ON THE SUPERSTABILITY OF THE GENERALIZED
ORTHOGONALITY EQUATION IN EUCLIDEAN SPACES

JACEK CHMIELIŃSKI

Abstract. We consider a class of approximate solutions of the generalized orthogonality equation in $\mathbb{R}^n$ ($n \geq 2$). We prove that this class coincides with the class of solutions of the equation, i.e., the superstability of the generalized orthogonality equation holds.

1. Introduction. Let E be an inner product space ("$x \circ y$" stands for the inner product of x and y). In 1931 E.P. Wigner [9] considered the functional equation

$$|T(x) \circ T(y)| = |x \circ y| \qquad \text{for} \quad x, y \in E$$

with the unknown function $T : E \to E$. This equation is referred to as the generalized orthogonality equation. In the present paper, however, we are not interested in *true* solutions of this equation (for them we refer to [7], [1], [2], [8]) but in *approximate* ones. Defining the class of approximate solutions of the generalized orthogonality equation we follow the method of D.H. Hyers applied originally to the Cauchy equation in [6]. Namely, for fixed $\varepsilon \geq 0$, we investigate the class of solutions of the functional inequality

$$\big|\, |T(x) \circ T(y)| - |x \circ y| \,\big| \leq \varepsilon \qquad \text{for} \quad x, y \in E.$$

It turns out (see [3]) that in the case where E is a real Hilbert space for each solution of the above inequality T, we may choose T_* – a true solution of the generalized orthogonality equation such that the difference between T and T_* is uniformly bounded by a constant (namely, by $\sqrt{\varepsilon}$). In other words, we may prove the stability of the generalized orthogonality equation in the

case of a real Hilbert space. In this paper we deal with the case where E is a finite–dimensional Euclidean space $\mathbf{R}^n$ (with $n \geq 2$). Using a method different from that applied in [3] we obtain, in this particular case, a stronger result. What we are going to prove is the *superstability* of the generalized orthogonality equation. It means that the class of solutions of the functional inequality

(1) $$\big|\, |T(x) \circ T(y)| - |x \circ y| \,\big| \leq \varepsilon \qquad \text{for} \quad x, y \in \mathbf{R}^n$$

coincides with the class of solutions of the equation

(2) $$|T(x) \circ T(y)| = |x \circ y| \qquad \text{for} \quad x, y \in \mathbf{R}^n.$$

2. Preliminary results. We begin with a lemma which apparently is not connected with the generalized orhogonality equation. However, we will strongly use this lemma in the proof of a proposition that follows.

LEMMA 1. *Fix $n \geq 2$ and $\varepsilon \geq 0$. For each $\eta > 0$ there exists $k_0 \in \mathbf{N}$ such that for any $k \geq k_0$, if the vectors $a, u_1, u_2, \ldots, u_{n-1} \in \mathbf{R}^n \setminus \{0\}$ satisfy the conditions*

(3) $$1 - \frac{\varepsilon}{k^2} \leq \|u_i\|^2 \leq 1 + \frac{\varepsilon}{k^2} \qquad \text{for} \quad i = 1, 2, \ldots, n-1, \cdot$$

(4) $$|u_i \circ u_j| \leq \frac{\varepsilon}{k^2} \qquad \text{for} \quad i, j = 1, 2, \ldots, n-1; \ i \neq j,$$

(5) $$|a \circ u_i| \leq \frac{\varepsilon}{k} \qquad \text{for} \quad i = 1, 2, \ldots, n-1,$$

then:

a) *vectors $u_1, \ldots, u_{n-1}$ are linearly independent and hence $H := \mathrm{lin}\{u_1, \ldots, u_{n-1}\}$ is an $(n-1)$-dimensional subspace in $\mathbf{R}^n$;*

b) *$|\cos A(a, \ell| \geq 1 - \eta$, where ℓ denotes the line in $\mathbf{R}^n$ which is the orthogonal complement of H and $A(\cdot, \cdot)$ stands for the angle.*

