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ON NON–NEGATIVE SOLUTIONS OF A CONVOLUTION EQUATION

WITOLD JARCZYK

Abstract. Some properties of non–negative measurable solutions of equation (1) are studied. The obtained results are stronger versions of those from [6] and their proofs are shorter and simpler.

Given a semigroup $(S,+)$, a solution $\varphi : S \to \mathbb{R}$ of the Cauchy equation

$$\varphi(x+y) = \varphi(x)\varphi(y)$$

and a measure ν on a set $E \subset S$ integrate (if possible) the above equality with respect to y. Then

$$\int\limits_E \varphi(x+y)d\nu(y) = \varphi(x)\int\limits_E \varphi(y)d\nu(y)$$

for every $x \in S$. Thus, assuming that the number $c = \int\limits_E \varphi(y)d\nu(y)$ is positive and finite and putting $\mu = \frac{1}{c}\nu$, we come to the equation

$$\varphi(x) = \int\limits_E \varphi(x+y)d\mu(y). \tag{1}$$

This equation originates from probability, especially from the theory of renewal processes and was intensively studied by many authors starting from G. Choquet and J. Deny [2] in 1960. There is a lot of results giving the form of non–negative solutions of equation (1) in various classes of functions and

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under various assumptions imposed on the semigroup S and the measure μ (cf. [10] and the references therein, also [8] for the infinite-dimensional case).

In [6], trying to find another way of solving equation (1), the author of the present paper proved a result describing a convexity property of its non-negative solutions in a pretty general, purely algebraic setting. This is only a step in the procedure but demands weaker assumptions concerning the semigroup and the solution as usual.

The aim of this paper is to give shorter and simpler proofs of more general versions of the results presented in [6]. The main one (Theorem 1) is an integral counterpart of the following result (see [5, Theorem 1.1]). Its special case was proved by K. Baron and the author in [1].

Denote by $\mathbf{e}_1, \ldots, \mathbf{e}_k$ the canonical zero-one basis of the k-dimensional Euclidean real space. Let $P_1, \ldots, P_k$ be sets of integers satisfying the conditions

$$P_i + 1 \subset P_i, \quad i = 1, \ldots, k, \tag{2}$$

and put $\mathbf{P} = P_1 \times \ldots \times P_k$.

THEOREM. *Let* $A_1, \ldots, A_k$ *be positive reals and let* $\varphi : \mathbf{P} \to \mathbb{R}$ *be a non-negative solution of the equation*

$$\varphi(\mathbf{n}) = \sum_{i=1}^{k} A_i \varphi(\mathbf{n} + \mathbf{e}_i). \tag{3}$$

Then

$$\varphi(\mathbf{n})^2 \leq \varphi(\mathbf{n} - \mathbf{m}) \varphi(\mathbf{n} + \mathbf{m})$$

for every vectors $\mathbf{n} \in \mathbf{P}$ *and* $\mathbf{m} \in \mathbf{Z}^k$ *such that* $\mathbf{n} - \mathbf{m}, \ \mathbf{n} + \mathbf{m} \in \mathbf{P}$.

The Theorem turned out to be very useful in solving quite a lot of problems not only in the theory of functional equations (for some of them see [4] and [5, Chapters II and IV]). In the present paper we are going to make use of it to prove Theorem 1.

Let $(S, +)$ be an Abelian semigroup. Given a non-void set $A \subset S$ denote by $S(A)$ the semigroup (with the neutral element denoted by θ) generated by A:

$$S(A) = \{n_1 a_1 + \ldots + n_k a_k : \ \mathbf{n} \in \mathbf{N}_0^k, \ a_1, \ldots, a_k \in A, \ k \in \mathbf{N}\}.$$

Fix a non-void set $E \subset S$ and assume that the semigroup $S(E)$ is cancellative, i.e.

$$x + z = y + z \quad \text{implies} \quad x = y$$

for every $x, y, z \in S(E)$. Due to a theorem of O. Ore [9] (see also [3, Section 1.10] or [7, Theorem 4.5.2]) it is known that there exists a group $(G(E), +)$ such that $(S(E), +)$ is a subsemigroup of $(G(E), +)$ and

$$G(E) = S(E) - S(E).$$

Moreover, the group $(G(E), +)$ is Abelian which follows almost immediately from the commutativity of $S(E)$.

