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ON APPROXIMATELY ADDITIVE FUNCTIONS

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Abstract. In the present paper we find a linear operator on a function space, essentially larger than the space of all bounded functions on an amenable semigroup, which behaves like an invariant mean. This leads to an extension of the Hyers–Ulam stability theorem for Cauchy's functional equation in the case of vector–valued mappings defined on amenable semigroups.

1. Introduction. This paper is a continuation of our study which we carried out in [1] and [2] . We shall formulate all our results in the left invariant version. It is quite obvious how to rephrase the theorems so as to obtain their right invariant analogues. The proofs of these alternative theorems require only minor changes, therefore, may also be omitted.

Let Y be a real linear space and let C be a non–empty class of subsets of Y. We say that the family C has the *binary intersection property* if every subfamily of C any two members of which intersect has a non–empty intersection. Moreover, a collection C is termed the *linear invariant family* if and only if C is invariant with respect to translations by vectors of Y, i.e.

$$A \in C,\ u \in Y \implies u + A \in C \tag{1}$$

and satisfies the following condition:

$$A,\ B \in C,\ t \in \mathbb{R} \implies A + B \in C,\ tA \in C. \tag{2}$$

Some examples of linear invariant families having the binary intersection property we presented below:

a) the family of all closed balls in the space $\mathbb{R}^n$ with the maximum norm;

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b) the family of all closed balls in the space $B(A,\mathbb{R})$ of all real bounded functions on a set A with the supremum norm;

c) the family of all closed intervals in a real boundedly complete linear lattice;

d) the family

$$\{y+[ay_0,by_0]:\ y\in Y,\ a,b\in\mathbb{R},\ a\le b\},$$

where y_0 is a fixed element of a real linear space Y and

$$[ay_0,by_0]:=\{tay_0+(1-t)by_0:\ t\in[0,1]\},\quad a,b\in\mathbb{R},\ a\le b.$$

Let $(S,\cdot)$ be a semigroup. For any $y\in S$ we define the *left* and *right translation* of a function $f:S\to Y$ as follows:

$$(3)\qquad {}_yf(x):=f(yx),\ \ f_y(x):=f(xy),\quad x\in S.$$

For a subset $\mathcal{F}$ of the space of all fuctions defined on the semigroup S with values in the space Y we say that $\mathcal{F}$ is the *left* (*right*) *invariant* if and only if

$$(4)\qquad f\in\mathcal{F},\ y\in S\implies {}_yf\in\mathcal{F}\ \ (f_y\in\mathcal{F}).$$

Now we recall the notion of invariant mean and amenability. If $B(S,\mathbb{R})$ denotes the space of all real bounded functions on a semigroup S, then a real linear functional M on $B(S,\mathbb{R})$ is called a *left* (*right*) *invariant mean* if and only if it is translation invariant, i.e.

$$M({}_yf)=M(f)\ \ (M(f_y)=M(f)),\quad y\in S,\ f\in B(S,\mathbb{R}),$$

and normalized

$$\inf\{f(x):\ x\in S\}\le M(f)\le\sup\{f(x):\ x\in S\},\quad f\in B(S,\mathbb{R}).$$

If a left (right) invariant mean exists, then we call S *left* (*right*) *amenable*. For example, every commutative semigroup is amenable (see e.g. F.P. Greenleaf [8]).

2. Generalized invariant mean. In [2] we have shown that the concept of an invariant mean can be extended to some function spaces larger than the space of all bounded functions. Precisely, we have proved the following

THEOREM 1. *Let $(S,\cdot)$ be a left (right) amenable semigroup and let Y be a real locally convex linear topological Hausdorff space. Let $\mathcal{F}$ be a left* *(right) invariant linear space of functions defined on S with values in Y and let C be a subfamily of the family of all bounded closed convex subsets of Y having the binary intersection property and invariant with respect to translations by vectors of Y. Assume that the map $F : \mathcal{F} \to C$ satisfies conditions:*

$$F(f+g) \subseteq F(f) + F(g), \qquad f, g \in \mathcal{F}, \tag{5}$$

$$F(tf) = tF(f), \qquad f \in \mathcal{F}, \ t \in \mathbb{R} \setminus \{0\}, \tag{6}$$

as well as

$$F({}_yf) \subseteq F(f) \ \ (F(f_y) \subseteq F(f)), \qquad f \in \mathcal{F}, \ y \in S. \tag{7}$$

Then there exists a linear operator $M : \mathcal{F} \to Y$ such that:

$$M(f) \in \mathcal{F}(f), \qquad f \in \mathcal{F}, \tag{8}$$

and

$$M({}_yf) = M(f) \ \ (M(f_y) = M(f)), \qquad y \in S, \ \ f \in \mathcal{F}. \tag{9}$$

This theorem implies the existence of a generalized invariant mean on the space of all essentially bounded functions (see [2]). Now, we shall present another example of a function space which admits a generalized invariant mean.