PROOF. 1. To begin with we prove a). Consider the Gram determinant for vectors $u_1, \ldots u_{n-1}$:

$$W(u_1, \ldots, u_{n-1}) = \det \begin{bmatrix} \|u_1\|^2 & u_1 \circ u_2 & \cdots & u_1 \circ u_{n-1} \\ u_2 \circ u_1 & \|u_2\|^2 & \cdots & u_2 \circ u_{n-1} \\ \vdots & \vdots & & \vdots \\ u_{n-1} \circ u_1 & u_{n-1} \circ u_2 & \cdots & \|u_{n-1}\|^2 \end{bmatrix}$$

$$= \overbrace{\|u_1\|^2 \cdot \ldots \cdot \|u_{n-1}\|^2}^{A} +$$

$$\overbrace{+ \sum_{\substack{f\in\mathrm{Per}\{1,\dots,n-1\}\\ f\neq\mathrm{id}}} (-1)^{I_f}(u_1\circ u_{f(1)})\cdot(u_2\circ u_{f(2)})\cdot\ldots\cdot(u_{n-1}\circ u_{f(n-1)}).}^{B}$$

Per $\{1,\dots,n-1\}$ is the set of all permutations of the set $\{1,\dots,n-1\}$ and I_f denotes the number of inversions of a permutation f. Suppose that k is large enough that $(1-\frac{\varepsilon}{k^2})\geq 0$. From (3) we have

(6)
$$|A|\geq\left(1-\frac{\varepsilon}{k^2}\right)^{n-1}\to 1 \qquad\text{as}\quad k\to\infty.$$

From (3) and (4) we get

(7)
$$|B|\leq((n-1)!-1)\left(1+\frac{\varepsilon}{k^2}\right)^{n-2}\cdot\frac{\varepsilon}{k^2}\to 0 \qquad\text{as}\quad k\to\infty.$$

Formulae (6) and (7) imply that, for a sufficiently large k, there is $|A|>|B|$ and hence
$$W(u_1,\dots,u_{n-1})=A+B\neq 0,$$

i.e., $u_1,\dots,u_{n-1}$ are linearly independent.

2. Take an $u_n\in\mathbb{R}^n$ such that $\ell=\mathrm{lin}\{u_n\}$ and $\|u_n\|=1$. We have a unique decomposition

$$a=h+l, \qquad\text{with}\quad h\in H,\; l\in\ell$$

and, moreover, there exist $\xi_1,\dots,\xi_{n-1},\xi_n\in\mathbb{R}$ such that

$$h=\xi_1 u_1+\dots+\xi_{n-1}u_{n-1}, \qquad l=\xi_n u_n.$$

Using (3) and (4) we obtain

$$\|a\|^2\geq\|h\|^2=h\circ h=\xi_1^2\|u_1\|^2+\dots+\xi_{n-1}^2\|u_{n-1}\|^2+\sum_{i=1}^{n-1}\sum_{\substack{j=1\\j\neq i}}^{n-1}\xi_i\xi_j u_i\circ u_j$$

$$\geq\xi_1^2\|u_1\|^2+\dots+\xi_{n-1}^2\|u_{n-1}\|^2-\sum_{i=1}^{n-1}\sum_{\substack{j=1\\j\neq i}}^{n-1}|\xi_i||\xi_j||u_i\circ u_j|$$

$$\geq(\xi_1^2+\dots+\xi_{n-1}^2)\left(1-\frac{\varepsilon}{k^2}\right)-\frac{\varepsilon}{k^2}\cdot\sum_{i=1}^{n-1}\sum_{\substack{j=1\\j\neq i}}^{n-1}|\xi_i||\xi_j|.$$

Defining $\xi := \max\{|\xi_1|, \ldots, |\xi_{n-1}|\}$ we get

$$(\xi_1^2 + \ldots + \xi_{n-1}^2)\left(1 - \frac{\varepsilon}{k^2}\right) \geq \xi^2\left(1 - \frac{\varepsilon}{k^2}\right)$$

and

$$\frac{\varepsilon}{k^2} \cdot \sum_{i=1}^{n-1} \sum_{\substack{j=1 \\ j \neq i}}^{n-1} |\xi_i||\xi_j| \leq (n-1)(n-2)\xi^2 \cdot \frac{\varepsilon}{k^2},$$

which enable us to continue the previous approximation so thus we get

$$\|a\|^2 \geq \xi^2\left(1 - \frac{\varepsilon}{k^2} - (n-1)(n-2)\frac{\varepsilon}{k^2}\right).$$