Let $\mathfrak{M}$ be a σ-algebra of subsets of E and let $\mu : \mathfrak{M} \to [0, \infty]$ be a σ-finite measure. Given a positive integer p denote by $\mathfrak{M}^{\otimes p}$ and $\mu^{\otimes p}$ the σ-products of p copies of $\mathfrak{M}$ and μ, respectively.

Fix a set $X \subset G(E)$ satisfying the condition

$$X + E \subset X. \tag{4}$$

In what follows if $\mathbf{n} \in \mathbb{Z}^k$ then $|\mathbf{n}|$ will stand for the number $n_1 + \ldots + n_k$.

THEOREM 1. *Let* $\varphi : X \to \mathbb{R}$ *be a non-negative solution of equation* (1) *and assume that the function*

$$E^p \ni (e_1, \ldots, e_p) \mapsto \varphi(x + e_1 + \ldots + e_p) \tag{5}$$

is $\mathfrak{M}^{\otimes p}$*-measurable for every* $x \in X$ *and* $p \in \mathbb{N}$.

If k *is a positive integer and* $U_1, \ldots, U_k \in \mathfrak{M}$ *are pairwise disjoint non-void sets then*

$$\Bigg(\int\limits_{U_1^{n_1} \times \ldots \times U_k^{n_k}} \varphi(x + t_1 + \ldots + t_{|\mathbf{n}|}) d\mu^{\otimes|\mathbf{n}|}(t_1, \ldots, t_{|\mathbf{n}|}) \Bigg)^2$$
$$\leq \int\limits_{U_1^{n_1 - m_1} \times \ldots \times U_k^{n_k - m_k}} \varphi(x + t_1 + \ldots + t_{|\mathbf{n}-\mathbf{m}|}) d\mu^{\otimes|\mathbf{n}-\mathbf{m}|}(t_1, \ldots, t_{|\mathbf{n}-\mathbf{m}|})$$
$$\cdot \int\limits_{U_1^{n_1 + m_1} \times \ldots \times U_k^{n_k + m_k}} \varphi(x + t_1 + \ldots + t_{|\mathbf{n}+\mathbf{m}|}) d\mu^{\otimes|\mathbf{n}+\mathbf{m}|}(t_1, \ldots, t_{|\mathbf{n}+\mathbf{m}|})$$

for every $x \in X$ *and vectors* $\mathbf{n} \in \mathbb{N}_0^k$ *and* $\mathbf{m} \in \mathbb{Z}^k$ *such that* $\mathbf{n} - \mathbf{m}$, $\mathbf{n} + \mathbf{m} \in \mathbb{N}_0^k$.

PROOF. Fix a point $x \in X$, a positive integer k, and pairwise disjoint non-void sets $U_1, \ldots, U_k \in \mathfrak{M}$. First assume additionally that $U_1 \cup \ldots \cup U_k = E$. For every $\mathbf{n} \in \mathbb{N}_0^k$ put

$$\psi(\mathbf{n}) = \int\limits_{U_1^{n_1} \times \ldots \times U_k^{n_k}} \varphi(x + t_1 + \ldots + t_{|\mathbf{n}|}) d\mu^{\otimes|\mathbf{n}|}(t_1, \ldots, t_{|\mathbf{n}|}).$$

Since φ is non-negative it follows from (1) that

$$\psi(\mathbf{n}) \leq \varphi(x), \qquad \mathbf{n} \in \mathbb{N}_0^k.$$

Therefore the values of ψ are finite and, evidently, non-negative. Moreover, due to the commutativity of S and by (1), we have

$$\begin{aligned}
&\sum_{i=1}^{k} \psi(\mathbf{n}+\mathbf{e}_i) \\
&= \sum_{i=1}^{k} \int\limits_{U_1^{n_1}\times\ldots\times U_k^{n_k}\times U_i} \varphi(x+t_1+\ldots+t_{|\mathbf{n}|}+t_{|\mathbf{n}|+1})d\mu^{\otimes(|\mathbf{n}|+1)} \\
&\qquad\qquad (t_1,\ldots,t_{|\mathbf{n}|},t_{|\mathbf{n}|+1}) \\
&= \int\limits_{U_1^{n_1}\times\ldots\times U_k^{n_k}} \left(\sum_{i=1}^{k}\int\limits_{U_i} \varphi(x+t_1+\ldots+t_{|\mathbf{n}|}+t)d\mu(t)\right) d\mu^{\otimes|\mathbf{n}|}(t_1,\ldots,t_{|\mathbf{n}|}) \\
&= \int\limits_{U_1^{n_1}\times\ldots\times U_k^{n_k}} \left(\int\limits_{E} \varphi(x+t_1+\ldots+t_{|\mathbf{n}|}+t)d\mu(t)\right) d\mu^{\otimes|\mathbf{n}|}(t_1,\ldots,t_{|\mathbf{n}|}) \\
&= \int\limits_{U_1^{n_1}\times\ldots\times U_k^{n_k}} \varphi(x+t_1+\ldots+t_{|\mathbf{n}|})d\mu^{\otimes|\mathbf{n}|}(t_1,\ldots,t_{|\mathbf{n}|}) = \psi(\mathbf{n})
\end{aligned}$$