LEMMA 1. *If a collection C of subsets of a real linear space Y is the linear invariant family having the binary intersection property, then the collection $\tilde{C}$ of all non–empty intersections of subfamilies of C is also the linear invariant family having the binary intersection property.*

PROOF. From [1, Lemma 1] we derive that the family $\tilde{C}$ has the binary intersection property.

Let A, $B \in \tilde{C}$ be fixed. Then

$$A = \bigcap \{A_i : \ i \in I\} \quad \text{and} \quad B = \bigcap \{B_j : \ j \in J\}$$

for some subfamilies $\{A_i : \ i \in I\}$ and $\{B_j : \ j \in J\}$ of the family C. Hence, by Lemma 2 from [1], we have

$$\begin{aligned} A + B &= \bigcap \{A_i : \ i \in I\} + \bigcap \{B_j : \ j \in J\} \\ &= \bigcap \{A_i + B_j : i \in I, \ j \in J\} \in \tilde{C}. \end{aligned}$$

Moreover,

$$tA = t\bigcap\{A_i:\ i \in I\} = \bigcap\{tA_i:\ i \in I\} \in \tilde{C}$$

for $t \in \mathbb{R}$.

The family $\tilde{C}$ is also invariant with respect to translations by vectors of Y. Indeed,

$$y + A = y + \bigcap\{A_i:\ i \in I\} = \bigcap\{y + A_i:\ i \in I\} \in \tilde{C}$$

for $y \in Y$. □

Let z be a fixed element of a semigroup S and let C be the linear invariant family of subsets of a real linear space Y having the binary intersection property. Let $\mathcal{P}^{C,z}$ be a collection of all functions

$$P : S \to C_0 := \{A \in C:\ 0 \in A\}$$

which satisfies the following condition:

$$\text{(10)}\qquad \begin{cases} \text{for every } y \in S \text{ there exists an } N \in \mathbb{N} \text{ such that} \\ \bigcap\limits_{n\geq N} P(z^n) = \{0\},\ \bigcap\limits_{n\geq N} P(yz^n) = \{0\}, \\ \bigwedge\limits_{n\geq N} \left(P(z^n) \neq \{0\} \Longrightarrow \bigvee\limits_{K\in\mathbb{N}} \bigwedge\limits_{k\geq K} P(z^k) \subseteq P(z^n)\right), \\ \bigwedge\limits_{n\geq N} \left(P(yz^n) \neq \{0\} \Longrightarrow \bigvee\limits_{K\in\mathbb{N}} \bigwedge\limits_{k\geq K} P(yz^k) \subseteq P(yz^n)\right). \end{cases}$$

For a function f from S into Y we define $\mathcal{P}^{C,z}(f)$ to be the family of all sets $A \in C$ such that there exists a map $P \in \mathcal{P}^{C,z}$ which fulfils the following condition:

$$f(x) \in A + P(x), \quad x \in S.$$

The space of all functions $f : S \to Y$ for which the family $\mathcal{P}^{C,z}(f)$ is non-empty will be denoted by $B^{C,z}(S,Y)$.

LEMMA 2. *The space $B^{C,z}(S,Y)$ is a left invariant linear space.*

PROOF. The fact that $B^{C,z}(S,Y)$ is closed under pointwise addition and scalar multiplication is a direct consequence of the following observation:

$$A_1, A_2 \in C,\quad P_1, P_2 \in \mathcal{P}^{C,z} \Longrightarrow A_1 + A_2 \in C,\quad P_1 + P_2 \in \mathcal{P}^{C,z}$$

and

$$A \in C,\quad P \in \mathcal{P}^{C,z},\quad t \in \mathbb{R} \Longrightarrow tA \in C,\quad tP \in \mathcal{P}^{C,z}.$$

For the proof of the left invariance of $B^{C,z}(S,Y)$ we fix arbitrarily an $f \in B^{C,z}(S,Y)$ and a $y \in S$. For any $A \in \mathcal{P}^{C,z}(f)$ there exists $P \in \mathcal{P}^{C,z}$ such that

$$f(x) \in A + P(x), \quad x \in S.$$

Moreover, for every $P \in \mathcal{P}^{C,z}$ we have ${}_yP \in \mathcal{P}^{C,z}$. Therefore,

$${}_yf(x) = f(yx) \in A + P(yx) = A + {}_yP(x), \quad x \in S,$$

and ${}_yf \in B^{C,z}(S,Y)$. □

By Theorem 1 we have the following theorem on the existence of a generalized invariant mean on the space $B^{C,z}(S,Y)$.

