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ON SOME SOLUTIONS OF THE SCHRÖDER EQUATION IN BANACH SPACES

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Abstract. The aim of this paper is to prove results on solutions of the Schröder equation (1) defined on cones in Banach spaces and having some properties connected with monotonicity and boundedness.

We consider the Schröder equation

$$\varphi(f(x)) = \rho\varphi(x) \tag{1}$$

in which φ is an unknown function and the function f is given. In [4; Ch. VI, §4] equation (1) is considered in the real case, among others in the class of functions such that the function "$x \to \varphi(x)/x$" is monotonic (see also [5; section 2.3F]). This class of functions is connected with classes of convex (concave) functions. In the paper we propose to study equation (1) for functions defined on cones in Banach spaces under an assumption which in the real case means that the function "$x \to \varphi(x) - x$" is either monotonic or bounded.

Let $(X, \|\cdot\|)$ be a Banach space and $K \neq \{\theta\}$ be a *closed cone* in X with non empty interior, i.e. (cf. [3; Definition 2.1]), K is a closed subset of X such that $K + K \subset K$, $tK \subset K$ for every $t \geq 0$, $K \cap (-K) = \{\theta\}$ and Int $K \neq \emptyset$. We define a (partial) order $\leq$ on X by

$$x \leq y \iff y - x \in K$$

and note the following simple lemma (cf. [3; p. 208]).

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LEMMA 1. *Suppose $x_n \in X$ for $n \in \mathbb{N}$. If $\lim_{n\to\infty} x_n = \theta$, then for every $a \in \operatorname{Int} K$ there exists an $N \in \mathbb{N}$ such that $x_n \leq a$ for $n \geq N$.*

Let $A : X \to X$ be a *completely continuous linear operator*, i.e. A is linear and maps bounded subsets of X into relatively compact ones. We assume additionally that

$$AK \subset K$$

and for every $x \in K \setminus \{\theta\}$ there exists a positive integer n such that $A^n x \in \operatorname{Int} K$. By the Krein-Rutman theorem [3; Theorem 6.3] there exist exactly one vector $u \in \operatorname{Int} K$ and exactly one continuous linear functional $g : X \to \mathbb{R}$ such that

$$Au = \rho u,$$

$$g(Ax) = \rho g(x), \quad x \in X,$$

$$g(x) > 0, \quad x \in K \setminus \{\theta\},$$

$$\|u\| = 1, \quad g(u) = 1,$$

where ρ denotes the spectral radius of A:

$$\rho = \lim_{n\to\infty} \|A^n\|^{1/n}.$$

Of course $\rho > 0$.

1. Assume

$$\rho \neq 1$$

and let $f : K \to K$ be a function such that $f(\theta) = \theta$ and

$$\lim_{n\to\infty} f^n(x) = \theta, \qquad x \in K. \tag{2}$$

LEMMA 2. *Let $\varphi : K \to \overline{\mathbb{R}}$ be a monotonic solution of (1) such that $\varphi(a) = 0$ for some $a \in \operatorname{Int} K$. If either φ is increasing and $\varphi(\theta) > -\infty$, or φ is decreasing and $\varphi(\theta) < \infty$, then $\varphi = 0$.*

PROOF. We may (and we do) assume, that φ is increasing. Then $\varphi(\theta) \leq \varphi(a) = 0$. In particular $\varphi(\theta)$ is finite. Hence, since φ is a solution of (1), $\varphi(\theta) = 0$. Now, if $x \in K$ is arbitrarily fixed then according to (2) and Lemma 1 there exists a positive integer n such that

$$f^n(x) \leq a$$

whence

$$0 = \rho^n \varphi(\theta) \le \rho^n \varphi(x) = \varphi(f^n(x)) \le \varphi(a) = 0$$

and $\varphi(x) = 0$. □

Arguing similarly we can prove also the following lemma.

LEMMA 3. *Let $\varphi : K \to \overline{\mathbb{R}}$ be a monotonic solution of (1) such that $|\varphi(a)| < \infty$ for some $a \in \operatorname{Int} K$. If either φ is increasing and $\varphi(\theta) > -\infty$, or φ is decreasing and $\varphi(\theta) < \infty$, then φ is finite-valued.*

Denoting

$$A_0 := A|_K, \quad g_0 := g|_K,$$

we have the following result.

THEOREM 1. *Assume that the function f is increasing and $f - A_0$ is monotonic. Then:*

(i) *For every $x \in K$ the sequence*

$$(g(f^n(x))/\rho^n)_{n\in\mathbb{N}} \tag{3}$$

is monotonic and the function $\varphi_0 : K \to [0, \infty]$ given by the formula

$$\varphi_0(x) := \lim_{n\to\infty} \frac{g(f^n(x))}{\rho^n} \tag{4}$$

is an increasing solution of (1).

