Annales Mathematicae Silesianae 8. Katowice 1994, 39–58

Prace Naukowe Uniwersytetu Śląskiego nr 1444

AN EQUATION ASSOCIATED WITH THE DISTANCE BETWEEN PROBABILITY DISTRIBUTIONS

PL. KANNAPPAN, P.K. SAHOO AND J.K. CHUNG (ZHONG JUKANG)

Dedicated to Professor Wolfgang Eichhorn on his 60 th birthday

Abstract. In this paper, we solve the functional equation

$$f_1(pr, qs) + f_2(ps, qr) = g(p,q)h(r,s) \qquad (p,g,r,s \in]0,1])$$

where f_1, f_2, g, h are complex-valued functions defined on $]0,1]$. This functional equation is a generalization of a functional equation which was instrumental in the characterization of symmetric divergence of degree α in [3]. This equation arises in the characterization of symmetric weighted divergence of degree α and symmetric inset divergence of degree α.

1. Introduction. Let $\Gamma_n^o = \{P = (p_1, p_2, \dots, p_k) \mid 0 < p_k < 1, \sum_{k=1}^n p_k = 1\}$ denote the set of all n-ary discrete probability distributions, that is, Γ_n^o is the class of discrete distributions on a finite set Ω of cardinality n. For P and Q in Γ_n^o, Kullback and Leibler [8]) (see also [7]) defined the directed divergence as

$$D_n(P\|Q) = \sum_{k=1}^n p_k \log\frac{p_k}{q_k}. \tag{1.1}$$

This measure is nonnegative and attains minimum when $P = Q$. Thus, it serves as a distance measure between the distributions P and Q. It is frequently used in statistics, pattern recognition, coding theory, signal processing and information theory. However, this directed divergence is neither

Received January 3, 1994 and, in final form, Septemer 9, 1994.

AMS (1991) subject classification: Primary 39B32, 94A17.

symmetric nor does it satisfy the triangle inequality and thus its application as a metric is limited. So, in [4] the notion of symmetric divergence between any two probability distributions P and Q in Γ_n^o, was introduced as

$$J_n(P,Q) = D_n(P\|Q) + D_n(Q\|P) \tag{1.2}$$

to restore the symmetry. In explicit form J_n is given by

$$J_n(P,Q) = \sum_{k=1}^{n}(p_k - q_k)\log\frac{p_k}{q_k}. \tag{1.3}$$

The measure (1.3) is called the *J-divergence* in honor of Jeffrey who first used this measure in connection with some estimation problems in [4]. A well known generalization of the J-divergence (see [3]) is the symmetric divergence of degree α and it is given by

$$J_{n,\alpha}(P,Q) = \frac{\sum_{k=1}^{n}(p_k^\alpha q_k^{1-\alpha} + q_k^\alpha p_k^{1-\alpha}) - 2}{2^{1-\alpha} - 1}, \tag{1.4}$$

where $\alpha \neq 1$. The J-divergence of degree α is a one parameter generalization of (1.3) since (1.4) tends to (1.3) as $\alpha \to 1$. This measure satisfies the composition law

$$\begin{aligned} &J_{nm,\alpha}(P\star R,\, Q\star S) + J_{nm,\alpha}(P\star S, Q\star R) \\ &\qquad = 2J_{n,\alpha}(P,Q) + 2J_{m,\alpha}(R,S) + \lambda J_{n,\alpha}(P,Q)J_{m,\alpha}(R,S) \end{aligned} \tag{1.5}$$

for all $P,Q \in \Gamma_n^o$ and $R,S \in \Gamma_m^o$ where

$$P\star R = (p_1r_1,\ldots,p_1r_m,p_2r_1,\ldots,p_2r_m,\ldots,p_nr_1,\ldots,p_nr_m)$$

and $\lambda = 2^{\alpha-1} - 1$. The measure (1.4) was characterized in [3] through the sum property and the composition law (1.5). The functional equation

$$f(pr,qs) + f(ps,qr) = f(p,q)f(r,s) \qquad (p,q,r,s \in]0,1]) \tag{1.6}$$

was instrumental in the characterization of (1.4). In this paper, we solve the functional equation

$$f_1(pr,qs) + f_2(ps,qr) = g(p,q)h(r,s) \qquad (p,q,r,s \in]0,1]), \tag{FE}$$

where f_1, f_2, g, h are complex-valued functions. The solutions of (FE) are obtained via a system of equations

$$F(pr,qs) + F(ps,qr) = g(p,q)h_1(r,s) \qquad (p,q,r,s \in]0,1]), \tag{SE}$$

$$f(pr,qs) - f(ps,qr) = g(p,q)h_2(r,s) \qquad (p,q,r,s \in]0,1]) \tag{DE}$$

obtained from (FE). The equation (FE) is useful in the characterizations of symmetric weighted divergence of degree α and symmetric inset divergence of degree α. For some other functional equations and inequalities related to characterization of distance measures between probabilities distributions see [3], [5] and [6].

2. Notation and terminology. Let I denote the open-closed unit interval $]0,1]$. Let $\mathbb{R}$ and $\mathbb{C}$ denote the set of real numbers and the set of complex numbers, respectively. A map $L : I \to \mathbb{C}$ is called *logarithmic* if and only if $L(xy) = L(x) + L(y)$ for all $x, y \in I$. A function M on I is called *multiplicative* if and only if $M(xy) = M(x)M(y)$ for all $x, y \in I$. For regular solutions of multiplicative or logarithmic Cauchy functional equations the interested reader should refer to [1]. The capital letters M and L along with their subscripts are used exclusively for multiplicative and logarithmic maps, respectively. For a map $f : I \to \mathbb{C}$, the notation $f \neq 0$ means that f is not identically zero on I; "f is nonzero" means $f \neq 0$.

