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ON BOUNDED SOLUTIONS OF A PROBLEM OF R. SCHILLING

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Abstract. It is proved that if

$$0 < q \le (1 - \sqrt[3]{2} + \sqrt[3]{4})/3,$$

then the zero function is the only solution $f:\mathbb{R} \to \mathbb{R}$ of (1) satisfying (2) and bounded in a neighbourhood of at least one point of the set (3).

The paper concerns bounded solutions $f:\mathbb{R} \to \mathbb{R}$ of the functional equation

$$f(qx) = \frac{1}{4q}[f(x-1) + f(x+1) + 2f(x)] \tag{1}$$

such that

$$f(x) = 0 \qquad \text{for} \qquad |x| > Q \tag{2}$$

where q is a fixed number from the open interval $(0,1)$ and

$$Q = \frac{q}{1-q}.$$

In what follows any solution $f:\mathbb{R} \to \mathbb{R}$ of (1) satisfying (2) will be called a *solution of Schilling's problem.* In the present paper we are interested in bounded solutions of Schilling's problem. The first theorem in this direction was obtained by K.Baron in [1]. This theorem reads as follows:

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If $q \in (0, \sqrt{2}-1]$, then the zero function is the only solution of Schilling's problem which is bounded in a neighbourhood of the origin.

This paper generalizes the above theorem in two directions. Namely, the interval $(0, \sqrt{2}-1]$ is replaced by the larger one $(0, \frac{1}{3} - \frac{\sqrt[3]{2}}{3} + \frac{\sqrt[3]{4}}{3}]$ and instead of the boundedness in a neighbourhood of the origin we have boundedness in a neighbourhood of at least one point of the set

$$\left\{ \varepsilon \sum_{i=1}^{n} q^i : \quad n \in \mathbf{N} \cup \{0, +\infty\}, \ \varepsilon \in \{-1, 1\} \right\}. \tag{3}$$

(To simplify formulas we adopt the convention $\sum_{i=1}^{0} a_i = 0$ for all real sequences $(a_i : i \in \mathbf{N})$.) In other words, we shall prove the following.

THEOREM. *If*

$$0 < q \leq \frac{1}{3} - \frac{\sqrt[3]{2}}{3} + \frac{\sqrt[3]{4}}{3}, \tag{4}$$

then the zero function is the only solution of Schilling's problem which is bounded in a neighbourhood of at least one point of the set (3).

The proof of this theorem is based on two lemmas. However, we start with the following simple remarks.

REMARK 1. *If f is a solution of Schilling's problem then so is the function $g : \mathbb{R} \to \mathbb{R}$ defined by the formula $g(x) = f(-x)$.*

REMARK 2. *Assume that f is a solution of Schilling's problem.*
(i) *If $q \neq \frac{1}{4}$, then $f(Q) = 0$. If $q = \frac{1}{4}$, then $f(Q) = 0$ iff $f(qQ) = 0$.*
(ii) *If $q < \frac{1}{2}$, then $f(0) = 0$.*

PROOF. It is enough to put in (1): $x = Q/q$, $x = Q$ and $x = 0$, respectively, and to use condition (2). □

LEMMA 1. *Assume $q \in (0, \frac{1}{2})$. If a solution of Schilling's problem vanishes either on the interval $(-q, 0)$ or on the interval $(0, q)$, then it vanishes everywhere.*

PROOF. Let f be a solution of Schilling's problem vanishing on the interval $[0, q)$. We shall prove that f vanishes on the interval $[0, Q)$. Define a sequence of sets $(A_n : n \in \mathbf{N})$ by the formula

$$A_n = [\, 0 \, , \, \sum_{i=1}^{n} q^i \,).$$

Fix a positive integer n and suppose that f vanishes on the set A_n. We shall show that f vanishes also on the set A_{n+1}. To this end fix an $x_0 \in A_{n+1} \setminus A_n$. Putting $x = x_0/q$ into (1) and taking into account that $x - 1 \in A_n$, whereas $x + 1 > x > 1 > Q$ we get

$$f(x_0) = \frac{1}{4q}[f(x-1) + f(x+1) + 2f(x)] = 0. \tag{5}$$

Consequently, f vanishes on the set $\bigcup_{n=1}^{\infty} A_n$ which equals to $[0, Q)$. This and Remark 2 (i) show that f vanishes on $[0, +\infty)$. Hence and from (1) we infer that f vanishes everywhere.

The case of the interval $(-q, 0)$ reduces to the previous one by using Remark 1. □

LEMMA 2. *Assume $q \in (0, \frac{1}{2})$. If f is a solution of Schilling's problem, then*

$$f\left(q^{m+n}x + \varepsilon \sum_{i=1}^{n} q^i\right) = \left(\frac{1}{2}\right)^n \left(\frac{1}{2q}\right)^{m+n} f(x) \tag{6}$$

for all $x \in (Q-1, 1-Q)$, for all $\varepsilon \in \{-1, 1\}$, and for all non-negative integers m and n.

PROOF. Fix an $x_0 \in (Q - 1, 1 - Q)$. First we shall show that

$$f(q^m x_0) = \left(\frac{1}{2q}\right)^m f(x_0) \tag{7}$$

for all non-negative integers m. Of course (7) holds for $m = 0$. Suppose that (7) holds for an m. Putting $x = q^m x_0$ into (1) and using (2) and (7) we have

$$\begin{aligned} f(q^{m+1}x_0) &= \frac{1}{4q}[f(x-1) + f(x+1) + 2f(x)] = \frac{1}{2q}f(x) \\ &= \left(\frac{1}{2q}\right)^{m+1} \cdot f(x_0). \end{aligned}$$

This proves that (7) holds for all non-negative integers m.