(ii) *Suppose $f - A_0$ is increasing. Then the function $\varphi_0 - g_0$ is increasing, and if $\varphi : K \to \overline{\mathbb{R}}$ is a solution of (1) such that $\varphi - g_0$ is increasing and $\varphi(\theta) > -\infty$* [resp. *$\varphi - g_0$ is decreasing and $\varphi(\theta) < \infty$*] *then $\varphi_0 \le \varphi$* [resp. *$\varphi \le \varphi_0$*] *and*

$$\varphi(a) = \varphi_0(a) < \infty \qquad \textit{for some} \quad a \in \operatorname{Int} K \ \textit{implies} \quad \varphi = \varphi_0.$$

(iii) *Suppose $f - A_0$ is decreasing. Then φ_0 is finite-valued, the function $\varphi_0 - g_0$ is decreasing and if $\varphi : K \to \overline{\mathbb{R}}$ is a solution of (1) such that $\varphi - g_0$ is increasing and $\varphi(\theta) > -\infty$* [resp. *$\varphi - g_0$ is decreasing and $\varphi(\theta) < \infty$*] *then $\varphi_0 \le \varphi$* [resp. *$\varphi \le \varphi_0$*] *and*

$$\varphi(a) = \varphi_0(a) \qquad \textit{for some} \quad a \in \operatorname{Int} K \ \textit{implies} \quad \varphi = \varphi_0.$$

PROOF. Denote

$$F := f - A_0.$$

Since $F(\theta) = \theta$ and F is monotonic, we have

$$\theta \le F \qquad \text{or} \qquad F \le \theta,$$

i.e.

$$A_0 \le f \qquad \text{or} \qquad f \le A_0.$$

In the first case

$$g(f(x)) \ge g(Ax) = \rho g(x), \qquad x \in K,$$

which shows that for every $x \in K$ the sequence (3) is increasing. In the second case it is a decreasing sequence. Moreover,

$$\varphi_0(f(x)) = \lim_{n\to\infty} \frac{g(f^{n+1}(x))}{\rho^n} = \rho \lim_{n\to\infty} \frac{g(f^{n+1}(x))}{\rho^{n+1}} = \rho\varphi_0(x)$$

for every $x \in K$, i.e. φ_0 is a solution of (1).

Of course the function φ_0 is increasing. Using induction it is easy to check that

$$\text{(5)} \qquad \frac{g(f^n(x))}{\rho^n} = g(x) + \sum_{k=0}^{n-1} \frac{g(F(f^k(x)))}{\rho^{k+1}}, \qquad x \in K, \quad n \in \mathbb{N}.$$

Therefore

$$\sum_{k=0}^{\infty} \frac{g(F(f^k(x)))}{\rho^{k+1}} = \varphi_0(x) - g(x), \qquad x \in K.$$

Consequently, if F is increasing [resp. decreasing] then so is $\varphi_0 - g_0$.

Suppose now that F is increasing and let $\varphi : K \to \overline{\mathbb{R}}$ be a solution of (1) such that $\varphi - g_0$ is increasing and $\varphi(\theta) > -\infty$. Then $g_0 \le \varphi$ and, consequently,

$$\text{(6)} \qquad \frac{g(f^n(x))}{\rho^n} \le \frac{\varphi(f^n(x))}{\rho^n} = \varphi(x), \qquad x \in K, \quad n \in \mathbb{N},$$

whence $\varphi_0 \le \varphi$. Assume now that $\varphi(a) = \varphi_0(a) < \infty$ for some $a \in \operatorname{Int} K$. According to Lemma 3, φ_0 is finite-valued. Denoting

$$\psi := \varphi - g_0$$

we have

$$\frac{\psi(f^n(x))}{\rho^n} = \varphi(x) - \frac{g(f^n(x))}{\rho^n}, \qquad x \in K, \quad n \in \mathbb{N},$$

whence

$$(7)\qquad \lim_{n\to\infty} \frac{\psi(f^n(x))}{\rho^n} = \varphi(x) - \varphi_0(x), \qquad x \in K.$$

In particular, $\varphi - \varphi_0$ is an increasing function. Since it is also a non-negative solution of (1) and vanishes at a, according to Lemma 2 it vanishes everywhere, i.e. $\varphi = \varphi_0$. In the case where $\varphi - g_0$ is decreasing we argue similarly.

