FUNCTIONAL EQUATIONS FOR HOMOGENEOUS POLYNOMIALS ARISING FROM MULTILINEAR MAPPINGS AND THEIR STABILITY

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Abstract. Some characterizations of homogeneous polynomials and of polynomials in general are given. This is done not only in the usual case but also for vector spaces over non archimedean valued fields. Moreover stability results in connection with these characterizations are given.

1. Introduction. Let A be a commutative ring with identity element 1 and let M and N be A-modules. For mappings $f : M \to N$ ($f \in N^M$) and elements $x \in M$ the difference operator $\Delta_x : N^M \to N^M$ is defined by

$$\underset{x}{\Delta} f(y) := f(x+y) - f(y).$$

$\Delta_{x_1,\ldots,x_n}$ is defined by

$$\underset{x_1,\ldots,x_n}{\Delta} := \underset{x_1}{\Delta} \circ \cdots \circ \underset{x_n}{\Delta}.$$

K. J. Heuvers ([4]) introduced the operators $\mathbf{K}_n$ defined by

$$\mathbf{K}_n f(x_1,\ldots,x_n) := \underset{x_1,\ldots,x_n}{\Delta} f(0) + (-1)^{n+1} f(0)$$

and used them to formulate functional equations whose solutions are the diagonalizations of n-linear mappings with domain M^n and range N. (It seems to be unclear whether in the most general case considered by K. J.

Heuvers these functional equations are characterizations of these homogeneous polynomials.)

The usual characterizations of homogeneous polynomials (see e.g. [5], [3]) use the difference operators $\Delta_h^n = \Delta_{h,\dots,h}$. But in this case one only gets a characterization of those homogeneous polynomials which are the diagonalization of n-additive (= $\mathbb{Z}$-n-linear) mappings.

In this paper we present a characterization of n-homogeneous polynomials which are diagonalizations of n-linear mappings when the ring A is such that $n!$ is a unit in this ring. This characterization is close to the usual one for diagonalizations of n-additive functions. Furthermore we investigate the stability properties of both those functional equations. This is done not only in the usual case (i.e. for normed or topological vector spaces over the reals) but also for normed vector spaces over the p-adic completion $\mathbb{Q}_p$ of $\mathbb{Q}$ with respect to the non-archimedean absolute value $|\ |_p$. Regarding the latter case (even for $A = \mathbb{Z}$) relatively little seems to be known (see [2]).

J. Tabor ([9]) considered stability theorems in connection with $\mathbb{R}$-linear functions. He presented his results at the 30th ISFE held in Oberwolfach in 1992. At that occasion G. L. Forti pointed out to me that one of the author's earlier results ([7]) could be used for investigating the stability properties of $\mathbb{R}$-linear mappings in the following way:

Let V be a real vector space and let E be a Banach space over $\mathbb{R}$. Then for any $f : V \to E$ the following holds. If

$$\|f(\alpha x + \beta y) - f(\alpha x) - f(\beta y)\| \le \varepsilon(\alpha, \beta) \quad (x, y \in V,\ \alpha, \beta \in \mathbb{R}),$$

where $\bar{\varepsilon} : \mathbb{R}^2 \to \mathbb{R}_+ := \{\gamma \in \mathbb{R}|\ \gamma > 0\}$, there is a unique $\mathbb{R}$-linear function $g : V \to E$ such that the difference $f - g$ is bounded.

Here one of the important points is that the bound ε may depend on α and β. In fact, Tabor's result was that for ε not depending on the scalars α and β "superstability" appears, i.e. in this case f itself necessarily has to be $\mathbb{R}$-linear. Our future investigations will follow this line. This means that our stability results are presented in a form where the bound ε may depend on the scalars involved.

2. The stability of Heuver's equation. In order to formulate the results from [4] and to consider the stability properties of the functional equation involved we need some preliminaries.