3. Some preliminary results. The following results are needed to establish the main results of this paper.

LEMMA 1 [2]. *The complete list of functions $f, g : I \to \mathbb{C}$ which satisfy*

$$f(xy) = f(x)g(y) + f(y)g(x) \tag{3.1}$$

is the following:

$$f = 0 \quad \text{and} \quad g \text{ arbitrary;} \tag{3.2}$$

$$\begin{cases} f(x) = cL(x)M(x), \\ g(x) = M(x); \end{cases} \tag{3.3}$$

$$\begin{cases} f(x) = c[M_1(x) - M_2(x)], \\ g(x) = \frac{1}{2}[M_1(x) + M_2(x)], \qquad M_1 \neq M_2, \end{cases} \tag{3.4}$$

where c is an arbitrary complex constants, M, M_1, M_2 are arbitrary nonzero multiplicative maps, and L is an arbitrary logarithmic function.

LEMMA 2. *Let $f, g_1, g_2 : I \to \mathbb{C}$ satisfy*

$$f(xy) = g_1(x)f(y) + g_2(y)f(x) \tag{3.5}$$

for all $x, y \in I$. Then f, g_1, g_2 are given by

$$f = 0 \quad \text{and} \quad g_1 \text{ and } g_2 \text{ are arbitrary;} \tag{3.6}$$

$$\begin{cases} f(x) = cM(x)L(x), \\ g_1(x) = M(x) - \alpha cM(x)L(x), \\ g_2(y) = M(y) + \alpha cM(y)L(y), \end{cases} \tag{3.7}$$

$$\begin{cases} f(x) = c[M_1(x) - M_2(x)], \\ g_1(x) = \frac{1}{2}[M_1(x) + M_2(x)] - \alpha c[M_1(x) - M_2(x)], \\ g_2(y) = \frac{1}{2}[M_1(y) + M_2(y)] + \alpha c[M_1(y) - M_2(y)], \end{cases} \tag{3.8}$$

where α ($\neq 0$), c are arbitrary complex constants, M, M_1, M_2 are arbitrary nonzero multiplicative maps, and L is an arbitrary logarithmic function.

PROOF. If $f = 0$, then any arbitrary maps g_1 and g_2 satisfy (3.5) and one obtains the solution (3.6). Henceforth, we suppose $f \neq 0$.

Interchanging x and y in (3.5), we get

$$[g_1(x) - g_2(x)]f(y) = [g_1(y) - g_2(y)]f(x). \tag{3.9}$$

If $g_1 = g_2$, then using Lemma 1 we get (3.7) and (3.8) with $\alpha = 0$. So, we assume that $g_1 \neq g_2$. Then from (3.9) we have

$$f(x) = c_1[g_2(x) - g_1(x)].$$

The constant $c_1 = 0$ implies $f(x) = 0$ which is not the case. So, $c_1 \neq 0$. Let $c_1 = \frac{1}{2\alpha}$ so that

$$g_2(x) = g_1(x) + 2\alpha f(x). \tag{3.10}$$

From (3.10) and (3.5), we obtain

$$f(xy) = g(x)f(y) + g(y)f(x),$$

where

$$g(x) = g_1(x) + \alpha f(x). \tag{3.11}$$

Now from Lemma 1, (3.11) and (3.10), we have the solutions (3.7) and (3.8) and the proof of the lemma is complete. □

Lemma 2 is used repeatedly in Sections 4 and 5.

4. Solution of the equation (SE). In this section, we determine the solution of the functional equation (SE).

THEOREM 1. *The functions $f, F : I^2 \to \mathbb{C}$ satisfy the functional equation*

$$f(pr, qs) + f(ps, qr) = f(p, q)F(r, s) \qquad (p, q, r, s \in I) \tag{4.1}$$

if and only if

$$f = 0 \quad \text{and} \quad F \text{ arbitrary;} \tag{4.2}$$

$$\begin{cases} f(p, q) = M(p)M(q)[\alpha + L(q) - L(p)], \\ F(r, s) = 2M(r)M(s); \end{cases} \tag{4.3}$$

$$\begin{cases} f(p, q) = \alpha M_1(p)M_2(q) + \beta M_1(q)M_2(p), \\ F(r, s) = M_1(r)M_2(s) + M_1(s)M_2(r), \end{cases} \tag{4.4}$$

where α, β are arbitrary complex constants, $M, M_1, M_2 : I \to \mathbb{C}$ are nonzero multiplicative functions, and $L : I \to \mathbb{C}$ is a logarithmic map.

