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ON APPROXIMATE SOLUTIONS OF AN ITERATIVE FUNCTIONAL EQUATION

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Abstract. We consider approximate solutions of the functional equation (1) in the class of functions which satisfy on compact sets the condition (2) with an increasing, subadditive, continuous at zero and vanishing at zero function $\gamma : [0,+\infty) \to [0,+\infty)$.

Continuing the study of approximate solutions of iterative functional equations (see [1; Sections 7.4 and 7.9.8]) we consider here approximate solutions of the functional equation

$$\varphi(x) = h(x, \varphi[f(x)]) \tag{1}$$

in the class of functions which satisfy on compact sets the condition

$$\|\varphi(x) - \varphi(x')\| \leq \gamma(\varrho(x,x')) \tag{2}$$

with an increasing, subadditive, continuous at zero and vanishing at zero function $\gamma : [0,+\infty) \to [0,+\infty)$.

In what follows (X,ϱ) is a metric space and Y is the Banach space $l_\infty(T)$ of all bounded functions $x : T \to \mathbb{R}$, defined on a non–void set T, with the supremum norm.

We start with the following form of E. J. McShane theorem [2; Theorem 2].

PROPOSITION. *Let $\gamma : [0,+\infty) \to [0,+\infty)$ be an increasing and subadditive function vanishing at zero. If U is a subset of X and $\varphi : U \to Y$ satisfies*

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the condition (2) *for all* $x, x' \in U$ *then there exists an extension* $\Phi : X \to Y$ *of* φ *such that*

$$\|\Phi(x) - \Phi(x')\| \leq \gamma(\varrho(x, x')) \quad \textit{for all} \quad x, x' \in X.$$

In fact the Proposition holds for more general spaces than the space $l_\infty(T)$ (see [4; Theorem 13.14]). However for the space $l_\infty(T)$ the proof is both very simple and constructive. Indeed, it is easy to observe that it is enough to consider the case of real-valued functions and then we have an explicit McShane's formula of the extension:

$$\Phi(x) = \sup\{\varphi(u) - \gamma(\varrho(x, u)) : \ u \in U\}$$

(cf. also [4; Theorem 13.16]).

REMARK. Let us observe that if in the Proposition the function φ is bounded by a constant c:

$$\|\varphi(u)\| \leq c \quad \text{for every} \quad u \in U,$$

then we can get an extension $\Phi : X \to Y$ of φ which satisfies (3) and is bounded by c as well. In fact, if an extension Φ obtained from the Proposition has the form $\Phi(x) = (\Phi_t(x))_{t\in T}$ for every $x \in X$, then we can define a new extension $\widetilde{\Phi} : X \to Y$ of φ by the formula

$$\widetilde{\Phi}_t(x) = \begin{cases} -c, & \text{if} \ \ \Phi_t(x) < -c, \\ \Phi_t(x), & \text{if} \ \ |\Phi_t(x)| \leq c, \\ c, & \text{if} \ \ \Phi_t(x) > c, \end{cases}$$

which is bounded by c and satisfies

$$\|\widetilde{\Phi}(x) - \widetilde{\Phi}(x')\| \leq \|\Phi(x) - \Phi(x')\| \leq \gamma(\varrho(x, x')) \quad \text{for all} \quad x, x' \in X.$$

Passing now to approximate solutions of (1) fix a family Γ of self-mappings of $[0, +\infty)$ which are increasing, subadditive, continuous at zero and vanishing at zero. We assume also that Γ fulfils the following condition:

(4) if γ_1 and γ_2 belong to Γ then $\gamma_1 + \gamma_2$ and $\gamma_1 \circ \gamma_2$ belong to Γ.

EXAMPLES. If $\Gamma = \{t \to Lt, \ \ t \in [0, +\infty) \mid L \in [0, +\infty)\}$ then (4) holds. The above assumed conditions are also fulfilled by the family of all increasing, concave and bounded functions $\gamma : [0, +\infty) \to [0, +\infty)$ such that

$$\gamma(t) \leq Lt^\delta \quad \text{for every} \quad t \in [0, +\infty),$$

with some $L \in [0, +\infty)$ and $\delta \in (0, 1]$ (depending on γ). In fact, it is well known (see e.g. [3; pp. 22–23]) that every concave function vanishing at zero is subadditive.