for every $\mathbf{n} \in \mathbb{N}_0^k$. So in this case the assertion immediately follows from the Theorem where we take $\mathbf{P} = \mathbb{N}_0^k$.

In the case where $U_1 \cup \ldots \cup U_k \neq E$ it is enough to put $U_{k+1} = E \setminus (U_1 \cup \ldots \cup U_k)$ and apply the part just proved of the theorem to the sets $U_1,\ldots,U_k,U_{k+1}$ and the vectors $(n_1,\ldots,n_k,0)$ and $(m_1,\ldots,m_k,0)$. ☐

Now we are interested in the situation where the semigroup S has a suitably rich topological structure.

REMARK 1. Assume that $(S,+)$ is an Abelian topological semigroup, E treated as a topological subspace of S has a countable base and μ is a σ-finite Borel measure on E.

Since E has a countable base it follows that for every $p \in \mathbb{N}$ the σ-algebra of Borel subsets of E^p coincides with the σ-product of p copies of the σ-algebra of Borel subsets of E. Therefore, if $\varphi : X \to \mathbb{R}$ is such that the function $E \ni e \mapsto \varphi(x+e)$ is Borel measurable then function (5) is product Borel measurable for every $p \in \mathbb{N}$.

In Theorem 2 we shall assume that the set E is additively independent. This means that if k, l are positive integers, $x_1, \dots, x_k, y_1, \dots, y_l \in E$ and $x_1 + \dots + x_k = y_1 + \dots + y_l$ then $k = l$ and there is a permutation π of the set $\{1, \dots, k\}$ such that $y_i = x_{\pi(i)}$ for each $i \in \{1, \dots, k\}$. Clearly the additive independence of E implies the cancellativity of the semigroup $S(E)$.

EXAMPLE ([6]). Fix a non–void set T and consider the set $S = \mathbb{R}^T$ endowed with the usual addition. The set E consisting of all the functions $e_t : T \to \mathbf{Z}$, $t \in T$, given by

$$e_t(u) = \begin{cases} 1 & \text{for} \quad u = t, \\ 0 & \text{for} \quad u \in T \setminus \{t\} \end{cases}$$

is additively independent. Moreover,

$$S(E) = \{x \in \mathbf{N}_0^T : \ \text{the set} \ \ \{t \in T : \ x(t) \neq 0\} \ \text{is finite}\}$$

and

$$G(E) = \{x \in \mathbf{Z}^T : \ \text{the set} \ \ \{t \in T : \ x(t) \neq 0\} \ \text{is finite}\}.$$

In particular, if $T = \{1, \dots, k\}$, where $k \in \mathbf{N}$, then $S(E) = \mathbf{N}_0^k$ and $G(E) = \mathbf{Z}^k$.

Under the assumptions imposed in Remark 1 on S, E, and μ we are going to prove the following result. Here supp μ stands for the support of the measure μ, i.e. the set of all points each neighbourhood of which has a positive measure μ. Observe that, in view of (4),

$$X + S(\operatorname{supp} \mu) \subset X.$$

THEOREM 2. *Assume that the set E is additively independent and each point of* supp μ *has a neighbourhood of finite μ measure. Let $\varphi : X \to \mathbb{R}$ be a non–negative solution of equation* (1) *such that the function $E \ni e \mapsto \varphi(x + e)$ is Borel measurable for every $x \in X$.*

If $x \in X$ then

$$\varphi(x+v)^2 \leq \varphi(x+u)\varphi(x+w) \tag{6}$$

for every $u, v, w \in S(\operatorname{supp} \mu)$ such that $x + u$, $x + v$, $x + w$ are points of continuity of φ and $2v = u + w$.