PROOF. First that $f = 0$ implies F is arbitrary. So, we assume from now on $f \neq 0$. Then $F \neq 0$. Interchanging r and s in (4.1), we see that F is symmetric, that is

$$F(r, s) = F(s, r). \tag{4.5}$$

Substituting $p = r = 1$ in (4.1), we have

$$f(s, q) = f(1, q)F(1, s) - f(1, qs). \tag{4.6}$$

Defining

$$g(q) := f(1, q) \quad \text{and} \quad h(s) := F(1, s) \tag{4.7}$$

for $q, s \in I$, and using (4.7) in (4.6), we obtain

$$f(s, q) = g(q)h(s) - g(qs). \tag{4.8}$$

Letting $p = q = 1$ in (4.1), we see that

$$f(r,s) + f(s,r) = aF(r,s) \tag{4.9}$$

where $a := f(1,1) = g(1)$ is a complex constant. Substituting $q = s = 1$ in (4.1) and then using (4.5) and (4.7), we get

$$f(p,r) = f(p,1)h(r) - f(pr,1). \tag{4.10}$$

Putting $q = 1$ in (4.8) and using (4.7), we obtain

$$f(s,1) = ah(s) - g(s). \tag{4.11}$$

Using (4.11) in (4.10) and (4.8), we get

$$f(p,r) = ah(p)h(r) - g(p)h(r) + g(pr) - ah(pr), \tag{4.12}$$

and

$$2g(pr) - ah(pr) = g(r)h(p) + g(p)h(r) - ah(p)h(r). \tag{4.13}$$

Defining

$$\phi(p) := g(p) - \frac{1}{2}ah(p), \tag{4.14}$$

and using (4.14) in (4.13), we have

$$\phi(pr) = \phi(p)\frac{h(r)}{2} + \phi(r)\frac{h(p)}{2} \qquad (p, r \in I). \tag{4.15}$$

The general solutions of (4.15) can be obtained from Lemma 1. Thus, we have the list of solutions:

$$\begin{cases} \phi(p) = 0, \\ h(p) \ \text{ arbitrary}, \end{cases} \tag{4.16}$$

$$\begin{cases} \phi(p) = cM(p)L(p), \\ h(p) = 2M(p), \end{cases} \tag{4.17}$$

$$\begin{cases} \phi(p) = c[M_1(p) - M_2(p)], \\ h(p) = M_1(p) + M_2(p), \quad M_1 \neq M_2, \end{cases} \tag{4.18}$$

where c is an arbitrary constant, $L : I \to \mathbb{C}$ is a logarithmic function, and $M, M_1, M_2 : I - \mathbb{C}$ are multiplicative functions.

Now we consider three cases corresponding to the above solutions.

CASE 1. Consider the solution (4.16). By (4.16) and (4.14), we have

$$g(p) = \frac{1}{2}ah(p). \tag{4.19}$$

SUBCASE 1.1. First we assume $a = 0$. Then (4.19) implies that $g = 0$, and (4.8) yields that $f = 0$, which is not the case. So $a \neq 0$.

SUBCASE 1.2. Next suppose, $a \neq 0$. Then using (4.19) in (4.8), we obtain

$$f(s,q) = \frac{2}{a}g(q)g(s) - g(qs). \tag{4.20}$$

Letting $s = 1$ in (4.1) and using (4.7), (4.19) and (4.20), we get

$$ag(q)g(pr) - a^2g(pqr) = 2g(p)g(q)p(r) - ag(p)g(qr) - ag(r)g(pq). \tag{4.21}$$

For fixed but arbitrary q in I, we define

$$\psi(p) := g(q)g(p) - ag(pq), \tag{4.22}$$

then (4.21) becomes

$$\psi(pr) = \psi(p)\frac{g(r)}{a} + \psi(r)\frac{g(p)}{a}. \tag{4.23}$$

Note that only ψ is dependent on q, and that g is independent of q. The general solution of (4.23) can be obtained from Lemma 1 as

$$\psi(p) = 0 \quad \text{and} \quad g \text{ arbitrary}, \tag{4.24}$$

$$\psi(p) = cL(p)M(p) \quad \text{and} \quad g(p) = aM(p), \tag{4.25}$$

$$\psi(p) = c[M_1(p) - M_2(p)] \quad \text{and} \quad g(p) = a[M_1(p) + M_2(p)] \tag{4.26}$$

for $M_1 \neq M_2$, where c is a function of q. Note the independence of g from q. To prove our assertions we are going to use mostly g. Using $\psi(p) = 0$ in (4.22) we get $ag(pq) = g(p)g(q)$ for some q. Since g is arbitrary, in this case, we can assume that $ag(pq) = g(p)g(q)$ holds for all p and q. Hence, we have

$$g(p) = aM(p), \qquad p \in I. \tag{4.27}$$

Using (4.27) into (4.20), we get

$$f(s,q) = aM(s)M(q). \tag{4.28}$$

Using (4.28) in (4.9), we obtain

$$F(r,s) = 2M(r)M(s). \tag{4.29}$$

This gives the solution (4.3) with $L = 0$.

Inserting (4.25) into (4.22), we get

$$\psi(p) = a^2 M(pq) - a^2 M(pq) = 0,$$

which reduces to the above case with $ag(pq) = g(p)g(q)$ for all p and q, that is, to the solution (4.3) (using only g).

Similarly, from (4.26) and (4.20), we obtain

$$f(p,q) = \frac{a}{2}M_1(p)M_2(q) + \frac{a}{2}M_1(q)M_2(p), \tag{4.30}$$

and then from (4.9), we see that

$$F(r,s) = M_1(r)M_2(s) + M_1(s)M_1(r). \tag{4.31}$$

This leads to the solution (4.4) with $\alpha = \beta = \frac{a}{2}$. Here again we made use of g only.

Case 2. Now we consider the case corresponding to (4.17). From (4.17), we get

$$h(p) = 2M(p) \quad \text{and} \quad g(p) = aM(p) + M(p)L(p) \tag{4.32}$$

after absorbing the constant c with the logarithmic function L. Now letting (4.32) into (4.8), we see that

$$f(p,q) = M(p)M(q)[a + L(q) - L(p)]$$

and this in (4.9) gives

$$F(r,s) = 2M(r)M(s).$$

This yields the solution (4.3).