Denoting

$$\varphi(k) := \left(1 - \frac{\varepsilon}{k^2} - (n-1)(n-2)\frac{\varepsilon}{k^2}\right)$$

we have, for k sufficiently large,

$$\xi \leq \frac{\|a\|}{\sqrt{\varphi(k)}}$$

and then

$$(8) \qquad |\xi_1| + \ldots + |\xi_{n-1}| \leq (n-1)\xi \leq \frac{(n-1)\|a\|}{\sqrt{\varphi(k)}}.$$

As $\|h\|^2 = a \circ h$ we get, using (5),

$$\|h\|^2 = \xi_1 a \circ u_1 + \ldots + \xi_{n-1} a \circ u_{n-1}$$

$$\leq |\xi_1||a \circ u_1| + \ldots + |\xi_{n-1}||a \circ u_{n-1}| \leq (|\xi_1| + \ldots + |\xi_{n-1}|)\frac{\varepsilon}{k}$$

which, together with (8), implies

$$(9) \qquad \|h\|^2 \leq \frac{\varepsilon(n-1)\|a\|}{\sqrt{\varphi(k)} \cdot k}.$$

We have

$$(a \circ u_n)^2 = (l \circ u_n)^2 = \|l\|^2 = \|a\|^2 - \|h\|^2$$

and then

$$\cos^2 A(a, \ell) = \cos^2 A(a, u_n) = \frac{(a \circ u_n)^2}{\|a\|^2} = 1 - \frac{\|h\|^2}{\|a\|^2}.$$

Now, (9) implies

$$\cos^2 A(a, \ell) \geq 1 - \frac{\varepsilon(n-1)}{\sqrt{\varphi(k) \cdot k \cdot \|a\|}}.$$

Since $\varphi(k) \to 1$ as $k \to \infty$, the above inequality means that $|\cos A(a, \ell)|$ can be arbitrarily close to 1 provided that k is sufficiently large. This is equivalent to the assertion of the lemma in point b). $\qquad \square$

PROPOSITION 1. *If* $T : \mathbf{R}^n \to \mathbf{R}^n$ $(n \geq 2)$ *satisfies* (1), *then:*

(i) $\|x\|^2 - \varepsilon \leq \|T(x)\|^2 \leq \|x\|^2 + \varepsilon$ *for* $x \in \mathbf{R}^n$,

(ii) $T(x) = 0 \iff x = 0$,

(iii) *for each* $x \in \mathbf{R}^n$ *there exists a function* $\mu_x : \mathbf{R} \to \mathbf{R}$ *such that, for each* $\lambda \in \mathbf{R}$, $T(\lambda x) = \mu_x(\lambda) \cdot T(x)$ *and* $|\mu_x(\lambda)| \to \infty$ *as* $|\lambda| \to \infty$ *for* $\lambda \in \mathbf{R}$,

(iv) $T(x) \perp T(y) \iff x \perp y$.

PROOF. 1. To prove (i) we need only to put $x = y$ in (1). Now suppose $T(x) = 0$. By (1) we have for an arbitrary $y \in \mathbf{R}^n$ that $|x \circ y| \leq \varepsilon$. Taking $y = kx$ we get $k\|x\|^2 \leq \varepsilon$, for an arbitrary k, which implies $x = 0$. Thus we have proven "$\Rightarrow$" in (ii).

2. Now, let us discus (iii). At first, consider the case where $x \neq 0$ and $\lambda \neq 0$. Suppose that $T(x)$ and $T(\lambda x)$ are linearly independent; thus we would be able to set

(10) $\qquad |\cos A(T(x), T(\lambda x))| = 1 - \omega$ $\qquad$ for some $\omega > 0$.

Take $v_1, \ldots, v_{n-1} \in \mathbf{R}^n$, $\|v_1\| = \ldots = \|v_{n-1}\| = 1$ such that $\{x, v_1, \ldots, v_{n-1}\}$ forms an orthogonal basis in $\mathbf{R}^n$. For any $k \in \mathbf{N}$ and $i, j = 1, 2, \ldots, n-1; i \neq j$ we have

$$kv_i \circ x = 0, \qquad kv_i \circ \lambda x = 0, \qquad kv_i \circ kv_j = 0$$

and so, by inequality (1), we get

(11)
$$\begin{aligned} |T(kv_i) \circ T(x)| &\leq \varepsilon, \\ |T(kv_i) \circ T(\lambda x)| &\leq \varepsilon, \\ |T(kv_i) \circ T(kv_j)| &\leq \varepsilon \end{aligned}$$