We say that a function $f : X \to X$ is Γ–*Lipschitzian on compacts* iff for every compact set $C \subset X$ there exists a $\gamma \in \Gamma$ such that

$$\varrho(f(x), f(x')) \leq \gamma(\varrho(x, x')) \quad \text{for all } x, x' \in C.$$

Similarly, if $A \subset X$, then we say that a function $\varphi : A \to Y$ is Γ –*Lipschitzian on compacts* iff for every compact set $C \subset A$ there exists a $\gamma \in \Gamma$ such that inequality (2) holds for all $x, x' \in C$. Finally, we say that a function $h : X \times Y \to Y$ is Γ–*Lipschitzian on compacts* iff for every compact set $K \subset X \times Y$ there are $\gamma_1, \gamma_2 \in \Gamma$ such that

$$\|h(x, y) - h(x', y')\| \leq \gamma_1(\varrho(x, x')) + \gamma_2(\|y - y'\|) \quad \text{for all } (x, y), (x', y') \in K.$$

Let us observe that any function which is Γ–Lipschitzian on compacts is necessarily continuous.

Our theorem reads.

THEOREM. *Assume that:*

(i) *functions $f : X \to X$ and $h : X \times Y \to Y$ are Γ–Lipschitzian on compacts;*

(ii) *a point $(\xi, \eta) \in X \times Y$ is given such that $h(\xi, \eta) = \eta$ and*

$$\lim_{n \to \infty} f^n(x) = \xi \quad \textit{for every } x \in X; \tag{5}$$

(iii) *every neighbourhood of ξ contains a neighbourhood U of ξ such that $f(U) \subset U$;*

(iv) *φ_0 is a function defined on a neighbourhood of ξ, taking values in Y, satisfying $\varphi_0(\xi) = \eta$, Γ–Lipschitzian on compacts and such that*

$$\|(\varphi_0(x) - h(x, \varphi_0[f(x)])) - (\varphi_0(x') - h(x', \varphi_0[f(x')]))\| \leq \gamma(\varrho(x, x'))$$

holds with a $\gamma \in \Gamma$ for all x, x' from a neighbourhood of ξ.

Then: for every $\varepsilon > 0$, there exists a function $\varphi : X \to Y$ equal to φ_0 on a neighbourhood of ξ, Γ–Lipschitzian on compacts and such that

$$\|(\varphi(x) - h(x, \varphi[f(x)])) - (\varphi(x') - h(x', \varphi[f(x')]))\| \leq \gamma(\varrho(x, x')) \tag{6}$$

for all $x, x' \in X$, and

$$\|\varphi(x) - h(x, \varphi[f(x)])\| \leq \varepsilon \quad \textit{for every } x \in X. \tag{7}$$

Proof. We can choose a neighbourhood U of ξ such that the function $\chi_0 : U \to Y$ given by the formula

$$\chi_0(x) = \varphi_0(x) - h(x, \varphi_0[f(x)])$$

is well defined and

$$\|\chi_0(x) - \chi_0(x')\| \le \gamma(\varrho(x, x')) \quad \text{for all} \quad x, x' \in U.$$

Since

$$\chi_0(\xi) = \eta - h(\xi, \eta) = 0,$$

there exists a neighbourhood $U_0 \subset U$ of ξ such that

$$\|\chi_0(x)\| \le \varepsilon \quad \text{for every} \quad x \in U_0$$

and $f(U_0) \subset U_0$. Applying now the Proposition and the Remark we get an extension $\chi : X \to Y$ of χ_0 such that

(8) $$\|\chi(x) - \chi(x')\| \le \gamma(\varrho(x, x')) \quad \text{for all} \quad x, x' \in X,$$