PROOF. Fix $x \in X$ and points $u, v, w \in S(\operatorname{supp} \mu)$ such that $2v = u + w$ and φ is continuous at $x + u$, $x + v$, and $x + w$. Then

$$u = \sum_{i=1}^{k} p_i s_i, \qquad v = \sum_{i=1}^{k} q_i s_i, \qquad \text{and} \qquad w = \sum_{i=1}^{k} r_i s_i$$

for some $k \in \mathbb{N}$, pairwise different $s_1, \dots, s_k \in \operatorname{supp} \mu$ and $\mathbf{p}, \mathbf{q}, \mathbf{r} \in \mathbb{N}_0^k$. Since

$$\sum_{i=1}^{k} 2q_i s_i = \sum_{i=1}^{k} (p_i + r_i) s_i$$

it follows from the additive independence of E that

$$2q_i = p_i + r_i, \qquad i = 1, \dots, k. \tag{7}$$

For every $l \in \mathbb{N}$ choose pairwise disjoint neighbourhoods $V_{1,l}, \dots, V_{k,l}$ of the points $s_1, \dots, s_k$ such that

$$|\varphi(x + t_1 + \ldots + t_{|\mathbf{p}|}) - \varphi(x + u)| < \frac{1}{l}, \qquad (t_1, \dots, t_{|\mathbf{p}|}) \in V(\mathbf{p}, l), \tag{8a}$$

$$|\varphi(x + t_1 + \ldots + t_{|\mathbf{q}|}) - \varphi(x + v)| < \frac{1}{l}, \qquad (t_1, \dots, t_{|\mathbf{q}|}) \in V(\mathbf{q}, l), \tag{8b}$$

and

$$|\varphi(x + t_1 + \ldots + t_{|\mathbf{r}|}) - \varphi(x + w)| < \frac{1}{l}, \qquad (t_1, \dots, t_{|\mathbf{r}|}) \in V(\mathbf{r}, l), \tag{8c}$$

where

$$V(\mathbf{n}, l) = V_{1,l}^{n_1} \times \ldots \times V_{k,l}^{n_k}, \qquad \mathbf{n} \in \mathbb{N}_0^k.$$

Since $s_1, \dots, s_k \in \operatorname{supp} \mu$ we can additionally assume that

$$0 < \mu(V_{i,l}) < \infty, \qquad i = 1, \dots, k, \quad l \in \mathbb{N}.$$

Thus $\mu^{\otimes|\mathbf{p}|}(V(\mathbf{p}, l))$, $\mu^{\otimes|\mathbf{q}|}(V(\mathbf{q}, l))$, and $\mu^{\otimes|\mathbf{r}|}(V(\mathbf{r}, l))$ are finite and positive numbers for each $l \in \mathbb{N}$.

For every $l \in \mathbb{N}$, by virtue of Theorem 1, Remark 1 and condition (7), we have

$$\begin{aligned} &\left(\frac{1}{\mu^{\otimes|\mathbf{q}|}(V(\mathbf{q}, l))} \int_{V(\mathbf{q},l)} \varphi(x + t_1 + \ldots + t_{|\mathbf{q}|}) d\mu^{\otimes|\mathbf{q}|}(t_1, \dots, t_{|\mathbf{q}|}) \right)^2 \\ &\leq \frac{1}{\mu^{\otimes|\mathbf{p}|}(V(\mathbf{p}, l))} \int_{V(\mathbf{p},l)} \varphi(x + t_1 + \ldots + t_{|\mathbf{p}|}) d\mu^{\otimes|\mathbf{p}|}(t_1, \dots, t_{|\mathbf{p}|}) \\ &\cdot \frac{1}{\mu^{\otimes|\mathbf{r}|}(V(\mathbf{r}, l))} \int_{V(\mathbf{r},l)} \varphi(x + t_1 + \ldots + t_{|\mathbf{r}|}) d\mu^{\otimes|\mathbf{r}|}(t_1, \dots, t_{|\mathbf{r}|}). \end{aligned} \tag{9}$$