Finally suppose that F is decreasing. As we noted, for every $x \in K$ the sequence (3) is then decreasing and, consequently, φ_0 is now finite-valued. Let $\varphi : K \to \overline{\mathbb{R}}$ be a solution of (1) such that $\varphi - g_0$ is increasing. Then (6) holds and this gives $\varphi_0 \le \varphi$. Assume now that $\varphi(a) = \varphi_0(a)$ for some $a \in \operatorname{Int} K$. As previously (cf. in particular (7)) we see that $\varphi - \varphi_0$ is a solution of (1) which is increasing, non-negative and vanishes at a. An application of Lemma 2 gives $\varphi = \varphi_0$ and ends the proof. □

REMARK 1. If $\rho \in (0,\infty)$ and $A : \mathbb{R} \to \mathbb{R}$ is given by the formula $Ax = \rho x$ then the vectors obtained from the Krein-Rutman theorem for this A (and $K = [0,\infty)$) are: $u = 1$, $g = \mathrm{id}_{\mathbb{R}}$. Consequently, we have the following corollary.

COROLLARY 1. *Let $\rho \in (0,1)$ and assume that $f : [0,\infty) \to [0,\infty)$ is an increasing function such that*

$$\lim_{n\to\infty} f^n(x) = 0, \qquad x \in [0,\infty),$$

and the function

$$(8)\qquad "x \longrightarrow f(x) - \rho x, \qquad x \in [0,\infty)"$$

is monotonic. Then:

(i) *For every $x \in [0,\infty)$ the sequence $(f^n(x)/\rho^n)_{n\in\mathbb{N}}$ is monotonic and the function φ_0: $[0,\infty) \to [0,\infty]$ defined by*

$$(9)\qquad \varphi_0(x) := \lim_{n\to\infty} \frac{f^n(x)}{\rho^n}$$

is an increasing solution of (1).

(ii) *Suppose (8) is increasing. Then the function "$x \longrightarrow \varphi_0(x) - x$, $x \in [0,\infty)$" is increasing, and if $\varphi : [0,\infty) \to \overline{\mathbb{R}}$ is a solution of (1) such that*

$$(10)\qquad "x \longrightarrow \varphi(x) - x, \qquad x \in [0,\infty)"$$

is increasing and $\varphi(0) > -\infty$ [resp. (10) *is decreasing and* $\varphi(0) < \infty$] *then* $\varphi_0 \leq \varphi$ [resp. $\varphi \leq \varphi_0$] *and*

$$\varphi(a) = \varphi_0(a) < \infty \qquad \textit{for some} \quad a \in (0,\infty) \;\; \textit{implies} \;\; \varphi = \varphi_0.$$

(iii) *Suppose* (8) *is decreasing. Then* φ_0 *is finite-valued, the function* "$x \longrightarrow \varphi_0(x) - x$, $x \in [0,\infty)$" *is decreasing, and if* $\varphi : [0,\infty) \to \overline{\mathbb{R}}$ *is a solution of* (1) *such that* (10) *is increasing and* $\varphi(0) > -\infty$ [resp. (10) *is decreasing and* $\varphi(0) < \infty$] *then* $\varphi_0 \leq \varphi$ [resp. $\varphi \leq \varphi_0$] *and*

$$\varphi(a) = \varphi(a) \qquad \textit{for some} \quad a \in (0,\infty) \;\; \textit{implies} \;\; \varphi = \varphi_0.$$

REMARK 2. *Let* $\rho \in (0,\infty)$. *If a function* $f : [0,\infty) \to [0,\infty)$ *is convex* [resp. *concave*], $f(0) = 0$ *and* $\rho x \leq f(x)$ [resp. $f(x) \leq \rho x$] *for* $x \in [0,\infty)$, *then the function* "$x \longrightarrow f(x) - \rho x$, $x \in [0,\infty)$" *is increasing* [resp. *decreasing*].

PROOF. Consider the case where f is convex, fix $x, y \in [0,\infty)$ such that $x < y$ and let $t = x/y$. Then

$$f(x) - \rho x = f(ty) - \rho ty \leq tf(y) - \rho ty = t(f(y) - \rho y) \leq f(y) - \rho y.$$

□

REMARK 3. In the real case we have a condition of Seneta (see [5; Theorem 1.3.2]) which guarantees that φ_0 is finite-valued.

The following example shows that the solution φ_0 need be neither convex nor concave.