for $k \in \mathbf{N}$; $i, j = 1, \ldots, n-1$; $i \neq j$. If we denote (with a fixed k)

$$a := T(x); \qquad a' := T(\lambda x); \qquad u_i := \frac{1}{k} T(kv_i) \qquad \text{for } i = 1, \ldots, n-1,$$

then we may write (11) in the form

$$|u_i \circ a| \leq \frac{\varepsilon}{k}, \qquad |u_i \circ a'| \leq \frac{\varepsilon}{k} \qquad |u_i \circ u_j| \leq \frac{\varepsilon}{k^2}$$

for $i, j = 1, \ldots, n - 1$; $i \neq j$. Moreover, from (i)

$$1 - \frac{\varepsilon}{k^2} \leq \|u_i\|^2 \leq 1 + \frac{\varepsilon}{k^2}.$$

Thus we may apply Lemma 1 for the system $\{a, u_1, \ldots, u_{n-1}\}$ and for $\{a', u_1, \ldots, u_{n-1}\}$ as well. If ℓ denotes 1-dimensional orthogonal complement of the subspace $\text{lin}\{u_1, \ldots, u_{n-1}\}$, then choosing k sufficiently large we will get $|\cos A(a, \ell)|$ and $|\cos A(a', \ell)|$ arbitrarily close to 1 which implies that $|\cos A(a, a')| = |\cos A(T(x), T(\lambda x))|$ is arbitrarily close to 1 as well. But this contradicts (10), whence $T(x)$ and $T(\lambda x)$ have to be linearly dependent. Having proved the implication "⇒" in (ii) we know that $T(x) \neq 0$; thus we can choose $\mu_x(\lambda) \in \mathbb{R}$ such that $T(\lambda x) = \mu_x(\lambda) \cdot T(x)$. If $|\lambda| \to \infty$, then $\|\lambda x\| \to \infty$ and – by (i) – $\|T(\lambda x)\| \to \infty$ as well which implies $|\mu_x(\lambda)| \to \infty$. We have proved (iii) for $x \neq 0$ and $\lambda \neq 0$.

3. Now we consider (iv) beginning with the part "$x \circ y = 0 \Rightarrow T(x) \circ T(y) = 0$". At first, we consider the case $x \neq 0$. If $x \circ y = 0$, then, for each $k \in \mathbb{N}$, we have $kx \circ y = 0$ so, from (1), we derive $|T(kx) \circ T(y)| \leq \varepsilon$. Since $x \neq 0$, by what we have proved above, $T(kx) = \mu_x(k) \cdot T(x)$. Thus we may write, for each $k \in \mathbb{N}$,

$$|T(x) \circ T(y)| \leq \frac{\varepsilon}{|\mu_x(k)|}$$

which yields $T(x) \circ T(y) = 0$.

4. Now we can prove $T(0) = 0$. Indeed, by (1) for any $0 \neq x \in \mathbb{R}^n$ and an arbitrary $k \in \mathbb{N}$ there is $|T(kx) \circ T(0)| \leq \varepsilon$. Since $T(kx) = \mu_x(k) \cdot T(x)$, we get

$$|T(x) \circ T(0)| \leq \frac{\varepsilon}{|\mu_x(k)|}$$

for an arbitrary $x \neq 0$ which implies $T(x) \circ T(0) = 0$. Now, if we take an arbitrary orthogonal basis $\{x_1, \ldots, x_n\}$ in $\mathbb{R}^n$, then the system of vectors $\{T(x_1), \ldots, T(x_n)\}$ forms and orthogonal basis as well (see: point 3. in the proof and "⇒" in (ii)). Thus we obtain that $\{T(0), T(x_1), \ldots, T(x_n)\}$ is a system of $n + 1$ orthogonal vectors in $\mathbb{R}^n$ which implies $T(0) = 0$. Thus the proof of (ii) has been completed.

5. Now, we easily complete the proof of "$x \circ y = 0 \Rightarrow T(x) \circ T(y) = 0$" in the remaining case $x = 0$. We may also finish the proof of (iii) taking $\mu_x(0) = 0$ and, in the case where $x = 0$, setting e.g., $\mu_0(\lambda) = \lambda$.