Case 3. By (4.18) and (4.14), we obtain

$$\begin{cases} h(p) = M_1(p) + M_2(p), \\ g(p) = c[M_1(p) - M_2(p)] + \frac{a}{2}[M_1(p) + M_2(p)] \end{cases} \tag{4.33}$$

for $M_1 \neq M_2$. Letting (4.33) into (4.8), we have

$$f(p,q) = \left(\frac{a}{2} - c\right) M_1(p)M_2(q) + \left(\frac{a}{2} + c\right) M_1(q)M_2(p). \tag{4.34}$$

Putting (4.34) in (4.9) and simplifying, we obtain

$$F(r,s) = M_1(r)M_2(s) + M_1(s)M_2(r). \tag{4.35}$$

Thus, we obtain from (4.34) and (4.35), the asserted solution (4.4) with $\alpha = \frac{a}{2} - c$, $\beta = \frac{a}{2} + c$.

Since there are no more cased left, the proof of the theorem is now complete. □

COROLLARY 1. *The functions $F, g, h_1 : I^2 \to \mathbb{C}$ satisfy the functional equation*

$$F(pr, qs) + F(ps, qr) = g(p,q)h_1(r,s) \qquad (p,q,r,s \in I) \tag{SE}$$

if and only if

$$F = 0, \qquad g = 0, \qquad \text{and} \quad h_1 \quad \text{arbitrary}; \tag{4.36}$$

$$F = 0, \qquad h_1 = 0, \qquad \text{and} \quad g \quad \text{arbitrary}; \tag{4.37}$$

$$\begin{cases} F(p,q) = M(p)M(q)[a + L(q) - L(p)], \\ g(p,q) = \dfrac{2}{b}F(p,q), \\ h_1(r,s) = bM(r)M(s); \end{cases} \tag{4.38}$$

$$\begin{cases} F(p,q) = \alpha M_1(p)M_2(q) + \beta M_1(q)M_2(p), \\ g(p,q) = \dfrac{2}{b}F(p,q), \\ h_1(r,s) = \dfrac{b}{2}[M_1(r)M_2(s) + M_1(s)M_2(r)], \end{cases} \tag{4.39}$$

where $a, b(\neq 0), \alpha, \beta$ are arbitrary constants, $L : I \to \mathbb{C}$ is a logarithmic function, and $M, M_1, M_2 : I \to \mathbb{C}$ are nonzero multiplicative functions.

PROOF. Letting $r = s = 1$ in (SE), we obtain

$$2F(p,q) = h_1(1,1)g(p,q). \tag{4.40}$$

Suppose $h_1(1,1) = 0$. Then from (4.40), we get $F \doteq 0$ and this in (SE) yields the solutions (4.36) and (4.37).

Suppose $h_1(1,1) \neq 0$. Then by (4.40), we have

$$g(p,q) = \frac{2}{b} F(p,q), \tag{4.41}$$

where $b = h_1(1,1)$. Letting (4.41) into (SE), we get

$$F(pr,qs) + F(ps,qr) = \frac{2}{b} F(p,q) h_1(r,s). \tag{4.42}$$

The general solution of (4.42) can be obtained from Theorem 1 and thus, we get the asserted solutions (4.37), (4.38) and (4.39). This completes the proof. □

5. Solution of the equation (DE).

THEOREM 2. *The functions $f, g, h_2 : I^2 \to \mathbb{C}$ satisfy the functional equation*

$$f(pr,qs) - f(ps,qr) = g(p,q) h_2(r,s) \qquad (p,g,r,s \in I) \tag{DE}$$

if and only if

$$\begin{cases} f(p,q) = \psi(pq), \\ g = 0, \\ h_2 \text{ arbitrary}; \end{cases} \tag{5.1}$$

$$\begin{cases} f(p,q) = \psi(pq), \\ g \text{ arbitrary}, \\ h_2 = 0; \end{cases} \tag{5.2}$$

$$\begin{cases} f(p,q) = \psi(pq) - a\alpha \left[\left(\frac{1}{2} - \alpha\beta\right) M_1(p) \right. \\ \qquad\qquad \left. + \left(\frac{1}{2} + \alpha\beta\right) M_2(p) \right] [M_1(q) - M_2(q)], \\ g(p,q) = a \left[\left(\frac{1}{2} - \alpha\beta\right) M_1(p) M_2(q) \right. \\ \qquad\qquad \left. + \left(\frac{1}{2} + \alpha\beta\right) M_1(q) M_2(q) \right], \\ h_2(r,s) = \alpha [M_1(r) M_2(s) - M_1(s) M_2(r)]; \end{cases} \tag{5.3}$$

$$(5.4) \qquad \begin{cases} f(p,q) = \psi(pq) - a\alpha M(p)M(q)[1-\beta L(p)]L(q), \\ g(p,q) = aM(p)M(q)[1-\beta(L(p)-L(q))], \\ h_2(r,s) = \alpha M(r)M(s)[L(r)-L(s)]; \end{cases}$$

$$(5.5) \qquad \begin{cases} f(p,q) = \psi(pq) - c^2\alpha M(p)M(q)L(p)L(q), \\ g(p,q) = c\alpha M(p)M(q)[L(p)-L(q)], \\ h_2(r,s) = cM(r)M(s)[L(r)-L(s)]; \end{cases}$$

$$(5.6) \qquad \begin{cases} f(p,q) = \psi(pq) - c^2\alpha[M_1(p)-M_2(p)][M_1(q)-M_2(q)], \\ g(p,q) = c\alpha[M_1(p)M_2(q)-M_1(q)M_2(p)], \\ h_2(r,s) = c[M_1(r)M_2(s)-M_1(s)M_2(r)], \end{cases}$$

where $\psi : I \to \mathbb{C}$ is an arbitrary function, $L : I \to \mathbb{C}$ is a logarithmic function, and $M, M_1, M_2 : I \to \mathbb{C}$ are the multiplicative maps, and α, β, a, b, c are arbitrary complex constants.