Let $M, N, A, \dots$ be as in the Introduction. Let n be an integer, $n \ge 2$. For $\alpha := (\alpha_1, \dots, \alpha_n) \in A^n$, $x := (x_1, \dots, x_n) \in M^n$, and $f \in N^M$ let

$$\mathbf{L}_{\alpha,n} f(x) := \mathbf{L}_\alpha f(x)$$

$$(1) \qquad := L_n \left[f\left(\sum_{i=1}^n \alpha_i x_i \right) - \sum_{i=1}^n \alpha_i^n f(x_i) \right] - \sum_{s \in S_n} \frac{L_n}{s!} \alpha^s\, \mathbf{K}_n f(x^s),$$

where

$$S_n := S := \left\{ s = (s_1, \ldots, s_n) \in \mathbb{N}_0^n \mid 0 \le s_i \le n-1, \ |s| := \sum_i s_i = n \right\},$$

$$x^s := (\underbrace{x_1, \ldots, x_1}_{s_1 \text{ times}}, \ldots, \underbrace{x_n, \ldots, x_n}_{s_n \text{ times}}),$$

$$\alpha^s := \prod_{i=1}^n \alpha_i^{s_i}, \quad s! := \prod_{i=1}^n s_i!, \quad \text{and} \quad L_n := \mathrm{lcm}\{s! \mid s \in S\}.$$

For convenience we also mention two formulae which will be useful in the future.

THE MULTINOMIAL THEOREM (see e.g. [8]). *Let* A, M, N *be as above. Let* $g : M^n \to N$ *be n-linear and symmetric. Then for* $x = (x_1, \ldots, x_k) \in M^k$ *and* $\alpha = (\alpha_1, \ldots, \alpha_k) \in A^k$ *we have*

$$g\left(\left(\sum_{i=1}^k \alpha_i x_i \right)^n \right) = \sum_{t \in T} \frac{n!}{t!} \alpha^t g(x^t),$$

where $T_k := T := \{ t = (t_1, \ldots, t_k) \in \mathbb{N}_0^k \mid \sum_{i=1}^k t_i = n \}$.

In [4] one finds the explicit expression

$$\underset{n}{\mathrm{K}} f(x_1, \ldots, x_n) = \sum_J (-1)^{n-|J|} f\left(\sum_{j \in J} x_j \right),$$

where the sum runs over all nonempty subsets J of $\mathbf{n} := \{1, 2, \ldots, n\}$ and $|J|$ denotes the cardinality of J.

Among others, in [4] the following is proved.

THEOREM 1. *Let* A *be a commutative ring with identity and let* M *be a free A-module. Let* A *contain elements* u *and* v *such that* $y \mapsto uv(u+v)y$ *is an injection from the A-module* N *into itself, and let, for some integer* $n \ge 2$, $(n-1)!$ *be a unit in* A. *Then, given a mapping* $f : M \to N$, *the condition*

(2) $$\mathrm{L}_\alpha f = 0 \qquad (\alpha \in A^n)$$

implies that there is some n-linear mapping $g : M^n \to N$ *such that* f *is the diagonalization of* g, *i.e.* $f = g \circ \delta_n$ *(with* $\delta_n(x) = (x, \ldots, x) \in M^n$ *for all* $x \in M$).

REMARK 1. In the case that $n!$ is a unit in A - which implies that also $(n-1)!$ is a unit - the converse is also true:

Let A be a commutative ring with identity such that for $n \in \mathbb{N}$, $n \geq 2$, the number $n!$ is a unit in A. Let M, N be arbitrary A-modules, and let $f : M \to N$ be the diagonalization of some n-linear mapping $g : M^n \to N$. Then f fulfils (2).

PROOF OF REMARK 1. In the case considered g may be supposed to be symmetric since $f = g \circ \delta_n$ implies $f = g' \circ \delta_n$ where g' defined by

$$g'(x_1, \ldots, x_n) = \frac{1}{n!} \sum_{\pi \in S_n} g(x_{\pi(1)}, \ldots, x_{\pi(n)})$$

(S_n denotes the group of all permutations of n), is n-linear and symmetric.