6. We need only to prove "$T(x) \circ T(y) = 0 \Rightarrow x \circ y = 0$". Notice, that if $T(x) \circ T(y) = 0$, then, for an arbitrary $k \in \mathbf{N}$, we have

$$|T(kx) \circ T(y)| = |\mu_x(k)| \cdot |T(x) \circ T(y)| = 0$$

and then, by (1), we have, for any $k \in \mathbf{N}$, $|kx \circ y| \leq \varepsilon$, whence

$$|x \circ y| \leq \frac{\varepsilon}{k} \to 0 \qquad \text{as} \quad k \to \infty$$

so $x \circ y = 0$. We have completed the proof of (iv) and the proof of the whole proposition as well. $\qquad \square$

Let us emphasize that in proofs of points (ii), (iii) and (iv) we essentially used the fact that we deal with the finite-dimensional space.

PROPOSITION 2. *If* $T : \mathbf{R}^n \to \mathbf{R}^n$ *satisfies* (1) *and* $2 \leq k \leq n$, *then:*

(a) *vectors* $x_1, \ldots, x_k$ *are linearly independent in* $\mathbf{R}^n$ *if and only if their images* $T(x_1), \ldots, T(x_k)$ *are linearly independent;*

(b) T *transforms a* k-*dimensional subspace* $\mathcal{P} = \lin\{x_1, \ldots, x_k\}$ *into the* k-*dimensional subspace* $\mathcal{P}' = \lin\{T(x_1), \ldots, T(x_k)\}$;

(c) *in particular, the image of a plane in* $\mathbf{R}^n$ *is contained in another plane and, similarly, the image of a line in* $\mathbf{R}^n$ *is contained in a line.*

The proof of this proposition runs exactly in the same way as the ones of Propositions 1 and 2 in [4]. What we essentially need to follow those proofs are properties (ii) and (iv) established in Proposition 1 of the present paper.

We end this section with two easy lemmas.

LEMMA 2. *Suppose that* $T_1, T_2 : \mathbf{R}^n \to \mathbf{R}^n$ ($n \geq 2$) *are solutions of* (1) *with constants on the right hand side* ε_1 *and* ε_2, *respectively. The composition* $T := T_2 T_1$ *satisfies the inequality* (1) *with the constant* $\varepsilon := \varepsilon_1 + \varepsilon_2$.

We omit an easy proof of this lemma as well as that one of the following

COROLLARY 1. *The composition of a solution of* (1) *with a solution of the (generalized) orthogonality equation is also a solution of* (1) *(with the same* ε).

LEMMA 3. *If* $T : \mathbf{R}^n \to \mathbf{R}^n$ ($n \geq 2$) *satisfies* (1), *then for any* $x, y \in \mathbf{R}^n$ *we have*

$$\lim_{k \to \infty} \frac{1}{k^2} \cdot \|T(kx)\| \cdot \|T(ky)\| = \|x\| \cdot \|y\|.$$

PROOF. On account of Proposition 1.(i) for an arbitrary $k \in \mathbb{N}$ we have

$$\sqrt{\|x\|^2 - \frac{\varepsilon}{k^2}} \cdot \sqrt{\|y\|^2 - \frac{\varepsilon}{k^2}} \leq \frac{1}{k^2} \|T(kx)\| \cdot \|T(ky)\|$$

$$\leq \sqrt{\|x\|^2 + \frac{\varepsilon}{k^2}} \cdot \sqrt{\|y\|^2 + \frac{\varepsilon}{k^2}}.$$

It is obvious that if $k \to \infty$, then expressions on the right and on the left tend to $\|x\| \cdot \|y\|$, whence so does the middle one. □

3. Superstability. In this section we prove the main result of the paper.

THEOREM 1. *If* $T : \mathbb{R}^n \to \mathbb{R}^n$ *($n \geq 2$) satisfies inequality (1), i.e., if*

$$\left| \, |T(x) \circ T(y)| - |x \circ y| \, \right| \leq \varepsilon \qquad \text{for} \quad x, y \in \mathbb{R}^n$$

(with a fixed $\varepsilon \geq 0$), then T satisfies the generalized orthogonality equation

$$|T(x) \circ T(y)| = |x \circ y| \qquad \text{for} \quad x, y \in \mathbb{R}^n.$$

In order to prove this theorem we need to state and to prove some partial results at first. We begin with the following