PROOF. If $g = 0$, then (DE) implies

$$(5.7) \qquad f(pr, qs) = f(ps, qr) \qquad (p,q,r,s \in I).$$

Hence, letting $r = q = 1$ in (5.7), we see that

$$f(p,s) = \psi(ps),$$

where $\psi(x) := f(x,1)$ is an arbitrary complex-valued function, and h_2 is arbitrary. This gives the solution (5.1). Similarly, if $h_2 = 0$, we obtain the asserted solution (5.2). From now on we assume that g and h_2 are not identically zero.

Interchanging r with s in (DE), we get

$$(5.8) \qquad f(ps, qr) - f(pr, qs) = g(p,q)h_2(s,r).$$

Adding (5.8) to (DE), we get

$$(5.9) \qquad g(p,q)[h_2(r,s) + h_2(s,r)] = 0.$$

Since $g \neq 0$, from (5.9) we get

$$h_2(r,s) = -h_2(s,r), \tag{5.10}$$

that is, h_2 is skew-symmetric.

Letting $q = s = 1$ in (DE), we obtain

$$f(p,r) = \psi(pr) - g(p,1)h_2(r,1) \tag{5.11}$$

where $\psi(x) := f(x,1)$. Replacing r by xr and s by ys in (DE), we get

$$f(pxr,qys) - f(pys,qxr) = g(p,q)h_2(xr,ys). \tag{5.12}$$

Similarly, replacing p by px and q by qy in (DE), we obtain

$$f(pxr,qys) - f(pxs,qyr) = g(px,qy)h_2(r,s). \tag{5.13}$$

Again, replacing p by ps, q by qr, r by x, and s by y in (DE), we get

$$f(pxs,qyr) - f(psy,qrx) = g(ps,qr)h_2(x,y). \tag{5.14}$$

Adding (5.13) to (5.14) and then applying (5.12) to it, we get

$$g(p,q)h_2(xr,ys) = g(px,qy)h_2(r,s) + g(ps,qr)h_2(x,y) \tag{5.15}$$

for all $p,q,r,s,z,y \in I$. Letting $p = q = 1$ in (5.15), we get

$$ah_2(xr,ys) = g(x,y)h_2(r,s) + g(s,r)h_2(x,y), \tag{5.16}$$

where

$$a := g(1,1). \tag{5.17}$$

CASE 1. Suppose $a \neq 0$. Then (5.16) reduces to

$$h_2(xr,ys) = g_1(x,y)h_2(r,s) + g_1(s,r)h_2(x,y), \tag{5.18}$$

where

$$g_1(x,y) := \frac{g(x,y)}{a}. \tag{5.19}$$

Letting $r = y = 1$ in (5.18), and then using (5.10), we get

$$h_2(x,s) = g_1(s,1)h_2(x,1) - g_1(x,1)h_2(s,1). \tag{5.20}$$

Now (5.11) becomes

$$f(p,r) = \psi(pr) - ag_1(p,1)h_2(r,1). \tag{5.21}$$

Next, substituting $y = s = 1$ in (5.18), we get

$$\phi(xr) = \phi_1(x)\phi(r) + \phi_2(r)\phi(x), \tag{5.22}$$

where

$$\begin{cases} \phi(x) = h_2(x,1), \\ \phi_1(x) = g_1(x,1), \\ \phi_2(x) = g_1(1,x). \end{cases} \tag{5.23}$$

The general solution of (5.22) can be obtained from Lemma 2 as

$$\begin{cases} h_2(x,1) = 0, \\ g_1(x,1) \text{ arbitrary}, \\ g_1(1,x) \text{ arbitrary}; \end{cases} \tag{5.24}$$

$$\begin{cases} h_2(x,1) = \alpha M(x)L(x), \\ g_1(x,1) = M(x) - \beta M(x)L(x), \\ g_1(1,x) = M(x) + \beta M(x)L(x); \end{cases} \tag{5.25}$$

$$\begin{cases} h_2(x,1) = \alpha[M_1(x) - M_2(x)], \\ g_1(x,1) = \dfrac{1}{2}[M_1(x) + M_2(x)] - \alpha\beta[M_1(x) - M_2(x)], \\ g_1(1,x) = \dfrac{1}{2}[M_1(x) + M_2(x)] + \alpha\beta[M_1(x) - M_2(x)]. \end{cases} \tag{5.26}$$

Now we consider several subcases.

Subcase 1.1. From (5.24) and (5.20) we see that $h_2 = 0$. Since $h_2 \neq 0$ by our assumption, this case is not possible.

Subcase 1.2. By (5.25) and (5.20), we get

$$h_2(x,s) = \alpha M(x)M(s)[L(x) - L(s)]. \tag{5.27}$$

Using (5.21) and (5.25), we get

$$f(p,r) = \psi(pr) - a\alpha M(p)M(r)[1 - \beta L(p)]L(r). \tag{5.28}$$

Letting (5.27) and (5.28) into (DE), we obtain

$$g(p,q) - aM(p)M(q)[1-\beta(L(p)-L(q))]. \tag{5.29}$$

Hence we have the solution (5.4).