THEOREM 2. *Let z be a fixed element of a left amenable semigroup $(S,\cdot)$ and let C be the linear invariant collection of bounded closed convex sets in a real locally convex linear topological Hausdorff space Y having the binary intersection property. Then there exists a linear operator $M : B^{C,z}(S,Y) \to Y$ such that*

$$M(f) \in \bigcap \mathcal{P}^{C,z}(f), \quad f \in B^{C,z}(S,Y) \tag{11}$$

and

$$M({}_yf) = M(f), \quad f \in B^{C,z}(S,Y), \quad y \in S. \tag{12}$$

PROOF. Let $\tilde{C}$ be the collection of all non-empty intersections of subfamilies of the family C and let

$$F(f) := \bigcap \mathcal{P}^{C,z}(f), \tag{13}$$

for $f \in B^{C,z}(S,Y)$. First, we shall show that the map F has values in the family $\tilde{C}$; that is $F(f) \neq \emptyset$ for $f \in B^{C,z}(S,Y)$.

Let $f \in B^{C,z}(S,Y)$ be fixed. The family

$$\mathcal{P}^{C,z}(f) = \{A \in C : \bigvee_{P \in \mathcal{P}^{C,z}} f(x) \in A + P(x), \quad x \in S\}$$

is a subfamily of C. Therefore, $F(f) = \bigcap \mathcal{P}^{C,z}(f) \neq \emptyset$ if any two members of $\mathcal{P}^{C,z}(f)$ intersect. Let $A, \tilde{A} \in \mathcal{P}^{C,z}(f)$ be fixed. Then there exist $P, \tilde{P} \in \mathcal{P}^{C,z}$ such that

$$f(x) \in [A + P(x)] \cap [\tilde{A} + \tilde{P}(x)], \quad x \in S. \tag{14}$$

Let

$$\mathcal{N} := \{n \in \mathbf{N} : \ P(z^n) = \{0\}\}$$

and

$$\tilde{\mathcal{N}} := \{n \in \mathbf{N} : \ \tilde{P}(z^n) = \{0\}\}.$$

If card $\mathcal{N} < \aleph_0$ and card $\tilde{\mathcal{N}} < \aleph_0$, then there exists an $N \in \mathbf{N}$ such that

$$\bigcap_{n \geq N} P(z^n) = \{0\}, \quad \bigcap_{n \geq N} \tilde{P}(z^n) = \{0\} \tag{15}$$

and

$$P(z^n) \neq \{0\}, \quad \tilde{P}(z^n) \neq \{0\}, \quad n \geq N.$$

We consider the subfamily

$$\mathcal{C}_1 := \{A + P(z^n) : \ n \geq N\} \cup \{\tilde{A} + \tilde{P}(z^n) : \ n \geq N\}$$

of the family $\mathcal{C}$. The intersection of any two elements of the family $\mathcal{C}_1$ is either of the form

$$A_1 = (A + P(z^n)) \cap (A + P(z^m))$$

or

$$A_2 = (\tilde{A} + \tilde{P}(z^n)) \cap (\tilde{A} + \tilde{P}(z^m))$$

or

$$A_3 = (A + P(z^n)) \cap (\tilde{A} + \tilde{P}(z^m))$$

for some n, $m \geq N$. Then

$$\emptyset \neq A = A + \{0\} \subseteq A_1 \qquad \text{and} \qquad \emptyset \neq \tilde{A} = \tilde{A} + \{0\} \subseteq A_2.$$

Condition (10) implies that there exists a $k \in \mathbf{N}$ such that

$$P(z^k) \subseteq P(z^n) \qquad \text{and} \qquad \tilde{P}(z^k) \subseteq \tilde{P}(z^m).$$

Moreover, by condition (14), we have

$$\emptyset \neq \{\ f(z^k)\ \} \subseteq (A+P(z^k)) \cap (\tilde{A}+\tilde{P}(z^k)) \subseteq (A+P(z^n)) \cap (\tilde{A}+\tilde{P}(z^m)) \subseteq A_3.$$