But then by the multinomial theorem (for the diagonalization of symmetric n-linear functions) and using the relation

$$\mathop{\mathbf{K}}_{n} f(y_1, \ldots, y_n) = \mathop{\Delta}_{y_1, \ldots, y_n} f(0)$$

as well as

$$\mathop{\Delta}_{y_1, \ldots, y_n} (g' \circ \delta_n)(0) = n! g'(y_1, \ldots, y_n)$$

(see [3]) we get

$$f\left(\sum_{i=1}^{n} \alpha_i x_i\right) = g'\left(\left(\sum_{i=1}^{n} \alpha_i x_i\right)^n\right) = \sum_{i=1}^{n} \alpha_i^n g'(x_i{}^n) + \sum_{s \in S} \frac{n!}{s!} \alpha^s g'(x^s)$$

$$= \sum_{i=1}^{n} \alpha_i^n f(x_i) + \sum_{s \in S} \frac{1}{s!} \alpha^s \mathop{\mathbf{K}}_{n} f(x^s),$$

which - when multiplied by L_n - is the desired result. $\qquad\square$

Now we consider a non trivial valuation $| \ |$ on the field $\mathbb{Q}$ of rational numbers. For the following facts on valuations we refer to [6].

It is well-known (Ostrowski's theorem) that such a valuation is either equivalent to the usual absolute value or to one of the (non-archimedean) valuations $| \ |_p$ where p is some prime and $|r|_p = p^{-k}$ if $\mathbb{Q} \ni r = p^k \frac{a}{b}, k \in \mathbb{Z}, a, b \in \mathbb{Z}, a, b \neq 0$, and $(a, p) = (b, p) = 1$ (and $|0|_p = 0$). Furthermore we may embed $(\mathbb{Q}, | \ |)$ in a complete valued field $(\hat{\mathbb{Q}}, | \ |)$. Normed spaces and Banach spaces may be defined as usual.

We have the following.

THEOREM 2. *Let A be a commutative ring containing $\mathbb{Q}$ as a subring. Let $\mathbb{Q} \ni 1$ be the identity of A. Let furthermore M, N be A-modules and let $(N, \| \ \|)$ at the same time be a Banach space over the completion $(\widehat{\mathbb{Q}}, | \ |)$ for some absolute value $| \ |$ of $\mathbb{Q}$. Then, for $f : M \to N$ and $n \geq 2$, the condition*

$$(3) \qquad \|\mathbf{L}_\alpha f(x)\| \leq \varepsilon(\alpha) \qquad (x \in M^n, \alpha \in A^n),$$

where $\varepsilon : A^n \to \mathbb{R}_+$ is some given function, implies that there is a (unique) A-n-linear and symmetric function $g : M^n \to N$, such that $f - g \circ \delta_n$ is bounded.

PROOF. Using (3) for $\alpha_1 = m \in A$ and $\alpha_i = 0$ for $2 \leq i \leq n$ gives

$$(4) \quad \|f(mx) - m^n f(x)\| \leq \varepsilon' \quad (\varepsilon' := \varepsilon(m, 0, \ldots, 0)/|L|, \ x \in M, m \in A).$$

Now we consider two cases.

CASE 1. $| \ |$ is the usual absolute value (and $\widehat{\mathbb{Q}} = \mathbb{R}$). In this case (4) for $m \in \mathbb{N}, m \geq 2$ reads as

$$(4') \qquad \left\| \frac{f(mx)}{m^n} - f(x) \right\| \leq \frac{\varepsilon'}{m^n} \qquad (x \in M, m \in \mathbb{N}).$$

Fix $m \geq 2$ and put $\varphi_k(x) := \frac{f(m^k x)}{m^{kn}}$. Then by $(4')$

$$\|\varphi_{k+1}(x) - \varphi_k(x)\| = \left\| \frac{f(mm^k x)}{m^{(k+1)n}} - \frac{f(m^k x)}{m^{kn}} \right\| \leq \frac{\varepsilon'}{m^{(k+1)n}}.$$