EXAMPLE 1. The functions

$$-F(x) := \begin{cases} 0 & \text{for} \quad x \in [0,2], \\ \frac{1}{2}x - 1 & \text{for} \quad x \in (2,4], \\ 1 & \text{for} \quad x \in (4,\infty) \end{cases}$$

and

$$f(x) := \frac{1}{2}x + F(x), \qquad x \in [0,\infty),$$

are increasing, and

$$0 \leq f(x) \leq \frac{1}{2}x, \qquad x \geq 0.$$

Since

$$f^n(x) = \begin{cases} 2^{-n}x & \text{for} \quad x \in [0,2], \\ 2^{-n+1} & \text{for} \quad x \in (2,4], \\ 2^{-n}x - 2^{-n+1} & \text{for} \quad x \in (4,6], \end{cases}$$

and $\rho = \frac{1}{2}$ the function $\varphi_0|_{[0,6]}$ given by (9) is of the form

$$\varphi_0(x) = \begin{cases} x & \text{for} \quad x \in [0,2], \\ 2 & \text{for} \quad x \in (2,4], \\ x-2 & \text{for} \quad x \in (4,6]. \end{cases}$$

Hence $\varphi_0|_{[0,6]}$ is neither convex nor concave. (Let us observe even more: the function "$x \longrightarrow \varphi_0(x)/x$, $x \in (0,6]$" is not monotonic.) Consequently also φ_0 is neither convex nor concave.

Our next example shows that equation (1) may have a lot of solutions φ such that the function $\varphi - g_0$ is increasing.

EXAMPLE 2. Let $\rho \in (0,1)$ and let $f(x) = \rho x$, $x \geq 0$. Then $\varphi_0(x) = x$, $x \geq 0$. Using a standard argument (see, e.g., [5; the proof of Theorem 2.2.3]) it is easy to prove that if $a > 0$ and $\tilde{\varphi} : [\rho a, a] \to \mathbb{R}$ is a function such that the function "$x \longrightarrow \tilde{\varphi}(x) - x$, $x \in [\rho a, a]$" is increasing,

$$\tilde{\varphi}(\rho a) = \rho\tilde{\varphi}(a)$$

and

$$x \leq \tilde{\varphi}(x), \qquad x \in [\rho a, a],$$

then there exists exactly one solution $\varphi : [0,\infty) \to \mathbb{R}$ of (1) such that $\varphi|_{[\rho a,a]} = \tilde{\varphi}$; moreover the function "$x \longrightarrow \varphi(x) - x$, $x \in [0,\infty)$" is increasing. In particular, there are solutions $\varphi_1, \varphi_2 : [0,\infty) \to \mathbb{R}$ of (1) such that $\varphi_1(a) = \varphi_2(a)$, functions "$x \longrightarrow \varphi_i(x) - x$, $x \in [0,\infty)$", $i \in \{0,1\}$, are increasing, but $\varphi_1 \neq \varphi_2$.

2. Now we pass to solutions φ of (1) such that $\varphi - g_0$ is bounded. Let $f : K \to K$ be an arbitrary function.

THEOREM 2. *Assume $\rho > 1$ and let $f - A_0$ be bounded. Then:*
(i) *For every $x \in K$ the sequence (3) converges.*
(ii) *The function $\varphi_0 : K \to [0,\infty)$ given by the formula (4) is a non-zero solution of (1) such that $\varphi_0 - g_0$ is bounded.*
(iii) *If $\varphi : K \to \mathbb{R}$ is a solution of (1) such that for some $\eta \in \mathbb{R}$ the function $\varphi - \eta g_0$ is bounded then $\varphi = \eta\varphi_0$.*

PROOF. Putting $F := f - A_0$ and taking into account boundedness of this function we infer that the series

$$\sum_{k=0}^{\infty} \frac{g \circ F \circ f^k}{\rho^{k+1}}$$

uniformly and absolutely converges and its sum is a bounded function. Hence and from (5) it follows that for every $x \in K$ the sequence (3) converges and $\varphi_0 - g_0$ is a bounded function. In particular, $\varphi_0 \neq 0$.

Assume now that $\varphi : K \to \mathbb{R}$ is a solution of (1) such that the function $\chi := \varphi - \eta g_0$ is bounded (by a constant M). Then

$$\left\| \varphi(x) - \eta \frac{g(f^n(x))}{\rho^n} \right\| = \left\| \frac{\chi(f^n(x))}{\rho^n} \right\| \leq M \frac{1}{\rho^n}, \qquad x \in K, \quad n \in \mathbb{N},$$

whence $\varphi = \eta\varphi_0$. □

We should mention here that the idea of examining the Schröder equation (1) with the aid of the Krein-Rutman theorem has come up while the author was thinking on generalization to the infinite-dimensional case of some results from the papers [1] by F. M. Hoppe and [2] by A. Joffe and F. Spitzer where the finite-dimensional case is considered with the aid of the Frobenius theory.

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