Annales Mathematicae Silesianae 8. Katowice 1994, 69–77

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# ON SOME CHARACTERIZATION OF THE ABSOLUTE VALUE OF AN ADDITIVE FUNCTION

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**Abstract.** Let $G$ be an abelian group, let $\mathbb{K}$ be the real or complex field, let $X$ be a normed space over $\mathbb{K}$, and let $u \in X$ such that $\|u\| = 1$ be given. We assume that there exists a subspace $X_1$ of $X$ such that

$$X = \operatorname{Lin}(u) \oplus X_1$$

and

$$\|\alpha u + x_1\| \geq \max(|\alpha|, \|x_1\|) \qquad \text{for} \quad \alpha \in \mathbb{K}, \quad x_1 \in X_1.$$

Then we prove that the general solution $f : G \to \mathbb{K}$ of the equation

$$f(x+y) + f(x-y) + \|f(x+y) - f(x-y)\| u = 2f(x) + 2f(y) \qquad \text{for} \quad x, y \in G$$

is given by the formula

$$f(x) = |a(x)| u \qquad \text{for} \quad x \in G,$$

where $a : G \to \mathbb{R}$ is an additive function.

Let $G$ be a group. A. Chaljub-Simon and P. Volkmann considered in [1] the functional equation

$$\max\{f(x+y), f(x-y)\} = f(x) + f(y) \qquad \text{for} \quad x, y \in G, \tag{1}$$

where $f$ maps $G$ into $\mathbf{R}$. They proved that if $G$ is abelian then the general solution of (1) is of the form

$$f(x) = |a(x)| \qquad \text{for} \quad x \in G, \tag{2}$$

*Received March 7, 1994.*

AMS (1991) subject classification: Primary 39B52.

where $a : G \to \mathbb{R}$ is an additive function. We are going to generalize this result. Since

$$\max\{x, y\} = \frac{x + y + |x - y|}{2} \qquad \text{for} \quad x, y \in \mathbb{R},$$

equation (1) can be rewritten as

$$(3)\quad f(x+y)+f(x-y)+|f(x+y)-f(x-y)| = 2f(x)+2f(y) \quad \text{for} \quad x, y \in G.$$

Equation (3) may be considered not only for functions of real values but for functions of complex values, as well. We shall generalize equation (3) to admit functions taking values in a normed space. But first we establish some denotations.

$\mathbb{R}$ and $\mathbb{C}$ denote the real or complex field, respectively, and we treat them as normed spaces with the absolute value as the norm. For $x \in \mathbb{C}$ the real part of $x$ is written as $\operatorname{Re} x$. $\mathbb{K}$ stands for the real or complex field. In a vector space by $\operatorname{Lin}(u)$ we denote the subspace generated by $u$. Symbol $x \circ y$ stands for an inner product of $x$ and $y$.

Let $X$ be a (real or complex) normed space and let $u \in X$, $u \neq 0$, be fixed. We consider the following equation

$$(4)\quad f(x+y)+f(x-y)+\|f(x+y)-f(x-y)\|u = 2f(x)+2f(y) \quad \text{for } x, y \in G,$$

where $f$ maps $G$ into $X$. Note that if $X = \mathbb{C}$ and $u = 1$ then (4) becomes (3).

THEOREM 1. *Let $G$ be a group, let $X$ be a normed space over $\mathbb{K}$, and let a $u \in X$ such that $\|u\| = 1$ be fixed. We assume that there exists a subspace $X_1$ of $X$ such that*

$$(5)\qquad X = \operatorname{Lin}(u) \oplus X_1$$

*and*

$$(6)\qquad \|\alpha u + x_1\| \geq \max\{|\alpha|, \|x_1\|\} \qquad \text{for} \quad \alpha \in \mathbb{K}, \quad x_1 \in X_1.$$

*Then the general solution of equation (4) is given by the formula*

$$(7)\qquad f(x) = f_u(x)u \qquad \text{for} \quad x \in G,$$

*where $f_u : G \to \mathbb{R}$ satisfies (3).*

PROOF. Let $f: G \to X$ satisfy (4). Due to (5) we have uniquely determined mappings $f_u : X \to \mathbb{K}$, $g_u : X \to X_1$ such that

$$f(x) = f_u(x)u + g_u(x) \qquad \text{for} \quad x \in G. \tag{8}$$

From (4) we get

$$f_u(x+y) + f_u(x-y) + \|f(x+y) - f(x-y)\| = 2f_u(x) + 2f_u(y) \tag{9}$$

$$\text{for } x, y \in G,$$

$$g_u(x+y) + g_u(x-y) = 2g_u(x) + 2g_u(y) \qquad \text{for} \quad x, y \in G. \tag{10}$$