Thus using standard arguments the sequence $(\varphi_k(x) \mid k \in \mathbb{N}_0)$ is Cauchy with

$$(5) \quad \begin{aligned} \|\varphi_{k+l}(x) - \varphi_k(x)\| &\leq \varepsilon'(m^{-n(k+1)} + \cdots + m^{-n(k+l)}) \\ &\leq \frac{\varepsilon'}{m^{n(k+1)}}(1 - m^{-n})^{-1}. \end{aligned}$$

Denoting the limit function by φ and putting $k = 0$ in (5) we get for $l \to \infty$

$$(6) \qquad \|\varphi(x) - f(x)\| \leq \frac{\varepsilon'}{m^n - 1} \leq \varepsilon'.$$

By the explicit formula for $\mathbf{K}_n f$ as given above we see that

$$\mathbf{K}_n \varphi_k(x_1, \ldots, x_n) = m^{-kn} \mathbf{K}_n f(m^k x_1, \ldots, m^k x_n).$$

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Thus (3) (with x replaced by $m^k x$) yields

(7)
$$\|\mathbf{L}_\alpha \varphi_k(x)\| \leq \frac{\varepsilon(\alpha)}{m^{kn}}.$$

Obviously $\lim_{k \to \infty} \mathbf{L}_\alpha \varphi_k(x) = \mathbf{L}_\alpha \varphi(x)$. Accordingly (7) implies $\mathbf{L}_\alpha \varphi(x) = 0$. By Theorem 1 this means that there is some symmetric and n-linear g such that $\varphi = g \circ \delta_n$. Moreover, using (6), we have that $f - g \circ \delta_n$ is bounded. The uniqueness is shown in the usual way. If g' is another n-linear and symmetric mapping such that $f - g' \circ \delta_n$ is bounded, then $(g - g') \circ \delta_n$ is bounded and thus (by homogeneity) 0. But then (using the relation mentioned in the proof of Remark 1)

$$0 = \underset{y_1, \ldots, y_n}{\Delta} (g - g') \circ \delta_n(0) = n!(g - g')(y_1, \ldots, y_n)$$

implying $g = g'$, as desired.

CASE 2. $|\ | = |\ |_p$ for some prime p. Then (3) for $\alpha_1 = 1/p$ and $\alpha_i = 0$ for $i > 1$ leads to

$$\|f(x/p) - p^{-n} f(x)\| \leq \varepsilon' \qquad (\varepsilon' := \varepsilon(1/p, 0, \ldots, 0)/|L|, \ x \in M).$$

Multiplying this by $|p^n|$ and observing that now $|p| = |p|_p = 1/p$ we get

(4″)
$$\|p^n f(x/p) - f(x)\| \leq \frac{\varepsilon'}{p^n} \qquad (x \in M).$$

In this case we define $\varphi_k(x) := p^{nk} f(x/p^k)$. Similarly as in the first case we have

$$\|\varphi_{k+1}(x) - \varphi_k(x)\| \leq \frac{\varepsilon'}{p^{n(k+1)}} \quad \text{and} \quad \|\varphi_{k+l}(x) - \varphi_k(x)\| \leq \frac{\varepsilon'}{p^{n(k+1)}} \frac{1}{1 - p^{-n}},$$

which shows that the sequence $(\varphi_k(x))$ is a Cauchy sequence and that for $k = 0$ and $l \to \infty$ (with $\varphi := \lim_{k \to \infty} \varphi_k$)

$$\|\varphi(x) - f(x)\| \leq \frac{\varepsilon'}{p^n - 1}.$$

From this point on we may argue as in the first case. □

Let us point out that restricting ourselves to the cases considered is no real restriction of generality since replacing a valuation by an equivalent one, up to a change in the function ε, does not affect the validity of the condition (3).

The case $n = 1$ (i.e., the linear case) is not contained in this theorem. But we have the following generalization of the result mentioned in the Introduction.