Fix now a non-negative integer n and suppose that (6) is satisfied for all $x \in (Q - 1, 1 - Q)$, for all $\varepsilon \in \{-1, 1\}$, and for all non-negative integers m. Putting $x = q^{m+n}x_0 + \varepsilon \sum_{i=1}^{n} q^i + \varepsilon$ into (1) and applying (2) and (6) with $x = x_0$ we obtain

$$\begin{aligned} f\left(q^{m+n+1}x_0 + \varepsilon \sum_{i=1}^{n+1} q^i\right) &= f(qx) = \frac{1}{4q}[f(x-1) + f(x+1) + 2f(x)] \\ &= \frac{1}{4q}f(x - \varepsilon) = \left(\frac{1}{2}\right)^{n+1} \left(\frac{1}{2q}\right)^{m+n+1} f(x_0). \end{aligned}$$

The proof is completed. □

Now we pass to the proof of the main theorem.

Proof of the theorem. It follows from (4) that $q < 1/2$.

Fix $n \in \mathbf{N} \cup \{0, +\infty\}$ and $\varepsilon \in \{-1, 1\}$ such that a solution f of Schilling's problem is bounded in a neighbourhood of $\varepsilon \sum_{i=1}^{n} q^i$. We may (and we do) assume that n is finite.

If $|x| < 1 - Q$ is fixed, then the left-hand-side of (6) is bounded with respect to m whereas $\lim_{m\to\infty}(1/2q)^{m+n} = +\infty$. This shows that

$$f(x) = 0 \qquad \text{for} \quad |x| < 1 - Q. \tag{8}$$

Consider two cases:

$$q \leq \frac{3-\sqrt{5}}{2} \tag{i}$$

and

$$\frac{3-\sqrt{5}}{2} < q \leq \frac{1}{3} - \frac{\sqrt[3]{2}}{3} + \frac{\sqrt[3]{4}}{3}. \tag{ii}$$

In the case (i) we have $q \leq 1 - Q$ which jointly with (8) and Lemma 1 gives $f = 0$.

So we assume now that (ii) holds. First we notice that putting $x = 1 - Q$ into (1) and applying (8), Remarks 1 and 2(i) and (2) we get

$$0 = f(q(1-Q)) = \frac{1}{4q}[f(-Q) + f(2-Q) + 2f(1-Q)] = \frac{1}{2q}f(1-Q).$$

Hence, from (8) and Remark 1 we obtain

$$f(x) = 0 \qquad \text{for} \quad |x| \leq 1 - Q. \tag{9}$$

Fix an $x_0 \in [qQ, q(2-Q)]$. Putting $x = x_0/q$ into (1) and using (9), (2) and Remark 2 we have (5). Similarly (cf. Remark 1), $f(x) = 0$ for $x \in [-q(2-Q), -qQ]$. Consequently,

$$f(x) = 0 \qquad \text{whenever} \quad qQ \leq |x| \leq q(2-Q). \tag{10}$$

Now we fix an $x_0 \in [q - q^2(2-Q), q^2(2-Q)]$. Putting $x = x_0/q$ into (1), taking into account the inequality $qQ < 1 - q(2-Q)$ and applying (10) and (2) we obtain (5) once again. Similarly $f(x) = 0$ for $x \in [-q^2(2-Q), -q + q^2(2-Q)]$ and so

$$f(x) = 0 \qquad \text{whenever} \quad q - q^2(2-Q) \leq |x| \leq q^2(2-Q). \tag{11}$$

As the function $3t^3-3t^2+3t-1$ increases and vanishes at $(1-\sqrt[3]{2}+\sqrt[3]{4})/3$, we have

$$q-q^2(2-Q)\leq 1-Q. \tag{12}$$

Relations (9), (12) and (11) give

$$f(x)=0 \qquad \text{for} \quad |x|\leq q^2(2-Q). \tag{13}$$

Now let us fix an $x_0\in[1-q(2-Q),1-qQ]$. Putting $x=x_0-1$ into (1) and using (13), (2) and (10) we have

$$0=f(qx)=\frac{1}{4q}[f(x-1)+f(x+1)+2f(x)]=\frac{1}{4q}f(x_0). \tag{14}$$

So we obtain

$$f(x)=0 \qquad \text{whenever} \quad 1-q(2-Q)\leq x\leq 1-qQ. \tag{15}$$

Since (cf. (12)) $q+q^2Q<1-qQ$ and $1-q(2-Q)<q(2-Q)$, (15) proves that

$$f(x)=0 \qquad \text{whenever} \quad q(2-Q)\leq x\leq q+q^2Q. \tag{16}$$

Finally assume that $1-Q\leq x_0\leq qQ$. Putting $x=x_0+1$ into (1) and using (16) and (2) we see that (14) holds. Hence

$$f(x)=0 \qquad \text{whenever} \quad 1-Q\leq x\leq qQ,$$

which jointly with (9) and (10) gives

$$f(x)=0 \qquad \text{whenever} \quad 0\leq x\leq q(2-Q).$$

In particular, since $q<q(2-Q)$, f vanishes on the interval $(0,q)$. This jointly with Lemma 1 completes the proof. □

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References

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