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ON SOME EXTENSIONS OF THE GOŁĄB–SCHINZEL FUNCTIONAL EQUATION

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Abstract. Composite functional equations (arising in applications) are presented that may be interpreted as extensions of the Gołąb–Schinzel equation and as modifications of d'Alembert's equation. Depending on the type of the considered equation, continuous, and finite rate of growth solutions are discussed. Geometric interpretations are given.

Introduction. The Gołąb–Schinzel equation has been the topic of many papers. The equation was introduced by J. Aczél [1] and was treated in considerable detail by S. Gołąb and A. Schinzel [14] and by J. Aczél and S. Gołąb [5]. The general solution and general continuous or measurable solutions have been dealt with by several authors (cf. [2], [4], [6], [10], [12], [14], [15], [17], [19]). Certain generalizations have been obtained recently (cf. [7], [8], [9], [10]). On the other hand, d'Alembert's equation has been a classical theme in theory and applications (cf. [2], [3], [4]).

In this paper we present some functional equations that are related both to Gołąb–Schinzel and d'Alembert's equation, but may be viewed properly as extensions of the Gołąb–Schinzel equation. In particular, in Section 3, we deal with the functional equation

$$f\left(x+y\sqrt{f(x)}\right)+f\left(x-y\sqrt{f(x)}\right)=2f(x)f(y),$$

and we prove that if $f:\mathbb{R}\to\mathbb{R}$ is a continuous solution then, either $f\equiv 0$ or $f(x)=cx^2+1$ for some $c\geq 0$. For each $c<0$ the function $f(x)=cx^2+1$ satisfies this functional equation for all x and y in a neighbourhood of zero. It seems to be of interest that, in contrast to the well known property of

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Gołąb–Schinzel equation, these local solutions cannot be extended to the global ones.

The solutions may be interpreted geometrically as describing families of conic sections: the graphs of the solutions contain either parabolas, or a combination of hyperbolas and ellipses.

The paper originated from a meteorological problem of interpolation (equations (7) and (10)).

1. A functional equation for parabolas. We consider the equation

$$f\left(x+yf(x)^2\right)^2 = f(x)^2 f(y)^2 \tag{1}$$

for $f:\mathbb{R}\to\mathbb{R}$ which is an obvious modification of the Gołąb–Schinzel's functional equation. Inspection of this equation shows immediately that there is a trivial solution $f\equiv 0$, and there exists (non-trivial) constant solutions $f\equiv 1$ and $f\equiv -1$. Taking $y=0$ in (1) gives $f(x)^2 = f(x)^2 f(0)^2$, yielding $f(0)=1$ or $f(0)=-1$ if the trivial solution is excluded.

Making use of the classical result on the Gołąb–Schinzel equation we have the following

THEOREM 1. *A continuous function $f:\mathbb{R}\to\mathbb{R}$ is a solution of equation* (1) *if and only if it has one of the mutually exclusive forms:*
$f\equiv 0$,
$f(x)=\sqrt{\sup(1+cx,0)}, \qquad x\in\mathbb{R}$,
$f(x)=-\sqrt{\sup(1+cx,0)}, \qquad x\in\mathbb{R}$,
where c is a fixed real number.

PROOF. Suppose that $f:\mathbb{R}\to\mathbb{R}$ is a continuous solution of (1) and define $h:\mathbb{R}\to\mathbb{R}_+$ writing $h(x)=f(x)^2$. Then h is a nonnegative continuous solution of the Gołąb–Schinzel functional equation

$$h\left(x+yh(x)\right)=h(x)h(y), \qquad x,y\in\mathbb{R}. \tag{2}$$

Hence we get (cf. [2, pp. 132-133], or [4, p. 312]) either $h\equiv 0$ or $h(x)=f(x)^2=\sup(1+cx,0)$ for all $x\in\mathbb{R}$. In the latter case the continuity of f implies that either $f(x)=\sqrt{\sup(1+cx,0)}$ for all $x\in\mathbb{R}$ or $f(x)=-\sqrt{\sup(1+cx,0)}$ for all $x\in\mathbb{R}$. This completes the proof. □

REMARK 1. Note that, contrary to the Gołąb–Schinzel equation, functional equation (1) has no non-trivial everywhere differentiable solution satisfying equation (1) in $\mathbb{R}$. Depending on the sign of c, each non-trivial solution coincides on one of the intervals $(-\infty,-1/c)$, $(-1/c,+\infty)$ with the trivial solution. In fact, each non-trivial continuous solution of (1) is not differentiable only at the zero point $x = -1/c$ of the non-trivial part.

REMARK 2. It seems to be a natural problem to find all the non-trivial solutions $f : \mathbb{R} \to \mathbb{R}$ of (1) which are differentiable at 0. From (1) we get the functional equation

$$f(x+y)^2 = f(x)^2 f\left(\frac{y}{f(x)^2}\right)^2, \tag{3}$$

which holds true for all $x \in A := \{x \in \mathbb{R} : \ f(x) \neq 0\}$ and $y \in \mathbb{R}$. Then $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) := f(x)^2$ satisfies equation (2). Because f is non-trivial (i.e. $f \not\equiv 0$), we have $h(0) = 1$. The differentiability of f at 0 implies that there is a $\delta > 0$ such that for every $x \in (-\delta, \delta)$ we have $h(x) > 0$. Consequently, $(-\delta, \delta) \subset A$. Subtracting $h(x)$ from (3), dividing by $y \in \mathbb{R}$, $y \neq 0$, we obtain

$$\frac{h(x+y) - h(x)}{y} = \frac{h(y/h(x)) - 1}{y/h(x)}.$$

Since $h'(0)$ exists, it follows that h is differentiable at every point of the set A. Moreover, letting $y \to 0$, we get