THEOREM 3. *Let A be a commutative ring containing $\mathbb{Q}$ as a subring. Let $\mathbb{Q} \ni 1$ be the identity of A. Let furthermore M, N be A-modules and let $(N, \| \ \|)$ at the same time be a Banach space over the completion $(\widehat{\mathbb{Q}}, | \ |)$ for some absolute value $| \ |$ of $\mathbb{Q}$. Then, for $f : M \to N$ the condition*

$$(8) \quad \|f(\alpha x + \beta y) - \alpha f(x) - \beta f(y)\| \leq \varepsilon(\alpha, \beta) \quad (\alpha, \beta \in A, \ x, y \in M),$$

where $\varepsilon : A^2 \to \mathbb{R}_+$ is some given function, implies that there is a (unique) A-linear $g : M \to N$, such that $f - g$ is bounded.

PROOF. In the case of the (usual) archimedean absolute value the substitution $\alpha = \beta = 1$ shows that the conditions for applying the well-known stability result of Hyers (see e.g. [5]) are fulfilled. Thus there is a unique additive function g such that $f - g$ is bounded. Moreover g is given by $g(x) = \lim_{k \to \infty}(f(kx)/k)$. We have to show that g is A-linear. But (8) for $\beta = 0$ gives

$$\|f(\alpha x) - \alpha f(x)\| \leq \varepsilon' := \varepsilon(\alpha, 0)$$

which holds true for all α and all x. Replacing x by kx, dividing by k and letting tend k to ∞ then yields $\|g(\alpha x) - \alpha g(x)\| = 0$. This shows that g is homogeneous and thus A-linear.

In the non-archimedean case $| \ | = | \ |_p$ we may use similar arguments. Again $\alpha = \beta = 1$ leads to

$$(8') \quad \|f(x + y) - f(x) - f(y)\| \leq \varepsilon' \quad (x, y \in M),$$

where $\varepsilon' = \varepsilon(1, 1)$. It is easy to see by induction that this implies $\|f(nx) - nf(x)\| \leq (n - 1)\varepsilon'$ for all $n \in \mathbb{N}$. For $n = p$ this means that

$$\|f(x) - pf(x/p)\| \leq (p - 1)\varepsilon' =: \varepsilon'' \quad (x \in M).$$

Using this for x/p^n yields

$$\|p^n f(x/p^n) - p^{n+1} f(x/p^{n+1})\| \leq |p^n| \varepsilon'' = \varepsilon''/p^n.$$

Thus the sequence $(g_n(x)), g_n(x) := p^n f(x/p^n)$ is Cauchy. Explicit bounds (as in the proof of the theorem above) show that the norm of $f - g_n$ is bounded by some constant independent of n. So $f - g$, $g := \lim_{n \to \infty} g_n$, is bounded. Finally $(8')$ for $p^n x$ and $p^n y$ shows that

$$\|g_n(x + y) - g_n(x) - g_n(y)\| \leq \varepsilon'/p^n$$

which for $n \to \infty$ implies that g is additive. This g is the only additive function such that $f - g$ is bounded. In fact, the boundedness of $f - h$, h additive, implies that $p^n f(x/p^n) - h(x) = p^n(f(x/p^n) - h(x/p^n))$ tends to 0 for $n \to \infty$. Thus $h = g$.

This additive function, g, is also homogeneous. This follows from equation (8) multiplied by $|p^n|$ for $\beta = 0$ and $n \to \infty$, when x is replaced by x/p^n. $\square$

REMARK 2. Inspecting the proof of the second part one sees that we have also proved the following:

Let G be an abelian group which, for p prime, is uniquely divisible by p. Let E be a Banach space over $\mathbb{Q}_p$, the completion of $\mathbb{Q}$ with respect to $|\ |_p$. Then for any $f : G \to E$ the condition

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon$$

implies that there is a (unique) additive mapping $g : G \to E$ such that $f - g$ is bounded.

This was shown for $G = \mathbb{Q}_p$ in [2].

3. A characterization of homogeneous functions arising from n-linear mappings. Corollary 3 from [3] may be formulated as follows.