(9) $$\|\chi(x)\| \le \varepsilon \quad \text{for every } x \in X.$$

Consider $h_0 : X \times Y \to Y$ defined by the formula

$$h_0(x, y) = \chi(x) + h(x, y).$$

It is clear that h_0 is Γ–Lipschitzian on compacts, $h_0(\xi, \eta) = h(\xi, \eta) = \eta$, and for every $x \in U_0$ we have

$$h_0(x, \varphi_0[f(x)]) = \chi_0(x) + h(x, \varphi_0[f(x)]) = \varphi_0(x).$$

To extend $\varphi_0|_{U_0}$ to a solution $\varphi : X \to Y$ of the equation

(10) $$\varphi(x) = h_0(x, \varphi[f(x)])$$

define the sequence $(U_n : n \in \mathbf{N})$ of open subsets of X by the formula

$$U_n = f^{-1}(U_{n-1}).$$

We have

(11) $$f(U_n) \subset U_{n-1} \subset U_n \quad \text{for every} \quad n \in \mathbf{N}$$

and, due to (5),

$$X = \bigcup_{n=0}^{\infty} U_n. \tag{12}$$

In particular, we may define the sequence $(\varphi_n : n \in \mathbf{N})$ of functions by the formula

$$\varphi_n(x) = h_0(x, \varphi_{n-1}[f(x)]) \quad \text{for} \quad x \in U_n.$$

It is easy to see that φ_n is an extension of φ_{n-1} for every $n \in \mathbf{N}$. This and (12) allow us to define the function $\varphi : X \to Y$ by the formula

$$\varphi(x) = \varphi_n(x) \quad \text{for} \quad x \in U_n \quad \text{and} \quad n \in \mathbf{N}.$$

Of course φ coincides with φ_0 on U_0 and it is a solution of equation (10). Hence

$$\varphi(x) - h(x, \varphi[f(x)]) = \varphi(x) + \chi(x) - h_0(x, \varphi[f(x)]) = \chi(x)$$

for every $x \in X$, which jointly with (8) and (9) gives (6) and (7) respectively. Finally, since each of φ_n, $n \in \mathbf{N} \cup \{0\}$, is Γ–Lipschitzian on compacts and (cf. (11) and (12)) every compact subset of X is contained in a set U_n, the function φ is also Γ–Lipschitzian on compacts. This ends the proof of the theorem. □

EXAMPLE. If $\varphi : [0, \infty) \to \mathbb{R}$ is a solution of the functional equation

$$\varphi(x) = \varphi\left(\frac{x}{1+x}\right) + \sqrt{x} \tag{13}$$

then

$$\varphi(x) = \varphi\left(\frac{x}{1+nx}\right) + \sum_{j=0}^{n-1} \sqrt{\frac{x}{1+jx}}$$

for every $n \in \mathbf{N}$ and $x \in [0, \infty)$, which shows that φ is discontinuous at zero. In other words, (13) has no continuous (at zero) solution $\varphi : [0, +\infty) \to \mathbb{R}$. However, it follows from the Theorem that given $\varepsilon > 0$ there exists a function $\varphi : [0, +\infty) \to \mathbb{R}$ such that:

1. for every $c \in (0, +\infty)$ there exist positive real numbers L and $\delta \leq 1$ such that

$$|\varphi(x) - \varphi(y)| \leq L|x-y|^{\delta} \quad \text{for all} \quad x, y \in [0, c];$$

2. φ vanishes on a (right–sided) neighbourhood of zero;

3. for every $x, y \in [0, +\infty)$ the following inequality holds

$$\left|\left(\varphi(x) - \varphi\left(\frac{x}{1+x}\right) - \sqrt{x}\right) - \left(\varphi(y) - \varphi\left(\frac{y}{1+y}\right) - \sqrt{y}\right)\right| \le \min\{\sqrt{|x-y|}, \varepsilon\};$$

in particular,

$$\left|\varphi(x) - \varphi\left(\frac{x}{1+x}\right) - \sqrt{x}\right| \le \varepsilon \quad \text{for every} \quad x \in [0, +\infty).$$

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