$$h'(x) = h'(0), \qquad x \in A.$$

Thus there are $c, d \in \mathbb{R}$ such that $h(x) = d + cx$ for all $x \in (-\delta, \delta)$. Substituting this function into equation (2) gives $h(x) = 1 + cx$ for all $x, y \in (-\delta, \delta)$. Let (a, b), $a < 0 < b$, be the maximal interval such that $h(x) = 1 + cx$ for all $x \in (a, b)$. Setting $x, y \in (a, b)$ in (2), we get $h(x + cxy + y) = c(x + cxy + y) + 1$ for all $x, y \in (a, b)$. Suppose first that $c \geq 0$. It follows that $h(z) = cz + 1$ for all $z \in [0, 2b + cb^2)$. Hence $b = +\infty$, and consequently, $a = -\infty$. If $c < 0$ we can argue in a similar way. As a corollary one gets:

PROPOSITION 1. *If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at 0 and satisfies equation (1) then either $f \equiv 0$ or $f(x)^2 = \sup(1 + cx, 0)$ for all $x \in \mathbb{R}$.*

To get from Proposition 1 all the solutions listed in Theorem 1 we have additionally to assume the continuity of f at least on the set $A := \{x \in \mathbb{R} : f(x) \neq 0\}$. The following example shows that there are differentiable at 0 and discontinuous on a large set solutions of equation (1).

EXAMPLE 1. Denote by $\mathbb{Q}$ the set of all rational real numbers. The function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1, & x \in (-1, 1) \cup \mathbb{Q}, \\ -1, & x \notin (-1, 1) \cup \mathbb{Q}, \end{cases}$$

is a differentiable at 0 solution of equation (1).

Thus the differentiability at 0 is not a sufficient condition to characterize all the non-trivial solutions of equation (1) which are continuous on $\mathbb{R}$ and which are of the forms:

$$
\begin{aligned}
f(x) &= \sqrt{\sup(1+cx,0)}, \quad x \in \mathbb{R}, \\
f(x) &= -\sqrt{\sup(1+cx,0)}, \quad x \in \mathbb{R},
\end{aligned} \tag{4}
$$

where c is a fixed real number.

Geometrically, apart from the trivial part of the graph, (4) describes a family of parabolas with horizontal axis; the individual curves being identified by the numerical value of the parameter c.

REMARK 3. If the zero point $(-1/c)$ of the non-trivial part of the graph of the solution goes either to $+\infty$ or to $-\infty$, then the solution approaches the constant solutions $f \equiv 1$ or $f \equiv -1$. This fact explains why we treat these two constant solutions as non-trivial ones.

Alternatively, the solutions of (1) can be obtained by considering the symmetry with respect to the variables x and y of the right-hand side of this equation.

PROPOSITION 2. *If $f : \mathbb{R} \to \mathbb{R}$ is a solution of equation* (1) *such that the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) := [f(x)]^2$ is one-to-one either in a neighbourhood of $+\infty$ or $-\infty$, then there exists a positive $c > 0$ such that*

$$f(x)^2 = \sup(1+cx,0), \qquad x \in \mathbb{R}.$$

If moreover f is monotonic then it is of form (4).

PROOF. Assume for instance that h is one-to-one in a neighbourhood of $+\infty$. Thus there is an $a > 0$ such that h is injective on $(a,+\infty)$. As h is nonnegative we have

$$x, y \in (a,+\infty) \Rightarrow x + yf(x),\ y + xf(y) \in (a,+\infty).$$

The right-hand side of (1) is symmetric in x and y, therefore the left-hand side must be symmetric too:

$$h(x+yh(x)) = h(y+xh(y)), \qquad x,y \in (a,+\infty).$$

Since h is one-to-one in $(a,+\infty)$, this implies that $x + yh(x) = y + xh(y)$, i.e.

$$\frac{h(x)-1}{x} = \frac{h(y)-1}{y}, \qquad x,y \in (a,+\infty).$$

It follows that there exists a $c > 0$ such that $h(x) = 1 + cx$ for all $x > a$. Now using the functional equation and the nonnegativity of h one can show that $h(x) = f(x)^2 = \sup(1 + cx, 0)$ for all $x \in \mathbf{R}$. The monotonicity of f gives form (4). This completes the proof. $\square$

To show that the assumption of the function h to be one-to-one is essential consider the following

EXAMPLE 2. The function $f : \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) := \begin{cases} 1 & \text{for rational } x, \\ 0 & \text{for irrational } x, \end{cases}$$

(Dirichlet's function) satisfies the equation (1) and is not of form (4).

The assumption of f to be monotonic is also indispensable:

EXAMPLE 3. The function $f : \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) := \begin{cases} \sqrt{1+x} & \text{for rational} \quad x \geq -1, \\ -\sqrt{1+x} & \text{for irrational} \quad x \geq -1, \\ 0 & \text{for} \quad x < -1 \end{cases}$$

satisfies equation (1), and the function h is one-to-one on $(0, +\infty)$. The function f is not of form (4) because it is not monotonic.