SUBCASE 1.3. By (5.26) and (5.20), we obtain

$$h_2(x,s) = \alpha[M_1(x)M_2(s) - M_1(s)M_2(x)]. \tag{5.30}$$

Using (5.21) and (5.26), we get

$$\begin{aligned} f(p,r) =& \psi(pr) - a\alpha\left[\left(\frac{1}{2}-\alpha\beta\right)M_1(p)\right.\\ &\left.+\left(\frac{1}{2}+\alpha\beta\right)M_2(p)\right][M_1(r)-M_2(r)]. \end{aligned} \tag{5.31}$$

Using (5.31) and (5.30) in (DE), we get

$$g(p,q) = a\left[\left(\frac{1}{2}-\alpha\beta\right)M_1(p)M_2(q)+\left(\frac{1}{2}+\alpha\beta\right)M_2(p)M_1(q)\right]. \tag{5.32}$$

This gives the solution (5.3). Now case one is complete.

CASE 2. Now we consider the case $a=0$. Putting $p=q=1$ in (DE), we see that

$$f(s,r) = f(r,s) \tag{5.33}$$

and (5.16) gives

$$g(x,y)h_2(r,s) + g(s,r)h_2(x,y) = 0. \tag{5.34}$$

Since $h_2 \neq 0$, (5.34) yields

$$g(x,y) = \alpha h_2(x,y), \tag{5.35}$$

where α is a complex constant. Note that $\alpha \neq 0$, otherwise $g=0$ contrary to our assumption. Letting (5.35) into (5.15), we get

$$h_2(p,q)h_2(xr,ys) = h_2(px,qy)h_2(r,s) + h_2(ps,qr)h_2(x,y), \tag{5.36}$$

where $p,q,r,s,x,y \in I$.

We claim that $h_2(p,1) \neq 0$. Suppose not. Letting $r = y = 1$ in (5.36), we obtain

$$h_2(p,q)h_2(x,s) = h_2(px,q)h_2(1,s) + h_2(ps,q)h_2(x,1).$$

By (5.10), we get

$$h_2(p,q)h_2(x,s) = 0.$$

Hence $h_2 = 0$ contrary to our assumption that $h_2 \neq 0$. Hence $h_2(p,1) \neq 0$. Let $p = p_o \in I$ such that $h_2(p_o,1) \neq 0$. Letting $p = p_o$ and $q = 1$ in (5.36), we obtain

$$h_2(xr,ys) = h_1(x,y)h_2(r,s) + h_1(s,r)h_2(x,y), \tag{5.37}$$

where

$$h_1(x,y) = \frac{h_2(p_o x, y)}{h_2(p_o,1)}. \tag{5.38}$$

Letting $y = s = 1$ in (5.37), we see that

$$\phi(xr) = \phi_1(x)\phi(r) + \phi_2(r)\phi(x), \tag{5.39}$$

where

$$\begin{cases} \phi(x) = h_2(x,1), \\ \phi_1(x) = h_1(x,1), \\ \phi_2(x) = h_1(1,x). \end{cases} \tag{5.40}$$

The general solution of (5.39) can be obtained from Lemma 2. Hence we have

$$\begin{cases} h_2(x,1) = 0, \\ h_1(x,1) \text{ arbitrary}, \\ h_1(1,x) \text{ arbitrary}; \end{cases} \tag{5.41}$$

$$\begin{cases} h_2(x,1) = \alpha M(x)L(x), \\ h_1(x,1) = M(x) - dM(x)L(x), \\ h_1(1,x) = M(x) + dM(x)L(x); \end{cases} \tag{5.42}$$

$$\begin{cases} h_2(x,1) = c[M_1(x) - M_2(x)], \\ h_1(x,1) = \frac{1}{2}[M_1(x) + M_2(x)] - dc[M_1(x) - M_2(x)], \\ h_1(1,x) = \frac{1}{2}[M_1(x) + M_2(x)] + dc[M_1(x) - M_2(x)]. \end{cases} \tag{5.43}$$

Letting $y = r = 1$ in (5.37) and using (5.10), we obtain

$$h_2(x,s) = h_2(x,1)h_1(s,1) - h_2(s,1)h_1(x,1). \tag{5.44}$$

Now we consider three subcases.

SUBCASE 2.1. From (5.41) and (5.44), we get $h_2 = 0$ contrary to our assumption. Hence this case is not possible.

SUBCASE 2.2. By (5.42) and (5.44), we get

$$h_2(x,s) = cM(x)M(s)[L(x) - L(s)] \tag{5.45}$$

and (5.35) gives

$$g(x,y) = \alpha cM(x)M(y)[L(x) - L(y)]. \tag{5.46}$$

Using (5.45) and (5.46) in (5.11), we obtain

$$f(p,r) = \psi(pr) - \alpha c^2 M(p)M(r)L(p)L(r). \tag{5.47}$$

Hence by (5.45) – (5.47), we have the solution (5.5).

SUBCASE 2.3. Finally, from (5.43) and (5.44), we get

$$h_2(x,s) = c[M_1(x)M_2(s) - M_1(s)M_2(x)] \tag{5.48}$$

and from (5.35),

$$g(x,y) = \alpha c[M_1(x)M_2(y) - M_1(y)M_2(x)]. \tag{5.49}$$

Using (5.48) and (5.49) in (5.11), we get

$$f(p,r) = \psi(pr) - c^2\alpha[M_1(p) - M_2(p)][M_1(r) - M_2(r)]. \tag{5.50}$$

Hence we have the asserted solution (5.6).