We are going to prove that $f_u$ takes only real values and next that $g_u = 0$. Inserting into (9) and (10) $x = y = 0$ we conclude that

$$f_u(0) = 0 \tag{11}$$

and

$$g_u(0) = 0, \tag{12}$$

whence by (8)

$$f(0) = 0. \tag{13}$$

Putting in (9) $x = y$ and applying (11) and (13) we obtain

$$f_u(2x) + \|f(2x)\| = 4f_u(x) \qquad \text{for} \quad x \in G. \tag{14}$$

Conditions (6) and (8) yield

$$\|f(x)\| \geq |f_u(x)| \qquad \text{for} \quad x \in G. \tag{15}$$

Now we prove that $f_u$ admits real values only. In the case $\mathbb{K} = \mathbb{C}$ we denote by $f_1, f_2$ the real part and the imaginary part of $f_u$, respectively. Then we obtain from (14)

$$f_1(2x) + \|f(2x)\| = 4f_1(x) \qquad \text{for} \quad x \in G, \tag{16}$$

$$f_2(2x) = 4f_2(x) \qquad \text{for} \quad x \in G. \tag{17}$$

By (15) we have

$$\|f(x)\| \geq |f_u(x)| \geq |f_1(x)| \qquad \text{for} \quad x \in G,$$

and hence in view of (16)

$$4f_1(x) \geq f_1(2x) + |f_1(2x)| \geq 0 \qquad \text{for} \quad x \in G,$$

i.e.

$$f_1(x) \geq 0 \qquad \text{for} \quad x \in G. \tag{18}$$

Making use of (15) again we get

$$\|f(x)\| \geq |f_u(x)| \geq |f_2(x)| \qquad \text{for} \quad x \in G,$$

which together with (16) and (17) yields

$$f_1(2x) + 4|f_2(x)| \leq 4f_1(x) \qquad \text{for} \quad x \in G. \tag{19}$$

This inequality and (18) imply that

$$4(f_1(x) - |f_2(x)|) \geq f_1(2x) \geq 0 \qquad \text{for} \quad x \in G$$

and further that

$$f_1(x) \geq |f_2(x)| \qquad \text{for} \quad x \in G. \tag{20}$$

We put

$$k := \sup\{\alpha \in \mathbb{R} : \ f_1(x) \geq \alpha|f_2(x)| \quad \text{for all} \quad x \in G\}.$$

By (20) $k \geq 1$. If we had $k < \infty$ then, by (17), we would get

$$f_1(2x) \geq k|f_2(2x)| = 4k|f_2(x)| \qquad \text{for} \quad x \in G,$$

and further by (17) and (19)

$$f_1(x) \geq k|f_2(x)| + |f_2(x)| = (k+1)|f_2(x)| \qquad \text{for} \quad x \in G,$$

which contradicts the definition of $k$. Thus $k = \infty$ which means that $f_2 = 0$, i.e. $f_u$ takes real values only.

Now we show that $g_u = 0$. Since $f_2 = 0$, we have by (18)

$$f_u(x) = f_1(x) \geq 0 \qquad \text{for} \quad x \in G. \tag{21}$$

Setting in (10) $x = y$ and applying (12) we get

$$g_u(2x) = 4g_u(x) \qquad \text{for} \quad x \in G. \tag{22}$$

We put

$$p := \sup\{\alpha \in \mathbb{R} : \ f_u(x) \geq \alpha\|g_u(x)\| \quad \text{for all} \quad x \in G\}.$$

With the aid of the subsequential use of (8), (6), (14) and (22) we obtain

$$f_u(2x) + 4\|g_u(x)\| \leq 4f_u(x) \qquad \text{for} \quad x \in G. \tag{23}$$

This inequality and (21) imply that

$$f_u(x) \geq \|g_u(x)\| \qquad \text{for} \quad x \in G,$$

which means that $p \geq 1$. We claim that $p = \infty$. On the contrary we would have by (22) that

$$f_u(2x) \geq 4p\|g_u(x)\| \qquad \text{for} \quad x \in G,$$

and further by (23)

$$f_u(x) \geq (p+1)\|g_u(x)\| \qquad \text{for} \quad x \in G,$$

which is impossible. Hence $p = \infty$, i.e. $g_u = 0$.

Now (7) results from (8). To complete the first part of the proof we need to show yet that $f_u$ satisfies (3). But since $g_u = 0$ and $\|u\| = 1$, it follows directly from (9).

It is obvious that the function $f$ of the form (7) satisfies equation (4). □

Setting in Theorem 1 $X = \mathbb{C}$, $\mathbb{K} = \mathbb{R}$, $u = 1$ and making use of Theorem 1 from [1] we obtain the following

COROLLARY. *Let $G$ be an abelian group. Then the general solution $f : G \to \mathbb{C}$ of equation* (3) *is given by formula* (2).