REMARK 4. Applying Theorem 1 one can easily observe that:

1^0 the general continuous solution $f : \mathbf{R} \to \mathbf{R}_+ = [0, +\infty)$ of equation (1) is

$$f(x) = \sqrt{\sup(1 + cx, 0)}, \qquad x \in \mathbf{R};$$

2^0 the general continuous solution $f : \mathbf{R} \to \mathbf{R}_- = (-\infty, 0]$ of equation (1) is

$$f(x) = -\sqrt{\sup(1 + cx, 0)}, \qquad x \in \mathbf{R}.$$

2. A second functional equation for parabolas. Formally taking the square root of (1) leads to a functional equation for $f : \mathbf{R} \to \mathbf{R}_+$:

$$f\left(x + yf(x)^2\right) = f(x)f(y) \tag{5}$$

for all $x, y \in \mathbf{R}$, and, respectively, for $f : \mathbf{R} \to \mathbf{R}_- = (-\infty, 0]$:

$$f\left(x + yf(x)^2\right) = -f(x)f(y). \tag{6}$$

The general continuous solution $f : \mathbb{R} \to \mathbb{R}_+$ of (5) coincides with the first formula of relation (4), which is in accord with a result of J. Brzdęk [9]. The general continuous solution $f : \mathbb{R} \to \mathbb{R}_-$ of (6) coincides with the second formula of relation (4).

3. A third functional equation for parabolas. In this section we consider the functional equation

$$(7) \qquad f\left(x + y\sqrt{f(x)}\right) + f\left(x - y\sqrt{f(x)}\right) = 2f(x)f(y), \qquad x, y \in \mathbb{R},$$

for $f : \mathbb{R} \to \mathbb{R}_+$. This equation may be interpreted as a modification of d'Alembert's equation, or as an extension of Gołąb–Schinzel's equation.

REMARK 5. Suppose that $f : \mathbb{R} \to \mathbb{R}_+$ satisfies equation (7). Setting $x = y = 0$ in (7) we get $f(0) = f(0)^2$ which implies that either $f(0) = 0$ or $f(0) = 1$. If $f(0) = 0$ then, with $y = 0$ in (7), we infer that $f \equiv 0$. Of course $f \equiv 0$ is a solution of (7). In the sequel this solution is said to be trivial. There exists a (non-trivial) constant solution $f \equiv 1$.

REMARK 6. Suppose that $f : \mathbb{R} \to \mathbb{R}_+$ is a non-trivial solution of equation (7). Putting $x = 0$ in (7) leads to $f(-y) = f(y)$ for all $y \in \mathbb{R}$. Thus each solution of (7) is an even function.

In the sequel $\mathbf{N}$ and $\mathbf{Z}$ denote, respectively, the set of positive integers and integers.

LEMMA 1. *Suppose that $f : \mathbb{R} \to \mathbb{R}_+$ is a solution of equation (7).*

1^0 *If there is an $x_0 \neq 0$ such that $f(x_0) = 1$ then $f(nx_0) = 1$ for all $n \in \mathbf{Z}$. If moreover f is bounded on $(0, x_0)$ then $0 \leq f(x) \leq 1$ for all $x \in \mathbb{R}$.*

2^0 *If $f \not\equiv 0$ is continuous on an interval $(-c, c)$, $0 < c < +\infty$, then*

$$\alpha := \inf\{f(x) : \; x \in (-c, c)\} > 0.$$

If f is continuous on $(-c, c)$, $0 < c < +\infty$, and $\alpha = 0$, then $f \equiv 0$.

PROOF. 1^0 By Remark 6 the function f is even. Therefore it is enough to show that $f(nx_0) = 1$ for all $n \in \mathbf{N}$. Setting $x = y = x_0$ in (7) gives $f(2x_0) + f(0) = 2$. Since f is non-trivial, by Remark 5, we have $f(0) = 1$. It follows that $f(2x_0) = 1$. For an inductive proof suppose that $f(kx_0) = 1$ for all $k \leq n$, for some $n \in \mathbf{N}$. Hence, taking $x = nx_0$ and $y = x_0$ in (7) we get

$$f((n+1)x_0) + 1 = f((n+1)x_0) + f((n-1)x_0) = 2f(nx_0)f(x_0) = 2.$$

Consequently, $f((n+1)x_0) = 1$. The induction completes the proof of the first statement of 1^0.

Setting $x = nx_0$ in (7) we get

$$f(nx_0 + y) + f(nx_0 - y) = 2f(y), \qquad n \in \mathbf{Z}, \quad y \in \mathbb{R}.$$

The boundedness of f in $[0, x_0]$ implies that $M_0 := \sup\{f(y) : y \in [0, x_0]\}$ is finite. Since f is nonnegative, we hence get

$$0 \le f(nx_0 + y) \le 2M_0, \qquad n \in \mathbf{Z}, \quad y \in [0, x_0],$$

which means that $0 \le f \le 2M_0$ on $\mathbb{R}$. Let in the sequel $M := \sup\{f(y) : y \in \mathbb{R}\}$. From (7) we have $2f(x)f(y) \le 2M$ for all $x, y \in \mathbb{R}$, and, consequently, $2M^2 \le 2M$. Now from $f(0) = 1$, it follows that $M = 1$. This completes the proof.

2^0 On the contrary, suppose that $\alpha = 0$. Since f is continuous in $(-c, c)$, making use of the part 1^0 of this lemma, we infer that

$$0 \le f(x) \le 1, \qquad x \in (-c, c).$$

Moreover, there exist a sequence x_n, $(n \in \mathbf{N})$, and $x_0 \in (0, c]$ such that

$$\lim_{n \to \infty} x_n = x_0, \qquad \lim_{n \to \infty} f(x_n) = 0.$$

Of course we have

$$-x_0 < x - x_0\sqrt{f(x)} < x_0, \qquad x \in (0, x_0).$$

Setting $y = x_n$ in equation (7) gives

$$f\left(x + x_n\sqrt{f(x)}\right) + f\left(x - x_n\sqrt{f(x)}\right) = 2f(x)f(x_n), \qquad x \in \mathbb{R}, \quad n \in \mathbf{N},$$

and, since f is nonnegative,

$$\lim_{n \to \infty} f\left(x - x_n\sqrt{f(x)}\right) = 0, \qquad x \in \mathbb{R}.$$