Therefore, by Lemma 2 from [1] and condition (15), we get

$$\begin{aligned}\emptyset \neq \bigcap \mathcal{C}_1 &= \bigcap_{n \geq N} (A + P(z^n)) \cap \bigcap_{m \geq N} (\tilde{A} + \tilde{P}(z^m)) \\ &= (A + \bigcap_{n \geq N} P(z^n)) \cap (\tilde{A} + \bigcap_{m \geq N} \tilde{P}(z^m)) = A \cap \tilde{A}.\end{aligned}$$

If card $\mathcal{N} = \aleph_0$, then there exists a sequence $(n_k)_{k\in\mathbb{N}}$ such that

$$P(z^{n_k}) = \{0\}, \qquad k \in \mathbb{N}.$$

Now we shall consider the subfamily

$$\begin{aligned} C_2 :=& \{A + P(z^{n_k}) : \ k \in \mathbb{N}\} \cup \{\tilde{A} + \tilde{P}(z^{n_k}) : \ k \in \mathbb{N}\} \\ =& \{A\} \cup \{\tilde{A} + \tilde{P}(z^{n_k}) : \ k \in \mathbb{N}\} \end{aligned}$$

of the family C. In the present case, the intersection of any two members of the family C_2 is either of the form

$$A_1 = (A + P(z^{n_m})) \cap (A + P(z^{n_l})) = A \neq \emptyset$$

or

$$A_2 = (\tilde{A} + \tilde{P}(z^{n_m})) \cap (\tilde{A} + \tilde{P}(z^{n_l})) \supseteq (\tilde{A} + \{0\}) \cap (\tilde{A} + \{0\}) = \tilde{A} \neq \emptyset$$

or

$$A_3 = (A + P(z^{n_m})) \cap (\tilde{A} + \tilde{P}(z^{n_l}))$$

for some l, $m \geq N$. Then, by condition (14), we get

$$\begin{aligned} \emptyset \neq& \{ f(z^{n_l}) \} \subseteq (A + P(z^{n_l})) \cap (\tilde{A} + \tilde{P}(z^{n_l})) = (A + \{0\}) \cap (\tilde{A} + \tilde{P}(z^{n_l})) \\ =& (A + P(z^{n_m})) \cap (\tilde{A} + \tilde{P}(z^{n_l})) = A_3. \end{aligned}$$

Hence, Lemma 2 from [1] and condition (10) imply that

$$\emptyset \neq \bigcap C_2 = A \cap \bigcap_{m\in\mathbb{N}} (\tilde{A} + \tilde{P}(x^{n_m})) = A \cap (\tilde{A} + \bigcap_{m\in\mathbb{N}} \tilde{P}(x^{n_m})) = A \cap \tilde{A}.$$

Therefore, for any A, $\tilde{A} \in \mathcal{P}^{C,z}(f)$, we have

$$A \cap \tilde{A} \neq \emptyset,$$

and so

$$F(f) = \bigcap \mathcal{P}^{C,z}(f) \neq \emptyset.$$

Now we shall prove that the map F satisfies conditions (5), (6) and (7).

Let $f, g \in B^{C,z}(S, Y)$ be fixed. Then, for any $A \in \mathcal{P}^{C,z}(f)$, $\tilde{A} \in \mathcal{P}^{C,z}(g)$ and any $P, \tilde{P} \in \mathcal{P}^{C,z}$ such that

$$f(x) \in A + P(x) \qquad \text{and} \qquad g(x) \in \tilde{A} + \tilde{P}(x), \quad x \in S,$$

we get

$$(f+g)(x) \in A + \tilde{A} + P(x) + \tilde{P}(x), \quad x \in S.$$

Hence, $A + \tilde{A} \in \mathcal{P}^{C,z}(f+g)$ and, by Lemma 2 from [1],

$$\begin{aligned} F(f+g) &= \bigcap \mathcal{P}^{C,z}(f+g) \subseteq \bigcap \mathcal{P}^{C,z}(f) + \bigcap \mathcal{P}^{C,z}(g) \\ &= F(f) + F(g), \end{aligned}$$

which proves (5).

