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# TRANSFORMING THE FUNCTIONAL EQUATION OF GOŁĄB–SCHINZEL INTO ONE OF CAUCHY

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**Abstract.** It is shown that Gołąb-Schinzel's equation may be transformed into one of Cauchy's equations by an embedding and limit process concerning the general continuous solution.

A meteorological problem of interpolation

$$f(x+h(x,y)) = f(x)g(y), \quad f,g:\mathbf{R}\to\mathbf{R},\ h:\mathbf{R}^2\to\mathbf{R},$$

where $f,g,h$ are functions to be determined, comprises Gołąb-Schinzel's functional equation ([1], [2], [4])

$$f(x+yf(x)) = f(x)f(y) \tag{1}$$

by choosing $g(y)=f(y)$, $h(x,y)=yf(x)$, but also one of Cauchy's functional equations ([1])

$$f(x+y) = f(x)f(y) \tag{2}$$

by specializing $g(y)=f(y)$, $h(x,y)=y$. Now the question may be raised if there exists a connection between (1) and (2) via an embedding.

Replacing in (1) the value $f(x)$ by $f(x)^{1/n}$ where $n$ is an odd positive integer, we obtain a family of modified Gołąb-Schinzel equations with a parameter $n\in 2\mathbf{N}_0+1=\{1,3,5,\dots\}$,

$$f\left(x+yf(x)^{1/n}\right) = f(x)f(y). \tag{$3_n$}$$

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Gołąb-Schinzel's original equation (1) is recovered for $n = 1$. Concerning the meteorological problem, $(3_n)$ is the special case $g(y) = f(y)$, $h(x,y) = yf(x)^{1/n}$. Nicole Brillouët-Belluot's generalization ([3]) of (1)

$$f\left(f(y)^k x + f(x)^l y\right) = F(x, y, f(x), f(y), f(xy)),$$

where $k, l$ are integers and $F$ is a given function, evidently does not comprise $(3_n)$. It is well known from [1, p. 312] that

*a continuous $f : \mathbb{R} \to \mathbb{R}$ is a solution of (1) if, and only if, $f$ has one of the mutually exclusive forms:*

$$f(x) = ax + 1 \qquad (a \in \mathbb{R}), \tag{4}$$

$$f(x) = 0, \tag{5}$$

$$f(x) = \max(ax + 1, 0) \qquad (a \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}). \tag{6}$$

Using this and the fact that a function $f$ solves $(3_n)$ if, and only if, $x \mapsto f(x)^{1/n}$ solves (1) we get

THEOREM 1. *A continuous $f : \mathbb{R} \to \mathbb{R}$ is a solution of $(3_n)$ ($n$ a positive odd integer) if, and only if, $f$ has one of the mutually exclusive forms:*

$$f(x) = (ax + 1)^n \qquad (a \in \mathbb{R}), \tag{$4_n$}$$

$$f(x) = 0, \tag{$5_n$}$$

$$f(x) = \max((ax + 1)^n, 0) \qquad (a \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}). \tag{$6_n$}$$

Now, let us consider a sequence $(f_n)_{n \in 2\mathbb{N}_0+1}$ of continuous solutions $f_n$ of $(3_n)$. Then we may ask for conditions implying that this sequence converges. We want to show that this is the case if, and only if, the limiting function is a continuous solution of (2). Suppose that the sequence $(f_n)$ converges and let

$$f(x) := \lim_{\substack{n \to \infty \\ n \in 2\mathbb{N}_0+1}} f_n(x).$$

Since $f_n(0) \in \{0, 1\}$ and $f_n(0) = 0$ if, and only if, $f_n = 0$, we have $f(0) = 0\,(1)$ if, and only if, $f_n = 0\,(\neq 0)$ for almost all $n$. Thus, excluding the case that $f_n = 0$ for almost all $n$, Theorem 1 gives a sequence $b_n$, $n = 1, 3, \ldots$ of real numbers such that

$$f_n(x) = (b_n x + 1)^n \qquad \text{or} \qquad f_n(x) = \max((b_n x + 1)^n, 0) \tag{7}$$

for almost all $n \in 2\mathbf{N}_0 + 1$. Then by (7) and Bernoulli's inequality, (and since $\max(f_n(x), f_n(-x)) \geq 1$)