The function $h : [0, x_0) \to \mathbb{R}$ defined by

$$h(x) := x - x_0\sqrt{f(x)}, \qquad x \in [0, x_0),$$

is continuous, and

$$h(0) = -x_0, \qquad \lim_{n \to \infty} h(x_n) = x_0.$$

By the Darboux property there is a point $z \in [0, x_0)$ such that $h(z) = 0$. Now the continuity of f at the point 0 implies

$$f(0) = f(h(z)) = f(z - x_0\sqrt{f(z)}) = \lim_{n\to\infty} f(z - x_n\sqrt{f(z)}) = 0,$$

and, by Remark 5, $f \equiv 0$. This contradiction completes the proof of 2^0. □

REMARK 7. It is easy to verify that the Dirichlet function $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) := \begin{cases} 1 & \text{for rational} \quad x, \\ 0 & \text{for irrational} \quad x, \end{cases}$$

satisfies equation (7). It shows that the assumption of the continuity of f in Lemma 1.2^0 is essential.

LEMMA 2. *Let $f : \mathbb{R} \to \mathbb{R}_+$ be a non-trivial solution of equation* (7).

1^0 *If f is continuous in a neighbourhood of* 0 *then it is continuous everywhere.*

2^0 *If f is continuous in a neighbourhood of a point $x_0 \neq 0$ such that $f(x_0) \neq 0$, then it is continuous everywhere.*

PROOF. 1^0 Let I be the maximal open interval of the continuity of f such that $0 \in I$. Since f is even there is a $c > 0$ such that $I = (-c, c)$. For an indirect argument suppose that $c < +\infty$. By Lemma 1.2^0 the number $\alpha := \inf\{f(x) : \ x \in (-c, c)\}$ is positive.

First consider the case $\alpha < 1$. Let us choose an arbitrary d, $0 < d < c$, and δ such that

$$0 < \delta < \frac{d\sqrt{\alpha}}{2}.$$

Fix $z \in [c, c + \delta)$. Then for an arbitrary $x_0 \in (c - \delta, c)$ the number

$$y := \frac{z - x_0}{\sqrt{f(x_0)}} \in (0, d),$$

and

$$-c < x_0 - y\sqrt{f(x_0)} < x_0.$$

For an arbitrary sequence z_n, $n \in \mathbf{N}$, such that $\lim_{n\to\infty} z_n = z$ put

$$y_n := \frac{z_n - x_0}{\sqrt{f(x_0)}}, \qquad n \in \mathbf{N}.$$

Then $\lim_{n\to\infty} y_n = y$ and, for sufficiently large n,

$$y_n \in (0,d), \qquad -c < x_0 - y_n\sqrt{f(x_0)} < x_0.$$

Now from (7), in view of the continuity of f in $(-c,c)$, we get

$$\begin{aligned}\lim_{n\to\infty} f(z_n) &= \lim_{n\to\infty} f\left(x_0 + y_n\sqrt{f(x_0)}\right)\\ &= \lim_{n\to\infty}\left[2f(x_0)f(y_n) - f(x_0 - y_n\sqrt{f(x_0)})\right]\\ &=2f(x_0)f(y) - f(x_0 - y\sqrt{f(x_0)}) = f(x_0 + y\sqrt{f(x_0)}) = f(z),\end{aligned}$$

which proves the continuity of f at the point $z \in [c, c+\delta)$. This contradicts to the definition of c, and proves that c must be equal ∞.

Now suppose that $\alpha \geq 1$. Then $f(x) \geq 1$ for all $x \in (-c,c)$. It follows that for every $x \in (-c,c)$ we have $x\left(\sqrt{f(x)}\right)^{-1} \in (-c,c)$. Substituting

$$y := \frac{x}{\sqrt{f(x)}}$$

in equation (7) gives

$$f(2x) + f(0) = 2f(x)f\left(\frac{x}{\sqrt{f(x)}}\right), \qquad x \in (-c,c).$$

It follows that f is continuous in the interval $(-2c, 2c)$ which contradicts to the definition of c. This completes the proof of 1^0.

2^0 Suppose that f is continuous on an interval (a,b) such that $x_0 \in (a,b)$. We may assume that $f(x) \neq 0$ for all $x \in (a,b)$. By the continuity of f on (a,b) there exist an interval $[c,d] \subset (a,b)$ and $\delta > 0$ such that for all $x \in [c,d]$ and $y \in (-\delta,\delta)$ we have

$$\frac{a+c}{2} \leq x + y\sqrt{f(x)} \leq \frac{b+d}{2}, \qquad \frac{a+c}{2} \leq x - y\sqrt{f(x)} \leq \frac{b+d}{2}.$$

Fix an $x \in [c,d]$. Taking an arbitrary $y \in (-\delta,\delta)$ and a sequence (y_n) such that $\lim_{n\to\infty} y_n = y$ and, making use of equation (7), we hence get

$$\begin{aligned}\lim_{n\to\infty} f(y_n) &= \lim_{n\to\infty} [2f(x)]^{-1}\left[f(x + y_n\sqrt{f(x)}) + f(x - y_n\sqrt{f(x)})\right]\\ &=[2f(x)]^{-1}\left[f(x + y\sqrt{f(x)}) + f(x - y\sqrt{f(x)}\right] = f(y),\end{aligned}$$

which proves the continuity of f on $(-\delta,\delta)$. By the first part of this lemma f is continuous everywhere. □