If $l \in \mathbb{N}$ then, using (8a), we obtain

$$\left|\frac{1}{\mu^{\otimes|\mathbf{p}|}(V(\mathbf{p},l))}\int\limits_{V(\mathbf{p},l)}\varphi(x+t_1+\ldots+t_{|\mathbf{p}|})d\mu^{\otimes|\mathbf{p}|}(t_1,\ldots,t_{|\mathbf{p}|})-\varphi(x+u)\right|$$
$$\leq\frac{1}{\mu^{\otimes|\mathbf{p}|}(V(\mathbf{p},l))}\int\limits_{V(\mathbf{p},l)}\left|\varphi(x+t_1+\ldots+t_{|\mathbf{p}|})-\varphi(x+u)\right|d\mu^{\otimes|\mathbf{p}|}(t_1,\ldots,t_{|\mathbf{p}|})$$
$$<\frac{1}{l},$$

whence

$$\lim_{l\to\infty}\frac{1}{\mu^{\otimes|\mathbf{p}|}(V(\mathbf{p},l))}\int\limits_{V(\mathbf{p},l)}\varphi(x+t_1+\ldots+t_{|\mathbf{p}|})d\mu^{\otimes|\mathbf{p}|}(t_1,\ldots,t_{|\mathbf{p}|})=\varphi(x+u).$$

Similarly, by (8b) and (8c),

$$\lim_{l\to\infty}\frac{1}{\mu^{\otimes|\mathbf{q}|}(V(\mathbf{q},l))}\int\limits_{V(\mathbf{q},l)}\varphi(x+t_1+\ldots+t_{|\mathbf{q}|})d\mu^{\otimes|\mathbf{q}|}(t_1,\ldots,t_{|\mathbf{q}|})=\varphi(x+v)$$

and

$$\lim_{l\to\infty}\frac{1}{\mu^{\otimes|\mathbf{r}|}(V(\mathbf{r},l))}\int\limits_{V(\mathbf{r},l)}\varphi(x+t_1+\ldots+t_{|\mathbf{r}|})d\mu^{\otimes|\mathbf{r}|}(t_1,\ldots,t_{|\mathbf{r}|})=\varphi(x+w).$$

Consequently, on account of (9), we get inequality (6). □

REMARK 2. It follows from the proof of Theorem 2 that if either $u = \theta$ or $w = \theta$ then the conclusion holds true without the assumption of the continuity of φ at $x+u$ or $x+w$, respectively (that is at x). Moreover, for the validity of the theorem it is enough to know that the points $s_1,\ldots,s_k$ used in the representations of u, v, and w have neighbourhoods of finite measure μ.

REMARK 3. Fix a positive integer k, sets $P_1,\ldots,P_k \subset \mathbb{Z}$ satisfying the conditions (2) and put $\mathbf{P} = P_1\times\ldots\times P_k$. The set $E = \{\mathbf{e}_1,\ldots,\mathbf{e}_k\}$ is an additively independent subset of the group $\mathbb{R}^k$; moreover, $S(E) = \mathbb{N}_0^k$ and $G(E) = \mathbb{Z}^k$ (cf. the Example).

Let $A_1,\ldots A_k$ be positive reals and consider the measure μ defined on 2^E by the formula

$$\mu(\{\mathbf{e}_i\}) = A_i, \quad i = 1,\ldots,k.$$

Clearly $S(\operatorname{supp}\mu) = S(E) = \mathbf{N}_0^k$.

Take a non-negative solution $\varphi : \mathbf{P} \to \mathbb{R}$ of equation (3) and fix vectors $\mathbf{n} \in \mathbf{P}$ and $\mathbf{m} \in \mathbf{Z}^k$ such that $\mathbf{n} - \mathbf{m},\ \mathbf{n} + \mathbf{m} \in \mathbf{P}$. The vector $\mathbf{x}$, defined by

$$x_i = \min\{n_i - m_i,\ n_i + m_i\}, \qquad i = 1, \ldots, k,$$

is an element of $\mathbf{P}$. Moreover, putting $\mathbf{u} = \mathbf{n} - \mathbf{m} - \mathbf{x}$, $\mathbf{v} = \mathbf{n} - \mathbf{x}$, and $\mathbf{w} = \mathbf{n} + \mathbf{m} - \mathbf{x}$, we get $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{N}_0^k = S(\operatorname{supp}\mu)$ and $2\mathbf{v} = \mathbf{u} + \mathbf{w}$. Consequently, by virtue of Theorem 2,

$$\varphi(\mathbf{x} + \mathbf{v})^2 \le \varphi(\mathbf{x} + \mathbf{u})\varphi(\mathbf{x} + \mathbf{w}),$$

i.e.

$$\varphi(\mathbf{n})^2 \le \varphi(\mathbf{n} - \mathbf{m})\varphi(\mathbf{n} + \mathbf{m}).$$

Therefore the Theorem can be deduced from Theorem 2.