$$1 + n|b_n| \leq (1 + |b_n|)^n \leq M, \qquad (n \geq n_0),$$

where (say) $M := \max(f(1), f(-1)) + 1$ $(> 1)$. Thus

$$|b_n| \leq \frac{M-1}{n} \qquad (n \geq n_0). \tag{8}$$

This implies

$$\lim_{\substack{n \to \infty \\ n \in 2\mathbf{N}_0+1}} b_n = 0$$

and (since $|b_n x| < 1$ for $n \geq n(x)$)

$$f(x) = \lim_{\substack{n \to \infty \\ n \in 2\mathbf{N}_0+1}} f_n(x) = \lim_{\substack{n \to \infty \\ n \in 2\mathbf{N}_0+1}} (1 + b_n x)^n$$

for all $x \in \mathbf{R}$. Next we want to show that $f$ is a solution of (2). Fixing $x$ and putting $M' := M - 1$ $(\geq 0)$ we see that for sufficiently large $n$ the value $1 + b_n x$ is positive and that

$$\left(1 - \frac{M'}{n}|x|\right)^n \leq (1 + b_n x)^n \leq \left(1 + \frac{M'}{n}|x|\right)^n.$$

Thus

$$0 < \exp(-M'|x|) \leq f(x) \leq \exp(M'|x|). \tag{9}$$

Put $g_n(x) := (1 + b_n x)^n$ and fix $x, y \in \mathbf{R}$. Then $g_n(x+y) \neq 0$ for almost all $n$. Thus we may form the expression $g_n(x) g_n(y) (g_n(x+y))^{-1} =: (1 + c_n)^n$, where

$$c_n = \frac{b_n^2 xy}{1 + b_n(x+y)}.$$

But $|c_n| \leq M'' n^{-2}$ for some constant $M''$ depending on $x$ and $y$. This implies $\lim_{n\to\infty}(1 + c_n)^n = 1$ as can be seen from the following

**Lemma.** *Let $(a_n)$ be a sequence of real numbers such that $|a_n| \leq Cn^{-2}$ for all $n$. Then $(1 + a_n)^n$ converges to 1 when $n$ tends to $\infty$.*

**Proof.** It is enough to use the estimate

$$\begin{aligned}|n \ln(1 + a_n)| &= |n (a_n - a_n^2/2 + a_n^3/3 - + \dots)| \\ &\leq |na_n|(1 + |a_n| + |a_n|^2 + \dots) \doteq |na_n| \frac{1}{1 - |a_n|}\end{aligned}$$

which is true for large $n$. It clearly shows that $\lim_{n\to\infty} n \ln(1 + a_n) = 0$, as desired. □

Thus $g_n(x)g_n(y)\left(g_n(x+y)\right)^{-1}$ tends to 1 for $n$ tending to $\infty$. But this means that $f$ is a solution of (2). By (9) this solution is bounded from above on (any) bounded interval. This implies ([1, p. 29]) that there is some constant $b$ such that

$$f(x) = \exp(bx) \qquad (x \in \mathbb{R}).$$

Moreover, by using the logarithmic series again, it can be seen that the sequence $(nb_n)$ converges (with $b$ as its limit).

We have the following.

THEOREM 2. *Let $(f_n)_{n\in 2\mathbb{N}_0+1}$ be a sequence of solutions of $(3_n)$. Then the limit*

$$f := \lim_{\substack{n\to\infty \\ n\in 2\mathbb{N}_0+1}} f_n$$

*exists if, and only if, either $f_n = 0$ for almost all $n$ (and $f = 0$) or if there is a sequence $(b_n)_{n\in 2\mathbb{N}_0+1}$ of real numbers such that*

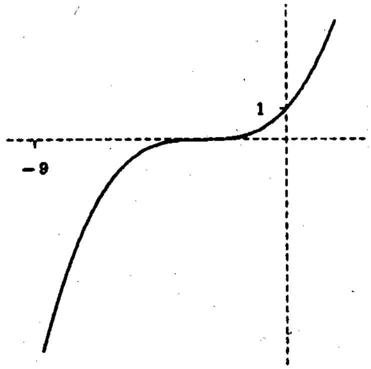
$$b := \lim_{\substack{n\to\infty \\ n\in 2\mathbb{N}_0+1}} nb_n$$

*exists and $f_n$ is of one of the forms* (7) *(and $f(x) = \exp(bx)$ for all $x$).*

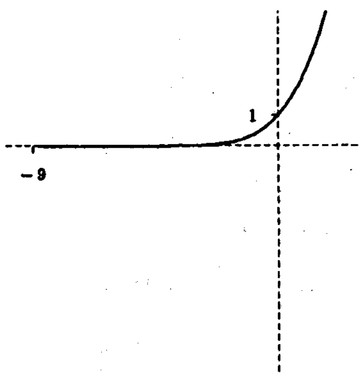
One part of the proof has been given above. The other part is obvious.