LEMMA 3. *Let $f:\mathbf{R}\to\mathbf{R}_+$ be a non-trivial solution of equation* (7). *If f is continuous in a neighbourhood of* 0 *then one of the following conditions holds true:*

(i) *there is a $\delta>0$ such that $f(x)>1$ for all $x\in(-\delta,\delta)$, $x\neq 0$;*
(ii) *there is a $\delta>0$ such that $0<f(x)<1$ for all $x\in(-\delta,\delta)$, $x\neq 0$;*
(iii) $$f\equiv 1.$$

PROOF. Since f is even, the interval $(-\delta,\delta)$ in the conditions (i) and (ii) may be replaced by $(0,\delta)$. Suppose for an indirect proof that none of the conditions (i), (ii), and (iii) is fulfilled. Then we could find a sequence $x_k>0$ such that

$$\lim_{k\to\infty} x_k = 0, \qquad f(x_k)=1, \qquad k\in\mathbf{N}.$$

By Lemma 1.1^0 we would have $f(nx_k)=1$ for all $k\in\mathbf{N}$, $n\in\mathbf{Z}$. Since the set

$$\{nx_k:\quad k\in\mathbf{N},\ \ n\in\mathbf{Z}\}$$

is dense in $\mathbf{R}$, and in view of Lemma 2.1^0 the function f is everywhere continuous, we have $f\equiv 1$. This contradiction completes the proof. □

LEMMA 4. *Let a function $f:\mathbf{R}\to\mathbf{R}_+$ be a non-trivial solution of equation* (7).

1^0 *If there is a $\delta>0$ such that $f(x)\geq 1$ for all $x\in(-\delta,\delta)$, then f is Jensen convex in $(-\delta,\delta)$. If moreover $f|_{(-\delta,\delta)}$, the restriction of f to $(-\delta,\delta)$, is bounded above in a neighbourhood of a point, then f is convex in $(-\delta,\delta)$.*

2^0 *If there are $\delta>0$ and $\alpha\in(0,1)$ such that $\alpha^2\leq f(x)\leq 1$ for all $x\in(-\delta,\delta)$, then f is concave in the interval $(-\alpha\delta,\alpha\delta)$.*

PROOF. 1^0 For arbitrary $s,t\in(-\delta,\delta)$ put

$$x:=\frac{s+t}{2}, \qquad y:=\frac{s-t}{2\sqrt{f\left(\frac{s+t}{2}\right)}}.$$

The number y is well defined because $\frac{s+t}{2}\in(-\delta,\delta)$. Moreover, since

$$-2\delta<s-t<2\delta \qquad \text{and} \qquad \sqrt{f\left(\frac{s+t}{2}\right)}\geq 1,$$

we have

$$y = \frac{s-t}{2\sqrt{f\left(\frac{s+t}{2}\right)}} \in (-\delta, \delta).$$

Writing (7) with the above defined x and y we get

$$f(s) + f(t) = 2f\left(\frac{s+t}{2}\right) f\left(\frac{s-t}{2\sqrt{f\left(\frac{s+t}{2}\right)}}\right).$$

Since $f(y) \geq 1$, it follows that

$$f(s) + f(t) \geq 2f\left(\frac{s+t}{2}\right), \qquad s, t \in (-\delta, \delta),$$

which means that f is Jensen convex in $(-\delta, \delta)$. The remaining part of (i) is now a consequence of the Bernstein-Doetsch theorem (cf. for instance M. Kuczma [16, p. 142, Theorem 2]). This completes the proof.

2^0 Put $\varepsilon := \alpha\delta$. We have $(-\varepsilon, \varepsilon) \subset (-\delta, \delta)$ and for all $s, t \in (-\varepsilon, \varepsilon)$

$$x := \frac{s+t}{2} \in (-\varepsilon, \varepsilon), \qquad -2\varepsilon \leq s - t \leq 2\varepsilon.$$

By the assumption

$$\alpha \leq \sqrt{f\left(\frac{s+t}{2}\right)} \leq 1, \qquad s, t \in (-\varepsilon, \varepsilon).$$

Therefore

$$y := \frac{s-t}{2\sqrt{f\left(\frac{s+t}{2}\right)}} \in \left(-\frac{\varepsilon}{\alpha}, \frac{\varepsilon}{\alpha}\right) = (-\delta, \delta).$$

Now, in the same way as in the previous part of the proof, we get

$$f(s) + f(t) = 2f\left(\frac{s+t}{2}\right) f\left(\frac{s-t}{2\sqrt{f\left(\frac{s+t}{2}\right)}}\right) \leq 2f\left(\frac{s+t}{2}\right)$$

for all $s, t \in (-\varepsilon, \varepsilon)$. Thus f is Jensen concave in $(-\varepsilon, \varepsilon)$. Because f is nonnegative, the Bernstein-Doetsch theorem implies that f is concave in $(-\varepsilon, \varepsilon)$. This completes the proof. □

Lemmas 3 and 4 show that rather weak regularity assumptions of a solution of equation (7) imply its convexity. It is well known that each convex function is two times differentiable almost everywhere. This fact allows us to prove the following

THEOREM 2. *Suppose that $f : \mathbb{R} \to \mathbb{R}_+$ is a solution of equation (7). If f is continuous in a neighbourhood of 0 then either $f \equiv 0$ or there exists a constant $c \geq 0$ such that*

$$f(x) = 1 + cx^2, \qquad x \in \mathbb{R}. \tag{8}$$

PROOF. By Lemma 2 the function f is continuous everywhere. If there is a point x_0 such that $f(x_0) = 0$ then, in view of Lemma 1, we get the trivial solution $f \equiv 0$. Thus, in the sequel, we can assume that $f(x) > 0$ for all $x \in \mathbb{R}$. By Remark 5 we have $f(0) = 1$. If $f \equiv 1$ there is nothing to prove. From now on we assume that $f \not\equiv 1$. We can write equation (7) in the form

$$f(x+y) + f(x-y) = 2f(x)f\left(\frac{y}{\sqrt{f(x)}}\right), \qquad x, y \in \mathbb{R}.$$