The final result deals with the equation

$$\psi(y) = \int_T \psi(f(t, y))d\nu(t) \tag{10}$$

more general than equation (1). Its proof does not differ essentially from that one of [6, Corollary 1].

Given sets Y and T and a function $f : T \times Y \to Y$ we shall write f_t instead of $f(t, \cdot)$ for any $t \in T$.

THEOREM 3. *Let $\mathfrak{A}$ and $\mathfrak{M}$ be σ–algebras of subsets of sets Y and T, respectively, and let $\nu : \mathfrak{M} \to [0, \infty]$ be a σ–finite measure. Assume that $f : T \times Y \to Y$ is such a function that*

$$f_s \circ f_t = f_t \circ f_s, \qquad s, t \in T, \tag{11}$$

and the function

$$T^p \ni (t_1, \ldots, t_p) \mapsto f_{t_1} \circ \ldots \circ f_{t_p}(y)$$

is $\mathfrak{M}^{\otimes p} - \mathfrak{A}$ – measurable for every $y \in Y$ and $p \in \mathbb{N}$.

Let $\psi : Y \to \mathbb{R}$ be a non–negative $\mathfrak{A}$–measurable solution of equation (10). If k is a positive integer and $V_1, \ldots, V_k \in \mathfrak{M}$ are pairwise disjoint non–void sets then

$$\left(\int\limits_{V_1^{n_1} \times \ldots \times V_k^{n_k}} \psi(f_{t_1} \circ \ldots \circ f_{t_{|\mathbf{n}|}}(y)) d\nu^{\otimes|\mathbf{n}|}(t_1, \ldots, t_{|\mathbf{n}|}) \right)^2$$
$$\le \int\limits_{V_1^{n_1 - m_1} \times \ldots \times V_k^{n_k - m_k}} \psi(f_{t_1} \circ \ldots \circ f_{t_{|\mathbf{n}-\mathbf{m}|}}(y)) d\nu^{\otimes|\mathbf{n}-\mathbf{m}|}(t_1, \ldots, t_{|\mathbf{n}-\mathbf{m}|})$$
$$\cdot \int\limits_{V_1^{n_1 + m_1} \times \ldots \times V_k^{n_k + m_k}} \psi(f_{t_1} \circ \ldots \circ f_{t_{|\mathbf{n}+\mathbf{m}|}}(y)) d\nu^{\otimes|\mathbf{n}+\mathbf{m}|}(t_1, \ldots, t_{|\mathbf{n}+\mathbf{m}|})$$

for every $y \in Y$ *and vectors* $\mathbf{n} \in \mathbf{N}_0^k$ *and* $\mathbf{m} \in \mathbf{Z}^k$ *such that* $\mathbf{n} - \mathbf{m}$, $\mathbf{n} + \mathbf{m} \in \mathbf{N}_0^k$.

PROOF. Define the group S and the set E as in the Example and let X be the set of all functions mapping T into $\mathbf{N}_0$ vanishing outside a finite subset of T. Clearly $X \subset G(E)$ and $X + E \subset X$. Since the function $F : T \to E$, given by $F(t) = e_t$, is a bijection, the formula

$$\mu(A) = \nu(F^{-1}(A))$$

defines a σ-finite measure μ on the σ-algebra $\mathfrak{M} = \{A \subset E : \ F^{-1}(A) \in \mathfrak{N}\}$.

Fix a $y \in Y$. For each $x \in X$ there is only a finite number of t's, say $t_1, \ldots, t_l \in T$, such that $x(t) \neq 0$. Then $x(t_1), \ldots, x(t_l) \in \mathbf{N}$, so we may take into account the iterates $f_{t_1}^{x(t_1)}, \ldots, f_{t_l}^{x(t_l)}$. Put

$$\varphi(x) = \psi(f_{t_1}^{x(t_1)} \circ \ldots \circ f_{t_l}^{x(t_l)}(y)).$$

(In the case $l = 0$ this means that $\varphi(x) = \psi(y)$.) The function $\varphi : X \to \mathbb{R}$ is non-negative and the function

$$E^p \ni (e^{(1)}, \ldots, e^{(p)}) \mapsto \varphi(x + e^{(1)} + \ldots + e^{(p)})$$

is $\mathfrak{M}^{\otimes p}$ – measurable for every $x \in X$ and $p \in \mathbf{N}$.

