b + c\|) \\ & \quad \cdot (\|a\| + \|b\| + \|c\| + \|a + b + c\|) \\ & = (\|a\| + \|b\| - \|a + b\|) \cdot (\|c\| + \|a + b + c\| - \|a + b\|) \\ & \quad + (\|a\| + \|c\| - \|a + c\|) \cdot (\|b\| + \|a + b + c\| - \|a + c\|) \\ & \quad + (\|b\| + \|c\| - \|b + c\|) \cdot (\|a\| + \|a + b + c\| - \|b + c\|), \end{aligned}$$

which implies (2). Accordingly, (2), (6) are called "Hlawka's inequality" and "Hlawka's identity", respectively, by Mitrinović [7]. In this book generalizations of (2) also may be found. Contrary to (1), the inequality (2) does not

characterize inner product spaces (Smiley and Smiley [9]): In fact, (2) holds in every two-dimensional real normed space (Kelly, Smiley, and Smiley [4]).

**4. Comments on the identity (3).** A slight generalization of (3) is

$$(7) \quad \begin{aligned} & \|p\|^2 + \|p+a+b\|^2 + \|p+a+c\|^2 + \|p+b+c\|^2 \\ & = \|p+a\|^2 + \|p+b\|^2 + \|p+c\|^2 + \|p+a+b+c\|^2. \end{aligned}$$

A proof is simple by using the well known parallelogram-identity

$$(8) \quad 2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2 \quad (x, y \in E).$$

Indeed, (7) is equivalent to

$$\begin{aligned} & \|2p+a+b\|^2 + \|a+b\|^2 + \|2p+a+b+2c\|^2 + \|a-b\|^2 \\ & = \|2p+a+b\|^2 + \|a-b\|^2 + \|2p+a+b+2c\|^2 + \|a+b\|^2. \end{aligned}$$

To see this, multiply (7) by two and use (8) four times in an obvious manner.

Let us give some geometric interpretation of (7): The endpoints

$$(9) \quad V_1, V_3, V_5, V_7, V_2, V_4, V_6, V_8$$

of the vectors  $p, p+a+b, \dots, p+a+b+c$  occurring there generate a parallelepiped  $\Pi$  (of dimension  $\leq 3$ ). The numbering in (9) is such that every edge of  $\Pi$  joins a vertex with an odd index to a vertex with an even index. Then (7) reads

$$\overline{OV}_1^2 - \overline{OV}_2^2 + \overline{OV}_3^2 - \overline{OV}_4^2 + \overline{OV}_5^2 - \overline{OV}_6^2 + \overline{OV}_7^2 - \overline{OV}_8^2 = 0,$$

where  $\overline{OV}_j$  denotes the distance of the vertex  $V_j$  to the origin of the space. Identity (3) has a similar interpretation:  $\|a\|, \|b\|, \|c\|$  are lengths of edges,  $\|a+b\|, \|a+c\|, \|b+c\|$  are lengths of face-diagonals, and  $\|a+b+c\|$  is the length of a 3-space-diagonal of a parallelepiped  $\Pi \subseteq E$ .

**5. Connections with functional equations.** There is something more general behind the here given proof (8)  $\Rightarrow$  (7): For a moment, let  $E$  only be a (real or complex) vector space (a vector space over the rationals would be sufficient). For functions  $f: E \rightarrow \mathbb{R}$  and elements  $h \in E$  define  $\Delta_h f: E \rightarrow \mathbb{R}$  by

$$(\Delta_h f)(x) = f(x+h) - f(x) \quad (x \in E).$$

Then it is well known that the functional equation

$$(10) \quad (\Delta_h \Delta_h f)(x) = 2f(h) \quad (x, h \in E)$$

implies

$$(11) \quad (\Delta_a \Delta_b \Delta_c f)(x) = 0 \quad (a, b, c, x \in E).$$

(More generally,  $(\Delta_h^n f)(x) = n!f(h)$  for  $x, h \in E$  implies

$$(\Delta_{a_1} \Delta_{a_2} \dots \Delta_{a_{n+1}} f)(x) = 0 \quad (a_1, a_2, \dots, a_{n+1}, x \in E).$$

This follows from results of Mazur and Orlicz [6]; cf. also Kuczma [5].) Now take  $f(x) = \|x\|^2$  in an inner product space  $E$ . Then (8) becomes (10) (with  $x+h, h$  replaced by  $x, y$ , respectively), and (7) becomes (11) (with  $x = p$ ). So (8)  $\Rightarrow$  (7) follows from the more general statement (10)  $\Rightarrow$  (11), but actually the proofs of both implications are the same.

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