Subtracting from the both sides $2f(x)$, and dividing by y^2 we hence get

$$\frac{f(x+y) + f(x-y) - 2f(x)}{y^2} = 2\frac{f\left(\frac{y}{\sqrt{f(x)}}\right) - 1}{\left(\frac{y}{\sqrt{f(x)}}\right)^2}, \qquad x, y \in \mathbb{R}, \;\; y \neq 0. \tag{9}$$

In view of Lemmas 4 and 3 there is a $\delta > 0$ such that f is either convex or concave in $(-\delta, \delta)$. Since every convex (concave) function is two times differentiable almost everywhere there is at least one $z \in (-\delta, \delta)$ such that $f''(z)$ exists. It follows that (cf. for instance W. Rudin [18, p. 97])

$$\lim_{y \to 0} \frac{f(z+y) + f(z-y) - 2f(z)}{y^2} = f''(z).$$

From (9) we infer that the limit

$$c := \lim_{y \to 0} \frac{f\left(\frac{y}{\sqrt{f(z)}}\right) - 1}{\left(\frac{y}{\sqrt{f(z)}}\right)^2} = \lim_{y \to 0} \frac{f(y) - 1}{y^2}$$

exists. Now letting $y \to 0$ in the relation (9) gives

$$\lim_{y \to 0} \frac{f(x+y) + f(x-y) - 2f(x)}{y^2} = 2c, \qquad x \in \mathbb{R},$$

showing that the second *symmetric derivative* is constant in $\mathbb{R}$. By the Schwarz Theorem (cf. for instance G. M. Fichtenholz [13, p.520]) the function f must be a polynomial of the degree at most 2. Hence we get $f(x) = cx^2 + bx + a$, $x \in \mathbb{R}$. By Remark 6 we have $b = 0$, and, since $f(0) = 1$, so $a = 1$. Consequently,

$$f(x) = cx^2 + 1, \qquad x \in \mathbb{R}.$$

Since f is positive on $\mathbb{R}$, the constant c must be nonnegative.

It is easy to verify that for each $c \geq 0$ this function is a solution of equation (7). This completes the proof. □

REMARK 8. Let us note that for each $c < 0$, the function $f(x) = cx^2 + 1$ is, in a sense, a *local solution* of equation (7). This function satisfies equation (7) for all points $(x, y) \in D$ where the set D is defined by

$$D := \{(x, y) \in \mathbb{R}^2 : \quad -c_0 \leq x \leq c_0, \quad y \in \mathbb{R}\},$$

where $c_0 := (-c)^{-1/2}$. We shall prove that this local solution cannot be extended to a global one. More precisely, we shall prove that:

For each fixed $c < 0$ and $\delta \in (0, c_0]$, the function $f(x) = cx^2 + 1, x \in (-\delta, \delta)$, cannot be extended to a continuous global solution.

For an indirect proof suppose that there is a continuous solution $f : \mathbb{R} \to (0, \infty)$ of (7) such that $f(x) = cx^2 + 1$, $x \in (-\delta, \delta)$ for a $c < 0$. Then, in view of Lemma 1, one of the following two cases would have to occur:

(i) there exists an $\alpha \in (0, 1)$ such that $\alpha^2 \leq f(x) \leq 1$ for all $x \in \mathbb{R}$;
(ii) $0 < f(x) \leq 1$ for all $x \in \mathbb{R}$, and $\inf\{f(x) : x \in \mathbb{R}\} = 0$.

CASE (i). The solution f satisfies the assumption of Lemma 4.2^0 with $\delta = +\infty$, and consequently, f would be concave on $\mathbb{R}$. This is impossible because f is bounded and non-constant.

CASE (ii). Since f is continuous and positive on $\mathbb{R}$, there exists a sequence (x_n), $\lim_{n \to +\infty} x_n = +\infty$ such that $\lim_{n \to +\infty} f(x_n) = 0$. Substituting in equation (7): $x = x_n$ and $y = x_n \left(\sqrt{f(x_n)}\right)^{-1}$ we get

$$f(2x_n) + f(0) = 2f(x_n) f\left(\frac{x_n}{\sqrt{f(x_n)}}\right), \qquad n \in \mathbb{N}.$$

Since f is bounded, the sequence on right hand side of this equality converges to 0. Hence $a := \lim_{n \to \infty} f(2x_n)$ exists, and letting $n \to \infty$, we obtain $a + f(0) = 0$, i.e. $a + 1 = 0$, with $a \geq 0$, which is a contradiction.

This remark shows that there is a significant difference between the behaviour of solutions of equation (7) and Gołąb–Schinzel's one.

Applying the same method we can prove the following two results:

THEOREM 3. *Suppose that $f : \mathbb{R} \to \mathbb{R}_+$ is a solution of equation (7). If there is a $\delta > 0$ such that $f(x) \geq 1$ for all $x \in (-\delta, \delta)$ and f is bounded above in a neighbourhood of a point, then there exists a constant $c \geq 0$ such that $f(x) = cx^2 + 1$ for all $x \in \mathbb{R}$.*

THEOREM 4. *Suppose that $f : \mathbb{R} \to (0, \infty)$ is a solution of equation (7). If there exists a finite limit*

$$\lim_{y \to 0} \frac{f(y) - 1}{y^2}$$

then there exists a constant $c \geq 0$ such that $f(x) = cx^2 + 1$, $x \in \mathbb{R}$.