THEOREM 4. *Let H be an abelian semigroup and G an abelian group satisfying the condition: for each $a \in G$ the equation $(n!)x = a$ has a unique solution $x = a/(n!)$. Then a necessary and sufficient condition that $f : H \to G$ has the form $f = g \circ \delta_n$, where g is symmetric and additive in each variable is that*

$$\underset{u}{\overset{n}{\Delta}} f = n! f(u) \quad \text{for all} \quad u \in H.$$

Our aim now is to get a characterization for functions f to be of the form $f = g \circ \delta_n$, where g is n-linear and symmetric.

We assume as above that A is a commutative ring with identity which is uniquely divisible by $n!$, where $n \geq 2$ is a fixed integer. M and N are A-modules. For $x = (x_1, \ldots, x_n) \in M^n$ and $s = (s_1, \ldots, s_n) \in \mathbb{N}^n$ we use the notion

$$\underset{x^s}{\Delta} := \underset{x_1}{\overset{s_1}{\Delta}} \circ \cdots \circ \underset{x_n}{\overset{s_n}{\Delta}} .$$

For $f : M \to N$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in A^n$, and $y \in M$ we consider the operator $\mathbf{H}_{\alpha,y}$ defined by

$$(9) \quad \mathbf{H}_{\alpha,y} f(x) := n! \left[f\left(\sum_{i=1}^n \alpha_i x_i \right) - \sum_{i=1}^n \alpha_i^n f(x_i) \right] - \sum_{s \in S_n} \frac{n!}{s!} \alpha^s \underset{x^s}{\Delta} f(y).$$

Then the following holds true.

THEOREM 5. *Let A, M, N, and n be as above. Then a necessary and sufficient condition that $f : M \to N$ has the form $f = g \circ \delta_n$, where $g : M^n \to N$ is n-linear, is that*

$$(10) \qquad \mathbf{H}_{\alpha,y} f(x) = 0 \qquad (\alpha \in A^n, x \in M^n, y \in M).$$

PROOF. At first we show that the condition is necessary. So, let $f = g \circ \delta_n$, where g is n-linear. As in the proof of Remark 1 we may assume that g is also symmetric. From [3] we know that

$$(11) \qquad \underset{x_1,\dots,x_n}{\Delta} (g \circ \delta_n)(y) = n! g(x_1, \dots, x_n).$$

Using (11), the definition of $\mathbf{H}_{\alpha,y}$, and the multinomial theorem one gets the desired result in a way similar to the corresponding part in the proof of Remark 1.

The condition is also sufficient. For, suppose that (10) holds. Then for $\alpha_1 = \beta$, $\alpha_i = 0$ for $i \geq 2$ we get

$$(12) \qquad n!(f(\beta x_1) - \beta^n f(x_1)) = 0 \qquad (\beta \in A, x_1 \in M)$$

since for this α we have $\alpha^s = 0$ if $s \in S_n$. Using (10) for $x = (a, \dots, a)$ with all the $x_i = a \in M$ we get (since $\Delta_{x^s} = \Delta_a^n$ for $s \in S_n$)

$$n! \left(f\left(\left(\sum_{i=1}^n \alpha_i \right) a \right) - \left(\sum_{i=1}^n \alpha_i^n \right) f(a) \right) - \left(\sum_{s \in S_n} \frac{n!}{s!} \alpha^s \right) \underset{a}{\overset{n}{\Delta}} f(y) = 0.$$

But $f((\sum_{i=1}^n \alpha_i)a) = (\sum_{i=1}^n \alpha_i)^n f(a)$ by (12) and

$$\left(\sum_{s \in S_n} \frac{n!}{s!} \alpha^s \right) = \sum_{t \in T} \frac{n!}{t!} \alpha^t - \sum_{t \in U_n} \alpha^t = \left(\sum_{i=1}^n \alpha_i \right)^n - \sum_{i=1}^n \alpha_i^n.$$

Thus

$$(13) \quad \left(\left(\sum_{i=1}^n \alpha_i \right)^n - \sum_{i=1}^n \alpha_i^n \right) \left(n! f(a) - \underset{a}{\overset{n}{\Delta}} f(y) \right) = 0 \quad (a, y \in M, \alpha \in A^n).$$