Condition (6) is a result of the fact that

$$A \in \mathcal{P}^{C,z}(f) \Longleftrightarrow tA \in \mathcal{P}^{C,z}(tf)$$

for all $f \in B^{C,z}(S,Y)$ and $t \in \mathbb{R} \setminus \{0\}$. Then

$$tF(f) = t\bigcap \mathcal{P}^{C,z}(f) = \bigcap \mathcal{P}^{C,z}(tf) = F(tf).$$

It remains to show that condition (7) holds true. Let $f \in B^{C,z}(S,Y)$ and $y \in S$ be fixed. Then from the following implication (see the proof of Lemma 2)

$$A \in \mathcal{P}^{C,z}(f) \Longrightarrow A \in \mathcal{P}^{C,z}({}_yf)$$

we have

$$F({}_yf) = \bigcap \mathcal{P}^{C,z}({}_yf) \subseteq \bigcap \mathcal{P}^{C,z}(f) = F(f).$$

Consequently, the space $B^{C,z}(S,Y)$ and the map F defined by (13) fulfil all the assumptions of Theorem 1. From Theorem 1 we get the existence of a generalized invariant mean on the space $B^{C,z}(S,Y)$ which has all the desired properties and the proof is finished. □

3. An application. In this part of our paper we are going to present an application of Theorem 2 to the study of the stability of Cauchy's functional equation

$$f(xy) = f(x) + f(y), \qquad x, y \in S \tag{16}$$

(a function f which satisfies equation (16) will be called *additive*).

The problem of the stability of equation (16) (for mappings transforming one Banach space into another) was raised by S. M. Ulam [11], and solved by D. H. Hyers [9]. Later on, the Hyers theorem has been generalized in various directions (see D. H. Hyers, Th. M. Rassias [10]). Applying the generalized invariant mean we have the following stability result.

THEOREM 3. *Let z be a fixed element of a left amenable semigroup $(S,\cdot)$ and let $\mathcal{C}$ be a linear invariant collection of bounded closed convex sets in a real locally convex linear topological Hausdorff space Y having the binary intersection property. Assume that*

$$P : S^2 \to \mathcal{C}_0 := \{A \in \mathcal{C} :\ 0 \in A\}$$

is a given mapping such that, for any $x \in S$, the map $P(x,\cdot) : S \to \mathcal{C}$ fulfils condition (10). Then for any function $f : S \to Y$ and any map $Q : S \to \mathcal{C}$ such that

$$f(xy) - f(x) - f(y) \in Q(x) + P(x,y), \quad x,y \in S, \tag{17}$$

there exists an additive function $a : S \to Y$ such that

$$a(x) - f(x) \in Q(x), \quad x \in S. \tag{18}$$

Moreover, if

$$\bigcap \frac{1}{n}(Q(x^n) - Q(x^n)) = \{0\}, \quad x \in S, \tag{19}$$

then the additive function a satisfying (18) is unique.

PROOF. By our assumption, for a fixed element $x \in S$, the function

$$S \ni y \longmapsto f(xy) - f(y) = {}_xf(y) - f(y) \in Y$$

belongs to the space $B^{\mathcal{C},z}(S,Y)$ with the set $f(x) + Q(x) \in \mathcal{P}^{\mathcal{C},z}({}_xf - f)$.

Let M stand for an operator on $B^{\mathcal{C},z}(S,Y)$ into Y which fulfils conditions (11) and (12). We define a function $a : S \to Y$ by the following formula:

$$a(x) := M({}_xf - f), \quad x \in S.$$

By the left invariance and the linearity of M we get the additivity of a and, by condition (11),

$$a(x) \in \bigcap \mathcal{P}^{\mathcal{C},z}({}_xf - f) \subseteq f(x) + Q(x), \quad x \in S.$$

Therefore

$$a(x) - f(x) \in Q(x), \quad x \in S.$$

To prove the uniqueness, we suppose that a and $\tilde{a}$ are two additive mappings on S into Y which satisfy condition (18). Then, for any $x \in S$, we get

$$\begin{aligned} a(x) - \tilde{a}(x) &= \frac{1}{n}\big(a(x^n) - f(x^n) - (\tilde{a}(x^n) - f(x^n))\big) \\ &\in \frac{1}{n}\big(Q(x^n) - Q(x^n)\big), \quad n \in \mathbb{N}. \end{aligned}$$

Hence

$$a(x) - \tilde{a}(x) \in \bigcap_{n \in \mathbb{N}} \frac{1}{n}\big(Q(x^n) - Q(x^n)\big) = \{0\}, \quad x \in S,$$

which shows that a and $\tilde{a}$ coincide. □

REMARK 1. Taking $P(x,y) := \{0\}$, $x, y \in S$, and $Q(x) := V$, $x \in S$, where V is a fixed element of a family $\mathcal{C}$ Theorem 2 reduces to the result of the type proved by D. H. Hyers:

If $f : S \to Y$ satisfies

$$f(xy) - f(x) - f(y) \in V, \quad x, y \in S,$$

then there exists exactly one additive function $a : S \to Y$ such that

$$a(x) - f(x) \in V, \quad x \in S.$$

Moreover, taking the family $\mathcal{C}$ as in example d) we obtain the following version of the result proved by G. L. Forti (see [3]):

If $f : S \to Y$ satisfies

$$f(xy) - f(x) - f(y) \in \{0, y_0, 2y_0, \dots, Ny_0\}, \quad x, y \in S,$$

where $y_0 \in Y$ and $N \in \mathbb{N}$ are fixed, then (putting $V := [0, Ny_0]$) *there exists exactly one additive function $a : S \to Y$ such that*

$$a(x) - f(x) \in [\,0, Ny_0\,], \quad x \in S.$$

Now we shall use Theorem 3 to the study of the following problem.

Let $(S, \cdot)$ be a semigroup and let $(Y, \|\cdot\|)$ be a real normed space. Which of the functions $p : S^2 \to [0, \infty)$ guarantee that for any functions $q : S \to [0, \infty)$ and $f : S \to Y$ fulfilling

$$\|f(xy) - f(x) - f(y)\| \le q(x) + p(x, y), \quad x, y \in S,$$

there exists an additive function $a : S \to Y$ and a constant $k \in \mathbf{R}$ such that

$$\|a(x) - f(x)\| \le kq(x), \quad x \in S?$$

Z. Gajda has proved the following

THEOREM 4 (Z. Gajda [5]). *Let $(S, \cdot)$ be a left amenable semigroup and let $(Y, \|\cdot\|)$ be a reflexive Banach space. Moreover, assume that $p, q : S \to [0, \infty)$ are two given functions the first of which satisfies the condition*

$$\inf\left\{\sum_{k=1}^{n} {}_{y_k}p(x) : \ x \in S_0\right\} = 0 \tag{20}$$

for all $n \in \mathbf{N}$ and $y_1, \dots, y_n \in S_0$ (where by S_0 we denote S enlarged, if necessary, by a single element e regarded as the unit of the semigroup $S_0 = S \cup \{e\}$), whereas the second is entirely arbitrary. Then for any mapping $f : S \to Y$ fulfilling the inequality

$$\|f(xy) - f(x) - f(y)\| \le q(x) + p(y), \quad x, y \in S,$$

there exists an additive function $a : S \to Y$ such that

$$\|a(x) - f(x)\| \le q(x), \quad x \in S.$$

REMARK 2. If $(S, \cdot) = (X, +)$, where $(X, \|\cdot\|)$ is a real normed space and $\alpha < 0$, then the function $p : X \to Y$ defined by

$$p(x) := \|x\|^{\alpha}, \quad x \in X \setminus \{0\},$$

satisfies condition (20).

For functions with values in a normed space Y having the binary intersection property (i.e. the family of all closed balls in Y has this property) Theorem 3 leads to the following

THEOREM 5. *Let z be a fixed element of a left amenable semigroup $(S, \cdot)$ and let $(Y, \|\cdot\|)$ be a normed space admitting an equivalent norm $\|\cdot\|_p$ such that the space $(Y, \|\cdot\|_p)$ has the binary intersection property. Assume that $p : S^2 \to [0, \infty)$ is a given function which satifies*

$$\lim_{n\to\infty} p(x, z^n) = \lim_{n\to\infty} p(x, yz^n) = 0 \tag{21}$$

for all $x, y \in S$. *Then for any mapping* $f : S \to Y$ *and any function* $q : S \to [0, \infty)$ *fulfilling*

$$\|f(xy) - f(x) - f(y)\| \leq q(x) + p(x, y), \quad x, y \in S, \tag{22}$$

there exist a constant $k(Y) \in \mathbb{R}$ *and an additive function* $a : S \to Y$ *such that*

$$\|a(x) - f(x)\| \leq k(Y)q(x), \quad x \in S. \tag{23}$$

PROOF. Let c and C be positive constants such that

$$c\|y\| \leq \|y\|_p \leq C\|y\|, \quad y \in Y. \tag{24}$$

For $y \in Y$ and $r \in [0, \infty)$ by $B(y, r)$ we denote the closed ball in the space $(Y, \|\cdot\|_p)$ with the center at y and the radius $r \geq 0$ ($B(y, 0) := \{y\}, \quad y \in Y$). Then

$$\mathcal{C} := \{B(y, r) : \ y \in Y, \ r \in [0, \infty)\}$$

is a linear invariant family of bounded closed convex sets in Y having the binary intersection property. Inequality (22) may be written as follows

$$f(xy) - f(x) - f(y) \in B(0, Cq(x) + Cp(x, y)), \quad x, y \in S,$$

which states (see [1, Remark 1]) that

$$f(xy) - f(x) - f(y) \in B(0, Cq(x)) + B(0, Cp(x, y)), \quad x, y \in S.$$

Now, we define functions $Q : S \to \mathcal{C}$ and $P : S^2 \to \mathcal{C}_0$ by the following formulae:

$$Q(x) := B(0, Cq(x)), \quad x \in S,$$
$$P(x, y) := B(0, Cp(x, y)), \quad x, y \in S.$$