PROPOSITION 3. *Suppose that* $T : \mathbb{R}^n \to \mathbb{R}^n$ *($n \geq 2$) satisfies (1). Then, for arbitrary $x, y \in \mathbb{R}^n \setminus \{0\}$, one has*

$$|\cos A(T(x), T(y))| = |\cos A(x, y)|.$$

PROOF. We fix $x, y \in \mathbb{R}^n \setminus \{0\}$. Of course, for any $k \in \mathbb{N}$,

$$\cos A(x, y) = \cos A(kx, ky).$$

Moreover, since T preserves linear dependence of vectors and maps a nonzero vector into a nonzero one (Proposition 1.(ii), (iii)), we may state that

$$|\cos A(T(x), T(y))| = |\cos A(T(kx), T(ky))|.$$

Using inequality (1) for kx and ky and dividing by k^2 we get

$$\|x\| \cdot \|y\| \cdot |\cos A(x, y)| - \frac{\varepsilon}{k^2} \leq \frac{1}{k^2} \|T(kx)\| \cdot \|T(ky)\| \cdot |\cos A(T(x), T(y))|$$

$$\leq \|x\| \cdot \|y\| \cdot |\cos A(x, y)| + \frac{\varepsilon}{k^2}.$$

Letting $k \to \infty$ we obtain, on account of Lemma 3,

$$\|x\| \cdot \|y\| \cdot |\cos A(x,y)| = \|x\| \cdot \|y\| \cdot |\cos A(T(x), T(y))|,$$

whence

$$|\cos A(x,y)| = |\cos A(T(x), T(y))|.$$

□

To facilitate further calculations we prove the following

LEMMA 4. *If* $T : \mathbb{R}^n \to \mathbb{R}^n$ $(n \geq 2)$ *satisfies inequality* (1), *then there exists an orthogonal automorphism* φ *such that*
  (i) *the composition* $T' := \varphi T$ *satisfies* (1) *(with the same* ε*);*
  (ii) *elements of the canonical basis in* $\mathbb{R}^n$ — $e_1, \ldots, e_n$ *are eigen vectors for* T', *namely*

(12) $T'(e_i) = \lambda_i e_i$ *for a certain* $\lambda_i \in (0, \sqrt{1+\varepsilon}], \quad i = 1, \ldots, n.$

PROOF. If $\{e_1, \ldots, e_n\}$ is the canonical basis in $\mathbb{R}^n$, then – by Proposition 1.(ii),(iv) – $\{T(e_1), \ldots, T(e_n)\}$ forms an orthogonal basis in $\mathbb{R}^n$. We define an automorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\varphi(T(e_i)) := \|T(e_i)\| \cdot e_i \qquad \text{for} \quad i = 1, \ldots, n.$$

It is easy to prove (using Corollary 1) that φT satisfies (1). From Proposition 1.(i), (ii) we get

$$0 < \|T(e_i)\| \leq \sqrt{1+\varepsilon} \qquad \text{for} \quad i = 1, \ldots, n.$$

□

Bearing in mind Proposition 2.(c), now we investigate the case $n = 2$.

PROPOSITION 4. *Let* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *satisfy* (1) *and the condition* (12) *as well. Then either*
$$T(x) = \pm x \qquad \text{for all} \quad x \in \mathbb{R}^2$$

or
$$T(x) = \pm \bar{x} \qquad \text{for all} \quad x \in \mathbb{R}^2.$$

(Here $\bar{x}$ *denotes the vector conjugate with a vector* x, *i.e.,* $\bar{x} = (x_1, -x_2)$ *for* $x = (x_1, x_2)$*).*

PROOF. Of course, $T(0) = 0$ (Proposition 1.(ii)) so for $x = 0$ the assertion is clear.

1. We fix $0 \neq x \in \mathbf{R}^2$. According to Proposition 3, using (12), we get

$$|\cos A(T(x), e_1)| = |\cos A(x, e_1)|.$$

Thus we have proved that for a vector $x \in \mathbf{R}^2$ there exists a $\lambda \in \mathbf{R}$ such that either (a): $T(x) = \lambda x$ or (b): $T(x) = \lambda \bar{x}$. We consider sets A and B consisting of those vectors in $\mathbf{R}^2$ for which the case (a) or (b) holds, respectively. Of course, vectors in $\lim e_1 \cup \lim e_2$ belong to the intersection of A and B. The set $\mathbf{R}^2_* := \mathbf{R}^2 \setminus (\lim e_1 \cup \mathrm{line}_2)$ is contained either in A or in B. Indeed, if there existed vectors $x, y \in \mathbf{R}^2_*$ and $\lambda, \mu \in \mathbf{R} \setminus \{0\}$ such that

$$T(x) = \lambda x \qquad \text{and} \qquad T(y) = \mu \bar{y},$$

then, by Proposition 3, we would have

$$|\cos A(x, \bar{y})| = |\cos A(T(x), T(y))| = |\cos A(x, y)|,$$

which means that x or y belongs to $\lim e_1 \cup \lim e_2$ – a contradiction.