Since no more cases left, this completes the proof of the theorem. □

6. The main result. In this section, we will present the general complex-valued solution of the functional equation (FE) without assuming any regularity condition on the unknown functions. The reasons for considering (SE) and (DE) in Sections 4 and 5 respectively are the following.

Interchanging r with s in (FE), we obtain

$$f_1(ps,qr) + f_2(pr,qs) = g(p,q)h(s,r). \tag{6.1}$$

Adding (6.1) to (FE), we get

$$F(pr,qs)+F(ps,qr)=g(p,q)h_1(r,s), \tag{SE}$$

where

$$\begin{cases} F(p,q):=f_1(p,q)+f_2(p,q), \\ h_1(r,s):=h(r,s)+h(s,r). \end{cases} \tag{6.2}$$

Similarly, subtracting (6.1) from (FE), we obtain

$$f(pr,qs)-f(ps,qr)=g(p,q)h_2(r,s), \tag{DE}$$

where

$$\begin{cases} f(p,q):=f_1(p,q)-f_2(p,q), \\ h_2(r,s):=h(r,s)-h(s,r). \end{cases} \tag{6.3}$$

The solutions of (SE) and (DE) are already given in Sections 4 and 5, respectively. Hence, by using the solutions of (SE), (DE), (6.2) and (6.3), we determine the solutions of (FE). To obtain the solution of (FE), one has to consider a total of twenty four cases. After, some tedious calculations, we have the following theorem.

THEOREM 3. *The functions $f_1, f_2, g, h : I^2 \to \mathbb{C}$ satisfy the functional equation*

$$f_1(pr,qs)+f_2(ps,qr)=g(p,q)h(r,s) \qquad (p,q,r,s\in]0,1]) \tag{FE}$$

if and only if

$$\begin{cases} f_1(p,q)=\psi(pq), \\ f_2(p,q)=-\psi(pq), \\ h(r,s) \text{ arbitrary}, \\ g(p,q)=0; \end{cases} \tag{6.4}$$

$$\begin{cases} f_1(p,q)=\psi(pq), \\ f_2(p,q)=-\psi(pq), \\ h(r,s)=0, \\ g(p,q) \text{ arbitrary}; \end{cases} \tag{6.5}$$

$$
\text{(6.6)} \qquad \begin{cases} f_1(p,q) = \psi(pq) - c\alpha M(p)M(q)[1-\beta L(p)]L(q), \\ f_2(p,q) = -\psi(pq) + c\alpha M(p)M(q)[1-\beta L(p)]L(q), \\ h(r,s) = \alpha M(r)M(s)[L(r) - L(s)], \\ g(p,q) = cM(p)M(q)[1-\beta(L(p)-L(q))]; \end{cases}
$$

$$
\text{(6.7)} \qquad \begin{cases} f_1(p,q) = \psi(pq) - c^2\alpha M(p)M(q)L(p)L(q), \\ f_2(p,q) = -\psi(pq) + c^2\alpha M(p)M(q)L(p)L(q), \\ h(r,s) = cM(r)M(s)[L(r) - L(s)], \\ g(p,q) = c\alpha M(p)M(q)[L(p) - L(q)]; \end{cases}
$$

$$
\text{(6.8)} \qquad \begin{cases} f_1(p,q) = \psi(pq) + M(p)M(q)[c + L(q) - L(p)], \\ f_2(p,q) = -\psi(pq) + M(p)M(q)[c + L(q) - L(p)], \\ h(r,s) = bM(r)M(s), \\ g(p,q) = \dfrac{2}{b}M(p)M(q)[c + L(q) - L(p)]; \end{cases}
$$

$$
\text{(6.9)} \qquad \begin{cases} f_1(p,q) = \psi(pq) - b\alpha M(p)M(q)[1-\beta L(p)]L(q) \\ \qquad\qquad + bM(p)M(q)\left[\dfrac{1}{\beta} + L(q) - L(p)\right], \\ f_2(p,q) = -\psi(pq) + b\alpha M(p)M(q)[1-\beta L(p)]L(q) \\ \qquad\qquad + bM(p)M(q)\left[\dfrac{1}{\beta} + L(q) - L(p)\right], \\ h(r,s) = \alpha M(r)M(s)\left[\dfrac{2}{b\beta} + \alpha\{L(r) - L(s)\}\right], \\ g(p,q) = bM(p)M(q)[1-\beta(L(p)-L(q))]; \end{cases}
$$

$$
\text{(6.10)} \qquad \begin{cases} f_1(p,q) = \psi(pq) - b^2\alpha M(p)M(q)L(p)L(q) \\ \qquad\qquad + M(p)M(q)[L(q) - L(p)], \\ f_2(p,q) = -\psi(pq) + b^2\alpha M(p)M(q)L(p)L(q) \\ \qquad\qquad + M(p)M(q)[L(q) - L(p)], \\ h(r,s) = M(r)M(s)\left[-\dfrac{2}{\alpha b} + c\{L(r) - L(s)\}\right], \\ g(p,q) = b\alpha M(p)M(q)[L(p) - L(q)]; \end{cases}
$$