Now fix an $x \in X$ and let $t_1, \ldots, t_l \in T$ be all t's for which $x(t) \neq 0$. Making use of the definition of φ and equalities (10) and (11) we obtain

$$\begin{aligned}\varphi(x) &= \psi(f_{t_1}^{x(t_1)} \circ \ldots \circ f_{t_l}^{x(t_l)}(y)) \\ &= \int_T \psi(f_{t_1}^{x(t_1)} \circ \ldots \circ f_{t_l}^{x(t_l)} \circ f_t(y))d\nu(t) \\ &= \int_T \varphi(x + e_t)d\nu(t) = \int_E \varphi(x + e)d\mu(e).\end{aligned}$$

Thus we have proved that φ satisfies equation (1).

Fix a positive integer k, pairwise disjoint non-void sets $V_1, \ldots, V_k \in \mathfrak{N}$ and vectors $\mathbf{n} \in \mathbf{N}_0^k$ and $\mathbf{m} \in \mathbf{Z}^k$ such that $\mathbf{n} - \mathbf{m}$, $\mathbf{n} + \mathbf{m} \in \mathbf{N}_0^k$. Then, by

Theorem 1,

$$\begin{aligned}
&\left(\int\limits_{V_1^{n_1}\times\ldots\times V_k^{n_k}} \psi(f_{t_1}\circ\ldots\circ f_{t_{|\mathbf{n}|}}(y))\,d\nu^{\otimes|\mathbf{n}|}(t_1,\ldots,t_{|\mathbf{n}|})\right)^2\\
&=\left(\int\limits_{V_1^{n_1}\times\ldots\times V_k^{n_k}} \varphi(e_{t_1}+\ldots+e_{t_{|\mathbf{n}|}})\,d\nu^{\otimes|\mathbf{n}|}(t_1,\ldots,t_{|\mathbf{n}|})\right)^2\\
&=\left(\int\limits_{F(V_1)^{n_1}\times\ldots\times F(V_k)^{n_k}} \varphi(s_1+\ldots+s_{|\mathbf{n}|})\,d\mu^{\otimes|\mathbf{n}|}(s_1,\ldots,s_{|\mathbf{n}|})\right)^2\\
&\le\int\limits_{F(V_1)^{n_1-m_1}\times\ldots\times F(V_k)^{n_k-m_k}} \varphi(s_1+\ldots+s_{|\mathbf{n}-\mathbf{m}|})\,d\mu^{\otimes|\mathbf{n}-\mathbf{m}|}(s_1,\ldots,s_{|\mathbf{n}-\mathbf{m}|})\\
&\quad\cdot\int\limits_{F(V_1)^{n_1+m_1}\times\ldots\times F(V_k)^{n_k+m_k}} \varphi(s_1+\ldots+s_{|\mathbf{n}+\mathbf{m}|})\,d\mu^{\otimes|\mathbf{n}+\mathbf{m}|}(s_1,\ldots,s_{|\mathbf{n}+\mathbf{m}|})\\
&=\int\limits_{V_1^{n_1-m_1}\times\ldots\times V_k^{n_k-m_k}} \psi(f_{t_1}\circ\ldots\circ f_{t_{|\mathbf{n}-\mathbf{m}|}}(y))\,d\nu^{\otimes|\mathbf{n}-\mathbf{m}|}(t_1,\ldots,t_{|\mathbf{n}-\mathbf{m}|})\\
&\quad\int\limits_{V_1^{n_1+m_1}\times\ldots\times V_k^{n_k+m_k}} \psi(f_{t_1}\circ\ldots\circ f_{t_{|\mathbf{n}+\mathbf{m}|}}(y))\,d\nu^{\otimes|\mathbf{n}+\mathbf{m}|}(t_1,\ldots,t_{|\mathbf{n}+\mathbf{m}|})
\end{aligned}$$

which completes the proof. □

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INSTITUTE OF MATHEMATICS
SILESIAN UNIVERSITY
BANKOWA 14
PL-40-007 KATOWICE, POLAND