By assumption (21), all the functions $P(x, \cdot) : S \to \mathcal{C}_0$, $x \in S$, satisfy condition (10). From Theorem 2 we have the existence of an additive function $a : S \to Y$ such that

$$a(x) - f(x) \in Q(x) = B(0, Cq(x)), \quad x \in S,$$

i.e.

$$\|a(x) - f(x)\|_p \leq Cq(x), \quad x \in S.$$

Hence

$$\|a(x) - f(x)\| \leq \frac{C}{c}q(x), \quad x \in S,$$

which means that the additive function a satisfies condition (23) with the constant $k(Y) := \frac{C}{c}$. $\square$

Now we shall present two examples of functions p which fulfil condition (21).

Assume that $(S, \cdot) = (X, +)$, where $(X, \|\cdot\|)$ is a real normed space. Then a function $p : X^2 \to [0, \infty)$ defined by

$$p(x,y) := \begin{cases} \|y\|^{\alpha(x)}, & x \in X, \quad y \in X \setminus \{0\}, \\ 0, & x \in X, \quad y = 0, \end{cases} \tag{25}$$

where $\alpha : X \to (-\infty, 0)$ is a given function, satisfies condition (21) for $z \in X \setminus \{0\}$. In this case Theorem 5 reduces to the following

THEOREM 6. *Let $(X, \|\cdot\|), (Y, \|\cdot\|)$ be two real normed spaces and let Y admit a norm $\|\cdot\|_p$ equivalent to $\|\cdot\|$ and such that $(Y, \|\cdot\|_p)$ has the binary intersection property. Then for any mapping $f : X \to Y$ and any function $q : X \to [0, \infty)$ fulfilling*

$$\|f(x+y) - f(x) - f(y)\| \le q(x) + \|y\|^{\alpha(x)}, \quad x, y \in X \setminus \{0\}, \tag{26}$$

with a function $\alpha : X \to (-\infty, 0)$, there exist a constant $k(Y)$ and an additive function $a : X \to Y$ such that

$$\|a(x) - f(x)\| \le k(Y) q(x), \quad x \in X. \tag{27}$$

An analogous theorem with the function $p : X^2 \to [0, \infty)$ defined by

$$p(x,y) = \|y\|^{\alpha}, \quad x, y \in X,$$

where $\alpha \in [0, 1)$, fails to hold. Indeed, let $X = Y = \mathbf{R}$ and let $q(x) = |x|$ for $x \in \mathbf{R}$. Then we can show, just as Z. Gajda in [4], that the function $f : \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{\phi(2^n x)}{2^n}, \quad x \in \mathbf{R},$$

where

$$\phi(x) = \begin{cases} -\dfrac{1}{6}, & x \le -1, \\ \dfrac{1}{6} x, & |x| < 1, \\ \dfrac{1}{6}, & x \ge 1, \end{cases}$$

satisfies the inequality

$$|f(x+y)-f(x)-f(y)| \le |x|+|y|^{\alpha}, \quad x,y \in \mathbb{R},$$

but there is no real constant k and no additive function $a:\mathbb{R}\to\mathbb{R}$ fulfilling

$$|a(x)-f(x)| \le k|x|, \quad x \in \mathbb{R}.$$

In the case where $\alpha \in [0,1)$ we have the following result of the type proved by R. Ger (see [6] and [7]).

THEOREM 7. *Let $(X,\|\cdot\|)$, $(Y,\|\cdot\|)$ be two real normed spaces and let Y admit a norm $\|\cdot\|_p$ equivalent to $\|\cdot\|$ and such that $(Y,\|\cdot\|_p)$ has the binary intersection property. Then for any mapping $f: X\to Y$ and any function $q: X\to[0,\infty)$ fulfilling*

$$\|f(x+y)-f(x)-f(y)\| \le q(x)+\big|\|y\|^{\alpha}-\|x+y\|^{\alpha}\big|, \quad x,y \in X, \tag{28}$$

with some constant $\alpha\in[0,1)$, there exist an additive function $a: X\to Y$ and a constant $k(Y)\in\mathbb{R}$ such that

$$\|a(x)-f(x)\| \le k(Y)q(x), \quad x \in X. \tag{29}$$

PROOF. Let $p: X^2\to[0,\infty)$ be a function defined by

$$p(x,y) := \big|\|y\|^{\alpha}-\|x+y\|^{\alpha}\big|, \quad y \in X, \tag{30}$$

where $0^0 := 1$.