$$
(6.11) \qquad \begin{cases} f_1(p,q) = \psi(pq) - \alpha c^2[M_1(p) - M_2(p)][M_1(q) - M_2(q)], \\ f_2(p,q) = -\psi(pq) + \alpha c^2[M_1(p) - M_2(p)][M_1(q) - M_2(q)], \\ h(r,s) = c[M_1(r)M_2(s) - M_1(s)M_2(r)], \\ g(p,q) = c\alpha[M_1(p)M_2(q) - M_1(q)M_2(p)]; \end{cases}
$$

$$
(6.12) \qquad \begin{cases} f_1(p,q) = \psi(pq) + \alpha M_1(p)M_2(q) + \beta M_1(q)M_2(p), \\ f_2(p,q) = -\psi(pq) + \alpha M_1(p)M_2(q) + \beta M_1(q)M_2(p), \\ h(r,s) = \dfrac{b}{2}[M_1(r)M_2(s) + M_1(s)M_2(r)], \\ g(p,q) = \dfrac{2}{b}[\alpha M_1(p)M_2(q) + \beta M_1(q)M_2(p)]; \end{cases}
$$

$$
(6.13) \qquad \begin{cases} f_1(p,q) = \psi(pq) - c^2\alpha[M_1(p) - M_2(p)][M_1(q) - M_2(q)] \\ \qquad\qquad + \dfrac{d}{2}c\alpha[M_1(p)M_2(q) - M_1(q)M_2(p)], \\ f_2(p,q) = -\psi(pq) + c^2\alpha[M_1(p) - M_2(p)][M_1(q) - M_2(q)] \\ \qquad\qquad + \dfrac{d}{2}c\alpha[M_1(p)M_2(q) - M_1(q)M_2(p)], \\ h(r,s) = \left(\dfrac{d}{2} + c\right) M_1(r)M_2(s) + \left(\dfrac{d}{2} - c\right) M_1(s)M_2(r), \\ g(p,q) = c\alpha[M_1(p)M_2(q) - M_1(q)M_2(p)]; \end{cases}
$$

$$
(6.14) \qquad \begin{cases} f_1(p,q) = \psi(pq) - c\alpha \left[\left(\dfrac{1}{2} - \alpha\beta\right) M_1(p) \right. \\ \qquad\qquad \left. + \left(\dfrac{1}{2} + \alpha\beta\right) M_2(p)\right] [M_1(q) - M_2(q)] \\ \qquad\qquad + \left(\dfrac{1}{2} - \alpha\beta\right) M_1(p)M_2(q) + \left(\dfrac{1}{2} + \alpha\beta\right) M_1(q)M_2(p), \\ f_2(p,q) = -\psi(pq) + c\alpha \left[\left(\dfrac{1}{2} - \alpha\beta\right) M_1(p) \right. \\ \qquad\qquad \left. + \left(\dfrac{1}{2} + \alpha\beta\right) M_2(p)\right] [M_1(q) - M_2(q)] \\ \qquad\qquad + \left(\dfrac{1}{2} - \alpha\beta\right) M_1(p)M_2(q) + \left(\dfrac{1}{2} + \alpha\beta\right) M_1(q)M_2(p), \\ h(r,s) = \left(\alpha + \dfrac{1}{c}\right) M_1(r)M_2(s) - \left(\alpha - \dfrac{1}{c}\right) M_1(s)M_2(r), \\ g(p,q) = c\left[\left(\dfrac{1}{2} - \alpha\beta\right) M_1(p)M_2(q) + \left(\dfrac{1}{2} + \alpha\beta\right) M_1(q)M_2(p)\right], \end{cases}
$$

where $\psi : I \to \mathbb{C}$ is an arbitrary function, $L : I \to \mathbb{C}$ is a logarithmic map, $M, M_1, M_2 : I \to \mathbb{C}$ are multiplicative functions, and $\alpha, \beta(\neq 0)$, $b(\neq 0), c, d$ are arbitrary complex constants.

Acknowledgement. The authors are thankful to the referee for several useful remarks which have been incorporated into the paper. This research is partially supported by a grant from Graduate Programs and Research of the University of Louisville.

References

[1] J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, Cambridge University Press, Cambridge, 1989.

[2] J.K. Chung, PL. Kannappan and C.T. Ng, *A generalization of the cosine-sine functional equation on groups*, Linear Algebra and Its Applications **66** (1985), 259-277.

[3] J.K. Chung, PL. Kannappan, C.T. Ng and P.K. Sahoo, *Measures of distance between probability distributions*, J. Math. Anal. Appl. **139** (1989), 280-292.

[4] H. Jeffreys, *An invariant form for the prior probability in estimation problems*, Proc. Roy. Soc. London Ser. A **186** (1946), 453-461.

[5] PL. Kannappan and P.K. Sahoo, *Kullback-Leibler type distance measures between probability distributions*, Jour. Math. Phy. Sci. **26** (1992), 443-454.

[6] PL. Kannappan, P.K. Sahoo and J.K. Chung, *On a functional equation associated with the symmetric divergence measures*, Utilitas Math. **44** (1993), 75-83.

[7] S. Kullback, *Information theory and statistics*, Peter Smith, Gloucester, MA, 1978.

[8] S. Kullback and R.A. Leibler, *On information and sufficiency*, Annals. Math. Statist. **22** (1951), 79-86.

Department of Pure Mathematics
University of Waterloo
Waterloo, Ontario
N2L 3G1, Canada

Department of Mathematics
University of Louisville
Louisville, Kentucky
40292, USA

Department of Applied Mathematics
South China University of Technology
Guangzhou, People's Republic of China