By Lemma 1 below this implies

$$n! f(a) = \underset{a}{\overset{n}{\Delta}} f(y) \qquad ,(a, y \in M),$$

and by previous statements that $f = g \circ \delta_n$ for some symmetric and n-additive function g (which in fact is given by $g(x_1, \dots, x_n) = (n!)^{-1} \Delta_{x_1, \dots, x_n} f(y)$). We still have to show that g is n-linear. Fix α and x and write $\alpha x :=$ $(\alpha_1 x_1, \dots, \alpha_n x_n)$. Let furthermore $m = (m_1, \dots, m_n) \in (\mathbb{Z} \cdot 1)^n \subseteq A^n$ and $ma := (m_1 \alpha_1, \dots, m_n \alpha_n)$. By (10) and (9)

$$n! \left(f\left(\sum_{i=1}^n \alpha_i x_i \right) - \sum_{i=1}^n \alpha_i^n f(x_i) \right) - \sum_{s \in S_n} \frac{n!}{s!} \alpha^s \mathop{\Delta}_{x^s} f(y) = 0$$

Since $f = g \circ \delta_n$ and $n! g(x^t) = \Delta_{x^t} f(y)$ for all $t \in T$ we obtain

$$n! \left(f\left(\sum_{i=1}^n \alpha_i x_i \right) - \sum_{t \in T} \frac{n!}{t!} \alpha^t g(x^t) \right) = 0.$$

Dividing by $n!$ and replacing α by $m\alpha$ leads to

$$f\left(\sum_{i=1}^n m_i \alpha_i x_i \right) - \sum_{t \in T} \frac{n!}{t!} (m\alpha)^t g(x^t) = 0.$$

Putting in this equality $\alpha = (1, 1, \dots, 1)$ and then replacing x by αx we get

$$f\left(\sum_{i=1}^n m_i \alpha_i x_i \right) = \sum_{t \in T} \frac{n!}{t!} m^t g((\alpha x)^t).$$

Equating the right-hand sides of these equations and using Lemma 1 again finally ends up with

$$\alpha^t g(x^t) = g((\alpha x)^t) \qquad (t \in T = T_n)$$

which for $t = (1, \dots, 1)$ implies the homogeneity of g in each variable. $\qquad \square$

LEMMA 1. *Let A be a commutative ring with identity containing a set U consisting of $n (\geq 2)$ units and the 0-element of A. Let U be such that the set $E := U - U$ of all differences $u - v$, $u, v \in U$ is contained in $U(A) \cup \{0\}$, where $U(A)$ is the group of units of A. Let M be an A-module. Then for any $k \geq 1$ the relation*

$$\sum_{l \in L_k} u^l x_l = 0 \quad (u = (u_1, \dots, u_k) \in E^k)$$

implies that all the $x_l \in M$ vanish. Here $L_k := \{l = (l_1, \cdots, l_k) \in \mathbb{N}_0^k \mid l_1, \dots, l_k \leq n\}$.

PROOF. The proof is by induction. For $k = 1$ the relation reads as

$$\sum_{l=0}^{n} u^l x_l = 0 \qquad (u \in E).$$

Next we choose (which is possible) $n + 1$ distinct elements $u_0, \ldots, u_n \in E$. Then we have a system of linear $n + 1$ linear equations

$$\sum_{l=0}^{n} u_0^l x_l = 0,$$

$$\sum_{l=0}^{n} u_1^l x_l = 0,$$

$$\ldots = 0,$$

$$\sum_{l=0}^{n} u_n^l x_l = 0.$$

The determinant of the matrix $C := (u_j^l)_{\substack{0 \leq j \leq n \\ 0 \leq l \leq n}}$ is the Vandermonde determinant $\prod_{0 \leq i < j \leq n} (u_j - u_i)$ which by our assumptions is a unit in A. Thus C has an inverse C^{-1} in the ring of $(n \times n)$-matrices over A. This implies that the above system of equations has only the trivial solutions. The induction process then uses that the original relation may be written as

$$\sum_{j=0}^{n} u^j \left(\sum_{l \in L_{k-1}} v^l x_{(j,l)} \right) = 0 \qquad (u \in E, \ v \in E^{k-1}).$$

Fixing v and varying u then yields that the inner sums vanish for all v. Then we may apply our induction hypothesis. □

Now we investigate the stability properties of (10).