EXAMPLE 4. Let $\mathbf{A} \subset \mathbb{R}$ be the field of algebraic numbers and $c \in \mathbf{A}$, $c \neq 0$, a fixed element. Let $B \subset \mathbb{R}$ be a base of the linear space $(\mathbb{R}, \mathbf{A}, +, \cdot)$. Then there exists a unique homomorphism $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi(x) = c$ for all $x \in B$ (cf. M. Kuczma [16, p. 82, Theorem 1]). Of course ϕ is additive and $\mathbf{A}$-homogeneous, i.e.

$$\phi(tx) = t\phi(x), \qquad t \in \mathbf{A}, \quad x \in \mathbb{R}.$$

Take an arbitrary $x \in \mathbb{R}$. There exist $n \in \mathbb{N}$, $b_1, \ldots, b_n \in B$, $t_1, \ldots, t_n \in \mathbf{A}$ such that $x = t_1 b_1 + \ldots + t_n b_n$. Since $c \in \mathbf{A}$,

$$\phi(x) = \phi(t_1 b_1 + \ldots + t_n b_n) = t_1 \phi(b_1) + \ldots + t_n \phi(b_n) = c(t_1 + \ldots + t_n) \in \mathbf{A},$$

i.e. $\phi(\mathbb{R})$, the range of ϕ, is contained in $\mathbf{A}$.

Define a function $f : \mathbb{R} \to \mathbb{R}_+$ by the formula

$$f(x) := 1 + [\phi(x)]^2, \qquad x \in \mathbb{R}.$$

Since the range of f is contained in $\mathbf{A}$, we have

$$\sqrt{1 + [\phi(x)]^2} \in \mathbf{A}, \qquad x \in \mathbb{R},$$

and, by the $\mathbf{A}$-homogeneity of ϕ,

$$\phi\left(y\sqrt{1 + [\phi(x)]^2}\right) = \sqrt{1 + [\phi(x)]^2}\,\phi(y), \qquad x, y \in \mathbb{R}.$$

Now, making use of the additivity of ϕ, we hence get

$$\begin{aligned}
&f(x+y\sqrt{f(x)})+f(x-y\sqrt{f(x)})\\
&=1+\left[\phi\left(x+y\sqrt{1+[\phi(x)]^2}\right)\right]^2+1+\left[\phi\left(x-y\sqrt{1+[\phi(x)]^2}\right)\right]^2\\
&=2+\left[\phi(x)+\phi\left(y\sqrt{1+[\phi(x)]^2}\right)\right]^2+\left[\phi(x)-\phi\left(y\sqrt{1+[\phi(x)]^2}\right)\right]^2\\
&=2+2[\phi(x)]^2+2\left[\phi\left(y\sqrt{1+[\phi(x)]^2}\right)\right]^2\\
&=2+2[\phi(x)]^2+2\left[\sqrt{1+[\phi(x)]^2}\phi(y)\right]^2\\
&=2\left[1+[\phi(x)]^2+(1+[\phi(x)]^2)[\phi(y)]^2\right]=2(1+[\phi(x)]^2)(1+[\phi(y)]^2)\\
&=2f(x)f(y)
\end{aligned}$$

for all $x, y \in \mathbb{R}$. Thus f is a solution of equation (7). Since ϕ is a discontinuous additive function, the function f is discontinuous at each point. Moreover, the graph of f is dense in the set

$$\{(x,y)\in\mathbb{R}^2:\quad y\geq 1\}.$$

Let us note that

$$\begin{aligned}
f\left(\frac{x+y}{2}\right)&=1+\left[\phi\left(\frac{x+y}{2}\right)\right]^2=1+\frac{1}{4}[\phi(x)+\phi(y)]^2\\
&\leq 1+\frac{1}{4}\left[2[\phi(x)]^2+2[\phi(y)]^2\right]=\frac{f(x)+f(y)}{2},
\end{aligned}$$

which proves that f is Jensen convex in $\mathbb{R}$.

REMARK 9. The above example shows that the functional equation (7) has a lot of very irregular solutions. It proves that the regularity assumptions in Theorems 2, 3 and 4 are indispensable.

REMARK 10. Geometrically, for $c > 0$ the formula (8) describes a family of parabolas with vertical axis, the individual curves being identified by the numerical value of the parameter c. For $c = 0$ we get the non-trivial constant solution $f \equiv 1$. By Remark 8, for $c < 0$, the graph of the local solution consists of a part of parabola $f(x) = 1 + cx^2$ for $x \in (-1/\sqrt{-c},\ 1/\sqrt{-c})$. The graphs of the local solutions cover the whole domain between the non--trivial constant solution $f \equiv 1$ and the trivial one $f \equiv 0$.

It turns out that the requirement of f to have a finite rate of growth at $+\infty$, allows us, without any additional regularity assumption, to characterize all the "regular" solutions of equation (7).

REMARK 11. If $f : \mathbb{R} \to \mathbb{R}_+$ is a solution of equation (7) and there exist $p \in \mathbb{R}$ and $c > 0$ such that

$$\lim_{x \to +\infty} \frac{f(x)}{x^p} = c,$$

then either $p = 2$ or $p = 0$. In fact, writing equation (7) in the form

$$\frac{f(x + y\sqrt{f(x)})}{(x + y\sqrt{f(x)})^p} \left(1 + y\frac{\sqrt{f(x)}}{x}\right)^p + \frac{f(x - y\sqrt{f(x)})}{(x - y\sqrt{f(x)})^p} \left(1 - y\frac{\sqrt{f(x)}}{x}\right)^p = 2\frac{f(x)}{x^p} f(y),$$

we easily infer the conclusion.