2. From the above we may state that for some $\lambda : \mathbf{R}^2 \to \mathbf{R}$ there is either

(α) $$T(x) = \lambda(x) \cdot x \qquad \text{for all} \quad x \in \mathbf{R}^2$$

or

(β) $$T(x) = \lambda(x) \cdot \bar{x} \qquad \text{for all} \quad x \in \mathbf{R}^2.$$

In the case (α), using (1), we get

$$\left| |\lambda(x)x \circ \lambda(x)x| - |x \circ x| \right| \leq \varepsilon \qquad \text{for any} \quad x \in \mathbf{R}^2,$$

whence

$$|\lambda^2(x) - 1| \cdot \|x\|^2 \leq \varepsilon \qquad \text{for} \quad x \in \mathbf{R}^2.$$

In particular, taking an $x \neq 0$ and an arbitrary $k \in \mathbf{N}$, we have

$$|\lambda^2(kx) - 1| \cdot \|x\|^2 \leq \frac{\varepsilon}{k^2},$$

whence

$$\lim_{k \to \infty} \lambda^2(kx) = 1 \qquad \text{for} \quad x \neq 0.$$

Now we fix arbitrary $x, y \in \mathbf{R}^2$ such that $x \circ y \neq 0$ and an arbitrary $k \in \mathbf{N}$. From (1) we get

$$\big|\, |\lambda(x)x \circ \lambda(ky)ky| - |x \circ ky|\,\big| \leq \varepsilon.$$

Dividing by k and letting $k \to \infty$ we get

$$\big|\, |\lambda(x)| - 1\big| \cdot |x \circ y| = 0,$$

which yields $|\lambda(x)| = 1$ for any $x \in \mathbf{R}^2$; in other words, in the case (α), $T(x) = \pm x$ for each $x \in \mathbf{R}^2$.

3. Proceeding analogously we may prove that in the case (β) we have $T(x) = \pm\overline{x}$ for each $x \in \mathbf{R}^2$. $\qquad\square$

The above proposition together with Lemma 4 imply (without assuming (12)) the following corollary.

COROLLARY 2. *If* $T : \mathbf{R}^2 \to \mathbf{R}^2$ *satisfies* (1), *then*

$$\|T(x)\| = \|x\| \qquad \text{for every} \quad x \in \mathbf{R}^2.$$

Now we are going to generalize the last corollary to the case of the spaces of higher dimensions.

PROPOSITION 5. *If* $T : \mathbf{R}^n \to \mathbf{R}^n$ $(n \geq 2)$ *satisfies the inequality* (1), *then*

$$\|T(x)\| = \|x\| \qquad \text{for each} \quad x \in \mathbf{R}^n.$$

PROOF. 1. For $x = 0$ the assertion holds trivially (Proposition 1.(ii)). We fix an $x \in \mathbf{R}^n \setminus \{0\}$ and take an arbitrary $0 \neq y \in (\text{lin } x)^{\perp}$. We have, in particular, $T(x) \neq 0$, $T(y) \neq 0$ and $T(y) \perp T(x)$ (Proposition 1.(ii),(iv)). Using Proposition 2.(c) we obtain

$$T\left(\text{lin}\left\{\frac{x}{\|x\|}, \frac{y}{\|y\|}\right\}\right) \subseteq \text{lin}\left\{\frac{T(x)}{\|T(x)\|}, \frac{T(y)}{\|T(y)\|}\right\},$$

whence, for a pair of reals (λ_1, λ_2), there exists a unique pair (λ'_1, λ'_2) such that

$$T\left(\lambda_1 \frac{x}{\|x\|} + \lambda_2 \frac{y}{\|y\|}\right) = \lambda'_1 \frac{T(x)}{\|T(x)\|} + \lambda'_2 \frac{T(y)}{\|T(y)\|}.$$

We define a mapping $T' : \mathbf{R}^2 \to \mathbf{R}^2$ by the formula

$$T'(\lambda) = \lambda',$$

where $\lambda = (\lambda_1, \lambda_2)$ and $\lambda' = (\lambda_1', \lambda_2')$ – according to the rule described above.