Let us point out the following: $(3_1)$ admits continuous nondifferentiable solutions. This is not the case for solutions of $(3_n)$ when $n \geq 3$. Each continuous solution in this case is $(n-1)$-times differentiable. But there are solutions which are not $n$-times differentiable.

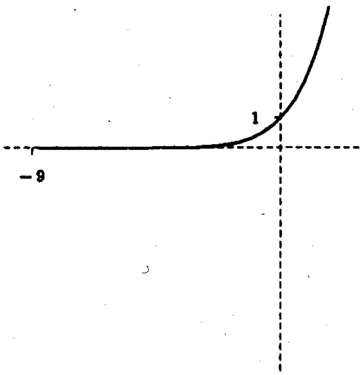
Furthermore we have the following.




$(1+x/3)^3$




$\max(0,(1+x/5)^5)$




$(1+x/7)^7$

Figure 1

REMARK. If, for $p > 0$ and $x \in \mathbb{R}$, we define $x^{[p]} := \operatorname{sign}(x)|x|^p$, we may consider

$$(3_p) \qquad f\left(x + yf(x)^{[1/p]}\right) = f(x)f(y).$$

Then, since $x \mapsto x^{[p]}$ is a strictly increasing power function mapping $\mathbb{R}$ bijectively onto itself and because of $x^{[pq]} = \left(x^{[p]}\right)^{[q]}$ Theorem 1 may be generalized to hold in this situation, too. Theorem 2 also holds in this new setting. In detail we may formulate the following.

THEOREM 1'. *A continuous $f : \mathbb{R} \to \mathbb{R}$ is a solution of $(3_p)$ ($p$ a positive real number) if, and only if, $f$ has one of the mutually exclusive forms:*

$$(4_p) \qquad f(x) = (ax+1)^{[p]} \qquad (a \in \mathbb{R}),$$

$$(5_p) \qquad f(x) = 0,$$

$$(6_p) \qquad f(x) = \max((ax+1)^{[p]}, 0) \qquad (a \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}).$$

THEOREM 2'. *Let $(p_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that $p_n \to \infty$ as $n \to \infty$, and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of solutions of $(3_{p_n})$. Then the limit $f := \lim_{n\to\infty} f_n$ exists if, and only if, either $f_n = 0$ for almost all $n$ (and $f = 0$) or if there is a sequence $(b_n)_{n\in\mathbb{N}}$ of real numbers such that*

$$b := \lim_{n\to\infty} p_n b_n$$

*exists and such that, for almost all $n$, $f_n(x)$ is of one of the forms $(b_n x+1)^{[p_n]}$ or $\max((b_n x + 1)^{[p_n]}, 0)$. Furthermore, in this case, $f$ is given by $f(x) := \exp(bx)$.*

To illustrate the behaviour of certain sequences $(f_n)$ we include two figures. In figures 1 and 2 we have $b_n = 1/n$ and $b_n = 1/n^2$ respectively resulting in the limit functions $\exp(x)$ and 1.

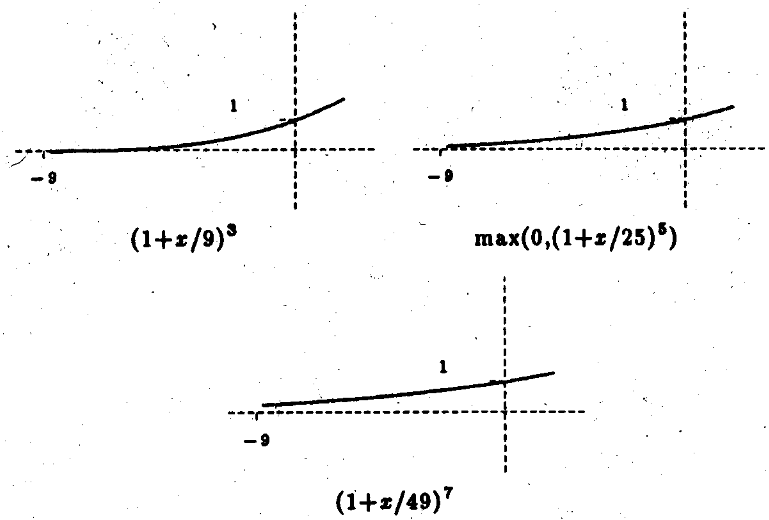



Figure 2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