Now we prove the following

THEOREM 5. *Suppose that $f : \mathbb{R} \to \mathbb{R}_+$ is a solution of equation* (7).
1^0 *If there exists a $c > 0$ such that*

$$\lim_{x \to \infty} \frac{f(x)}{x^2} = c,$$

then $f(x) = 1 + cx^2$ for all $x \in \mathbb{R}$.
2^0 *If there exists a $c > 0$ such that*

$$\lim_{x \to \infty} f(x) = c,$$

then $c = 1$ and $f(x) \equiv 1$.

PROOF. 1^0 Writing (7) in the form

$$\frac{f(x + y\sqrt{f(x)})}{(x + y\sqrt{f(x)})^2} \left(1 + y\frac{\sqrt{f(x)}}{x}\right)^2 + \frac{f(x - y\sqrt{f(x)})}{(x - y\sqrt{f(x)})^2} \left(1 - y\frac{\sqrt{f(x)}}{x}\right)^2 = 2\frac{f(x)}{x^2} f(y),$$

and letting $x \to \infty$, gives $c(1 + y\sqrt{c})^2 + c(1 - y\sqrt{c})^2 = 2cf(y)$, i.e. $f(y) = 1 + cy^2$ for all $y \in \mathbb{R}$.
2^0 Letting $x \to \infty$ in

$$f\left(x + y\sqrt{f(x)}\right) + f\left(x - y\sqrt{f(x)}\right) = 2f(x)f(y)$$

we get $2c = 2cf(y)$ for all $y \in \mathbb{R}$, i.e. $f \equiv 1$. This completes the proof. □

4. A functional equation for hyperbolas and ellipses. In this part we consider the equation

$$f\left(x+yf(x)\right)^2+f\left(x-yf(x)\right)^2=2f(x)^2f(y)^2, \qquad x,y\in\mathbb{R}, \tag{10}$$

for $f:\mathbb{R}\to\mathbb{R}$. This equation may be interpreted as a modification of d'Alembert's equation, or as an extension of Gołąb–Schinzel's equation. Taking $y=0$ in (10) gives $2f(x)^2=2f(x)^2f(0)^2$ for all $x\in\mathbb{R}$, yielding either $f(0)=0$ or $f(0)=1$ or $f(0)=-1$. Inspection of (10) shows immediately that if $f(0)=0$ then f must be the trivial solution $f\equiv 0$. Moreover there exist (non-trivial) constant solutions $f\equiv 1$ and $f\equiv -1$. Putting $x=0$ in (10) leads to $f(-y)^2=f(y)^2$ for all $y\in\mathbb{R}$, which implies the symmetries

$$f(-y)=f(y) \qquad \text{or} \qquad f(-y)=-f(y), \qquad y\in\mathbb{R}.$$

In a similar way as Theorem 2 one can prove the following

THEOREM 6. *Suppose that $f:\mathbb{R}\to\mathbb{R}$ is a non-trivial solution of equation* (10). *If f is continuous in a neighbourhood of* 0 *then there exists a constant $c\geq 0$ such that either*

$$f(x)=\sqrt{cx^2+1}, \qquad x\in\mathbb{R}, \tag{11}$$

or

$$f(x)=-\sqrt{cx^2+1}, \qquad x\in\mathbb{R}. \tag{12}$$

REMARK 12. For a $c<0$ put $I:=\left(-(-c)^{-1/2},\ (-c)^{-1/2}\right)$. Analogously as in Remark 8 one can observe that for each $c<0$ the functions

$$f(x)=\sqrt{cx^2+1}, \quad x\in I, \qquad \text{and} \qquad f(x)=-\sqrt{cx^2+1}, \quad x\in I,$$

are *local solutions* of equation (10).

Geometrically, for $c>0$, the formulas (11) and (12) describe a family of hyperbolas. Taking $c=0$ one gets the non-trivial constant solutions $f\equiv 1$ and $f\equiv -1$. For $c<0$ we get the local solutions. Their graphs are ellipses. The individual curves being identified by the numerical value of the parameter c. For $c>0$ (*hyperbolas*) and large argument, the solutions tend to their asymptotes: $f(x)\simeq \pm x\sqrt{c}$ for $x\to-\infty$ or $x\to+\infty$.

REMARK 13. Applying Theorem 6 one can easily observe that:

1^0 the general continuous solution $f : \mathbb{R} \to \mathbb{R}_+$, $f \not\equiv 0$, of equation (10) is

$$f(x) = \sqrt{cx^2 + 1}, \qquad x \in \mathbb{R},$$

2^0 the general continuous solution $f : \mathbb{R} \to \mathbb{R}_-$, $f \not\equiv 0$, of equation (10) is

$$f(x) = -\sqrt{cx^2 + 1}, \qquad x \in \mathbb{R}.$$

5. Final remarks. We have introduced some composite functional equations that are related both to Gołąb–Schinzel's equation and to d'Alembert's equation. Theorem 2, the main result of this paper, deals with the functional equation

$$f\left(x + y[f(x)]^p\right) + f\left(x - y[f(x)]^p\right) = 2f(x)f(y), \qquad x, y \in \mathbb{R},$$

for $p = 1/2$. It is not difficult to observe that Lemmas 1, 2, 3 and 4 remain true for this equation when $p > 0$, as well as for some more general functional equation

$$f\left(x + y\phi[f(x)]\right) + f\left(x - y\phi[f(x)]\right) = 2f(x)f(y),$$

where ϕ is a given function satisfying some general conditions. However, the method used in the proof of Theorem 2 is not applicable in the case $0 \neq p \neq 1/2$.

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