If $\alpha=0$, then $p(x,y)=0$, $x,y\in X$, and p satisfies condition (21) for any fixed element $z\in X\setminus\{0\}$.

If $\alpha\in(0,1)\cap\mathbb{Q}$, then $\alpha=\frac{m}{k}$, where $m,k\in\mathbb{N}$, $m<k$ ($k\ge 2$) and

$$\begin{aligned}
&\lim_{n\to\infty}\big|\|x+nz\|^{\frac{m}{k}}-\|nz\|^{\frac{m}{k}}\big| \\
&\quad=\lim_{n\to\infty}\frac{\big|\|x+nz\|-\|nz\|\big|\left(\|x+nz\|^{m-1}+\ldots+\|nz\|^{m-1}\right)}{\|x+nz\|^{\frac{m(k-1)}{k}}+\ldots+\|nz\|^{\frac{m(k-1)}{k}}} \\
&\quad\le\lim_{n\to\infty}\frac{\|x\|\left(\|x+nz\|^{m-1}+\ldots+\|nz\|^{m-1}\right)}{\|x+nz\|^{\frac{m(k-1)}{k}}+\ldots+\|nz\|^{\frac{m(k-1)}{k}}} \\
&\quad=\lim_{n\to\infty} n^{-1+\frac{m}{k}}\frac{\|x\|\left(\|\frac{1}{n}x+z\|^{m-1}+\ldots+\|z\|^{m-1}\right)}{\|\frac{1}{n}x+z\|^{\frac{m(k-1)}{k}}+\ldots+\|z\|^{\frac{m(k-1)}{k}}}=0
\end{aligned}$$

$(-1 + \frac{m}{k} < 0)$ for $x \in X$ and $z \in X \setminus \{0\}$.

Let $x \in X$ and $z \in X \setminus \{0\}$ be fixed. Then there exists an $n_0 \in \mathbf{N}$ such that

$$\|x + nz\| \geq \|nz\|$$

or

$$\|x + nz\| \leq \|nz\|$$

for $n \geq n_0$. Indeed, if

$$\|x + n_0 z\| \leq \|n_0 z\|$$

for some $n_0 \in \mathbf{N}$ then, for $n \geq n_0$, we get

$$\|x + nz\| \leq \|x + n_0 z\| + \|(n - n_0)z\| \leq \|n_0 z\| + \|(n - n_0)z\| = \|nz\|.$$

Hence we can assume that

$$\|x + nz\| > 1 \qquad \text{and} \qquad \|nz\| > 1, \quad n \geq n_0.$$

Then, for $\alpha \in (0, 1)$, there exist $u, w \in (0, 1) \cap \mathbf{Q}$ such that

$$\|x + nz\|^u - \|nz\|^u \leq \|x + nz\|^\alpha - \|nz\|^\alpha \leq \|x + nz\|^w - \|nz\|^w$$

for $n \geq n_0$, because the map

$$(0, 1) \ni x \mapsto a^x - b^x \in \mathbf{R} \quad (a,\ b \in (1, \infty))$$

is a monotonic function.

Hence

$$\lim_{n \to \infty} p(x, nz) = \lim_{n \to \infty} |\|x + nz\|^\alpha - \|nz\|^\alpha| = 0$$

for all $x \in X$ and $z \in X \setminus \{0\}$. Moreover,

$$\begin{aligned}
\lim_{n \to \infty} p(x, y + nz) &= \lim_{n \to \infty} |\|x + y + nz\|^\alpha - \|y + nz\|^\alpha| \\
&\leq \lim_{n \to \infty} (|\|x + y + nz\|^\alpha - \|nz\|^\alpha| + |\|y + nz\|^\alpha - \|nz\|^\alpha|) \\
&= 0 + 0 = 0
\end{aligned}$$

for all $x,\ y \in X$ and $z \in X \setminus \{0\}$, which means that the function p defined by (30) satisfies condition (21). Theorem 5 implies the existence of an additive mapping $a : X \to Y$ and a constant $k(Y)$ fulfilling condition (29). □

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