From Theorem 1 and Corollary we obtain directly the following theorem which is the main result of the paper.

THEOREM 2. *Let $G$ be an abelian group, let $X$ be a normed space over $\mathbb{K}$, and let $u \in X$ such that $\|u\| = 1$ be fixed. We assume that there exists a subspace $X_1$ of $X$ such that* (5) *and* (6) *hold. Then the general solution of equation* (4) *is given by the formula*

$$f(x) = |a(x)|u \qquad \text{for} \quad x \in G, \tag{24}$$

*where $a : G \to \mathbb{R}$ is an additive function.*

The subspace $X_1$ (satisfying (5) and (6)) plays the crucial role in the proof of Theorem 1. Therefore the existence and the uniqueness of such a subspace become important problems. One notices immediately that in a Hilbert space $X$ we may take $X_1 = (\mathrm{Lin}(u))^{\perp}$ - the orthogonal complement of $\mathrm{Lin}(u)$. It occurs that in this case the subspace $X_1$ is determined uniquely by $u$. Suppose, for contradiction, that there exists such a subspace $X_1$ satisfying (5) and (6) that $u \circ x \neq 0$ for some $x \in X_1$. We may assume, multiplying if necessary $x$ by a suitable scalar, that $\mathrm{Re}(u \circ x) < 0$. Then we have for sufficiently small positive real $\alpha$

$$\|u + \alpha x\|^2 = 1 + \alpha^2\|x\|^2 + 2\alpha\ \mathrm{Re}(u \circ x) = 1 + \alpha(\alpha\|x\|^2 + 2\ \mathrm{Re}(u \circ x)) < 1,$$

which contradicts (6).

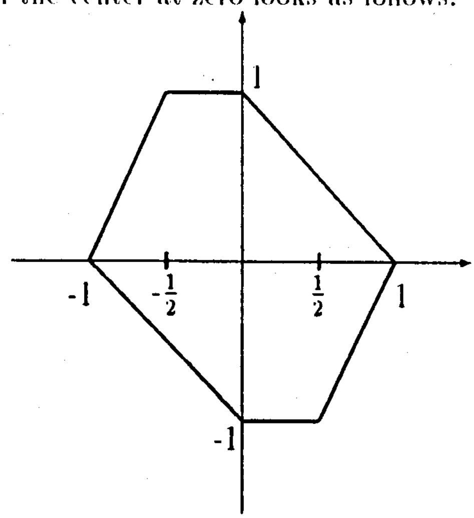
However in general case it may happen that in the same normed space for a given $u$ with $\|u\| = 1$ may be no such a subspace, may be exactly one and may be infinitely many, as well. Before presenting an appropriate example (suggested to me by Jacek Tabor) we rewrite (6) in a more convenient form. Making use of the absolute homogeneity of a norm we notice that condition (6) is equivalent to the following one

$$(25) \qquad \|u + tv\| \geq \max\{1, |t|\} \qquad \text{for} \quad t \in \mathbb{K},\ v \in X_1,\ \|v\| = 1.$$

EXAMPLE 1. Let $\mathbb{K} = \mathbb{R}$, $X = \mathbb{R}^2$ and

$$\|(x_1, x_2)\| := \begin{cases} |x_1| + |x_2| & \text{if} \quad x_1x_2 \geq 0, \\ |x_2| & \text{if} \quad x_1x_2 < 0 \ \text{ and } \ \left|\dfrac{x_1}{x_2}\right| \leq \dfrac{1}{2}, \\ |x_1| + \dfrac{1}{2}|x_2| & \text{if} \quad x_1x_2 < 0 \ \text{ and } \ \left|\dfrac{x_1}{x_2}\right| > \dfrac{1}{2}. \end{cases}$$

The unit circle with the center at zero looks as follows.



Obviously the mapping $||\cdot||$ is absolutely homogeneous. Since, moreover, the unit circle with the center at the origin is convex and contains zero in its interior, $||\cdot||$ is a norm.

Let $u_1 = (1,0)$, $v_1 = (x,1)$, $-\frac{1}{2} < x < 0$. Then we have for $t \in \mathbb{R}$

$$||u_1+tv_1|| = ||(1+tx,t)|| = \begin{cases} 1+(x+1)t & \text{if } 0 \le t \le -\dfrac{1}{x}, \\ |t| & \text{if } t < \dfrac{-1}{x+\frac{1}{2}} \text{ or } t > -\dfrac{1}{x}, \\ 1+\left(x-\dfrac{1}{2}\right)t & \text{if } \dfrac{-1}{x+\frac{1}{2}} \le t < 0. \end{cases}$$

Simple calculation shows that condition (25) is valid.