2. We prove that the function T' is a solution of (1). Let $\lambda = (\lambda_1, \lambda_2)$, $\mu = (\mu_1, \mu_2) \in \mathbf{R}^2$ be fixed and let

$$u = \lambda_1 \frac{x}{\|x\|} + \lambda_2 \frac{y}{\|y\|}, \qquad v = \mu_1 \frac{x}{\|x\|} + \mu_2 \frac{y}{\|y\|}.$$

Then

$$|u \circ v| = |\lambda_1 \mu_1 + \lambda_2 \mu_2| = |\lambda \circ \mu|$$

and

$$|T(u) \circ T(v)| = \left| \left(\lambda_1' \frac{T(x)}{\|T(x)\|} + \lambda_2' \frac{T(y)}{\|T(y)\|} \right) \circ \left(\mu_1' \frac{T(x)}{\|T(x)\|} + \mu_2' \frac{T(y)}{\|T(y)\|} \right) \right|$$

$$= |\lambda_1' \mu_1' + \lambda_2' \mu_2'| = |\lambda' \circ \mu'| = |T'(\lambda) \circ T'(\mu)|.$$

Since T satisfies (1),

$$\big| |T'(\lambda) \circ T'(\mu)| - |\lambda \circ \mu| \big| = \big| |T(u) \circ T(v)| - |u \circ v| \big| \le \varepsilon,$$

whence T' satisfies (1) as well.

3. Now we may use Corollary 2. We have $\|T'(\lambda)\| = \|\lambda\|$ for each $\lambda \in \mathbf{R}^2$. Let $\lambda = (\|x\|, 0) \in \mathbf{R}^2$; then, by the definition of function T', we have $T'(\lambda) = (\|T(x)\|, 0)$ and so

$$\|x\| = \|\lambda\| = \|T'(\lambda)\| = \|T(x)\|.$$

$\square$

PROOF OF THEOREM 1. All we need to do now, is to combine the assertion of Proposition 3 with the one of Proposition 5. $\square$

4. Final remarks. The following example shows that the superstability of the generalized orthogonality equation is no longer true in the general case.

EXAMPLE 1. We consider the Hilbert space l^2 with the usual inner product and define a mapping $T : l^2 \to l^2$ by the formula

$$T(x) = T(x_1, x_2, \dots) := (\sqrt{\varepsilon}, x_1, x_2, \dots) \qquad \text{for} \quad x \in l^2.$$

We have $T(x) \circ T(y) = x \circ y + \varepsilon$ so T satisfies (1). On the other hand, $T(0, 0, \dots) = (\sqrt{\varepsilon}, 0, 0, \dots)$, whence T cannot be a solution of the generalized orthogonality equation as for each such solution should be $T(0) = 0$.

If we consider the *orthogonality* equation

$$(13) \qquad \dot{T}(x) \circ T(y) = x \circ y \qquad \text{for} \quad x, y \in \mathbb{R}^n$$

and the approximate solutions of it, given by

$$(14) \qquad |T(x) \circ T(y) - x \circ y| \leq \varepsilon \qquad \text{for} \quad x, y \in \mathbb{R}^n$$

we obtain immediately that (14) implies (1) and then, as a corollary from Theorem 1, that T is a solution of (2). However, from (2) and (14) one can derive (13). That means the superstability of the orthogonality equation. We omit the details of the proof.

Finally, let us remark that the superstability phenomenon is sometimes considered as something unnatural and caused by an improper definition of the class of approximate solutions (see [5, pp. 109–110]). Comparing both sides of (2) it would be, probably, more adequate to the structure of the inner product space, to deal with the division of values of those sides instead of the difference when defining approximate solutions. The author has made such an attempt and proved the stability, in the new sense, of the generalized orthogonality equation in the case of the Euclidean space $\mathbb{R}^n$ (see [4]). Superstability does not hold in that case.

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INSTITUTE OF MATHEMATICS
PEDAGOGICAL UNIVERSITY IN CRACOW
PODCHORĄŻYCH 2
PL-30-084 KRAKÓW, POLAND