Now we prove that for $u_2 = \left(\frac{3}{4}, \frac{1}{4}\right)$ there is no such $v$ that $||v|| = 1$ and (25) holds true. Suppose that there exists a $v = (x_1, x_2)$, $||v|| = 1$ such that (25) is valid. Then, for sufficiently small $t \in \mathbb{R}$, we obtain

$$||u_2 + tv|| = \left\|\left(\frac{3}{4} + tx_1,\ \frac{1}{4} + tx_2\right)\right\| = 1 + t(x_1 + x_2).$$

From this equality and (25) we infer that $x_1 + x_2 = 0$, i.e. $x_1 = \frac{2}{3}, x_2 = -\frac{2}{3}$ or $x_1 = -\frac{2}{3}, x_2 = \frac{2}{3}$. Let e.g. $x_1 = \frac{2}{3}, x_2 = -\frac{2}{3}$. Then

$$\left\|u_2 - \frac{9}{8}v\right\| = ||(0,1)|| = 1 < \frac{9}{8},$$

contradictory to (25).

Finally we consider $u_3 = \left(\frac{1}{4}, -1\right)$, $v_3 = (1,0)$. For an arbitrary $t \in \mathbb{R}$, we have

$$||u_3 + tv_3|| = \left\|\left(\frac{1}{4} + t,\ -1\right)\right\| \ge 1.$$

For $t > 1$ or $t < -1$ we obtain, respectively

$$||u_3 + tv_3|| = \left\|\left(\frac{1}{4} + t,\ -1\right)\right\| \ge \frac{1}{4} + t > t = |t|,$$

$$||u_3 + tv_3|| = \left\|\left(\frac{1}{4} + t,\ -1\right)\right\| = -\frac{1}{4} - t + 1 > -t = |t|.$$

Hence (25) is valid.

To prove the uniqueness of $X_1$ (for $u_3$) suppose that there exists a $v = (x_1, x_2)$ with $x_2 \neq 0$ and $\|v\| = 1$ satisfying (25). We consider a $t \in \mathbb{R}$ such that

$$0 < \frac{1}{4} + tx_1 < 1, \quad -1 < -1 + tx_2 < 0, \quad \frac{\frac{1}{4} + tx_1}{1 - tx_2} < \frac{1}{2}.$$

Then

$$\|u_3 + tv\| = \left\| \left( \frac{1}{4} + tx_1, \ -1 + tx_2 \right) \right\| = 1 - tx_2 < 1.$$

This shows that (25) does not hold, and hence that for $u_3 = (\frac{1}{4}, -1)$ the subspace $X_1$ is determined uniquely.

Now we show that the condition $\|u\| = 1$ is an essential assumption in Theorem 1 and Theorem 2.

EXAMPLE 2. Let $\mathbb{K} = \mathbb{R}$, $X = \mathbb{R}^2$ with the Euclidean norm, $G = (\mathbb{R}, +)$, $u = (\sqrt{2}, \sqrt{2})$. We define a function $f_u : \mathbb{R} \to \mathbb{R}$ by the formula

$$f_u(x) := |x| \qquad \text{for} \quad x \in \mathbb{R}.$$

It is obvious that $f_u$ satisfies (3). However function $f$ defined by (7) does not satisfy (4). In fact, for $x = y = 1$

$$f(x+y) + f(x-y) + \|f(x+y) - f(x-y)\|u = (6\sqrt{2}, \ 6\sqrt{2}),$$

but

$$2f(x) + 2f(y) = (4\sqrt{2}, \ 4\sqrt{2}).$$

This proves that the condition $\|u\| = 1$ is an essential assumption in Theorem 1. Furthermore the function $f$ is of the form (24) however it does not satisfy equation (4), which means that the condition $\|u\| = 1$ is also an essential assumption in Theorem 2.

The following problems arising naturally from our considerations seem to be interesting.

PROBLEM 1. Find the general solution of the equation (4) without assuming that $\|u\| = 1$.

To solve Problem 1 one may find useful to get first an answer to the following

PROBLEM 2. Find the general solution of the equation

$$f(x+y) + f(x-y) + |f(x+y) - f(x-y)|c = 2f(x) + 2f(y) \qquad \text{for} \quad x, y \in G,$$

where $f$ maps $G$ into $\mathbb{R}$ or $\mathbb{C}$ and $c$ is a given positive real constant.

PROBLEM 3. Characterize normed spaces $X$ such that for each $u \in X$ with $\|u\| = 1$ there exists a subspace $X_1$ satisfying (5) and (6).

## REFERENCES

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