THEOREM 6. *Let A be a commutative ring containing $\mathbb{Q}$ as a subring. Let $\mathbb{Q} \ni 1$ be the identity of A. Let furthermore M, N be A-modules and let $(N, \| \ \|)$ at the same time be a Banach space over the completion $(\widehat{\mathbb{Q}}, | \ |)$ for some absolute value $| \ |$ of $\mathbb{Q}$. Then, for $f : M \to N$ and $n \geq 2$, the condition*

(14) $$\|H_{\alpha,y} f(x)\| \leq \varepsilon(\alpha) \qquad (x \in M^n, \alpha \in A^n, y \in M),$$

where $\varepsilon : A^n \to \mathbb{R}_+$ is some given function, implies that there is a (unique) A-n-linear and symmetric function $g : M^n \to N$, such that $f - g \circ \delta_n$ is bounded.

PROOF. The proof completely follows the lines given in the proof of Theorem 2. For some estimates one has to use the following explicit formula for $\Delta_{x_1,\ldots,x_n} f$ which may be found in [5, p. 367]:

$$\mathop{\Delta}_{x_1,\ldots,x_n} f(y) = \sum_{J \subseteq n} (-1)^{n-|J|} f(y + \textstyle\sum_{j \in J} x_j).$$

$\square$

4. The stability of some functional equations for polynomials and homogeneous polynomials in the non-archimedean case.

In [5, Ch. XVII] the stability properties of the equations

$$(15) \qquad \mathop{\Delta}_{x_1,\ldots,x_{n+1}} f(y) = 0, \quad \mathop{\Delta}_{x}^{n+1} f(y) = 0, \quad \text{and} \quad \mathop{\Delta}_{x}^{n} f(y) = n! f(x)$$

are investigated when the underlying Banach spaces are real spaces, i.e., the archimedean situation is considered. Here we want to do the same in the non-archimedean case. In doing so we follow the lines as given in the above reference.

Let $| \ | = | \ |_p$ be a fixed non-archimedean valuation of $\mathbb{Q}$ and $\mathbb{Q}_p$ the completion thereof. Let furtheremore M be a vector space over $\mathbb{Q}$ and N a Banach space over $\mathbb{Q}_p$. This notations are held fixed in this chapter.

Then we have the following generalization of Remark 2.

THEOREM 7. *Let $n \geq 2$ be a fixed integer. If $f : M^n \to N$ satisfies for every $x_1, \ldots, x_n, y_1, \ldots, y_n \in M$ the system of inequalities*

$$(16) \qquad \begin{cases} \|f(x_1, \ldots, x_i + y_i, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n) \\ \qquad - f(x_1, \ldots, y_i, \ldots, x_n)\| \leq \varepsilon_i \quad (1 \leq i \leq n), \end{cases}$$

where ε_i, $1 \leq i \leq n$ are positive numbers, then there exists a unique n-additive function $g : M^n \to N$ such that $f - g$ is bounded. Moreover, if f is symmetric, then g also is symmetric. The bound for $f - g$ may be chosen to depend on ε only.

PROOF. By (the proof of) Theorem 3, for every $x_1, \ldots, x_n \in M$ the limit

$$g(x_1, \ldots, x_n) = \lim_{k \to \infty} p^k f(x_1/p^k, x_2, \ldots, x_n)$$

exists, and with fixed $x_2, \ldots, x_n$ is additive in x_1. Moreover, for some ε'

$$\|f(x_1, \ldots, x_n) - g(x_1, \ldots, x_n)\| \leq \varepsilon'$$

for all $x_1, \ldots, x_n \in M$. For every $i = 2, \ldots, n$ and all $x_1, \ldots, x_n, y_i \in M$ we have by (16)

$$\| f(x_1/p^k, \ldots, x_i + y_i, \ldots, x_n) - f(x_1/p^k, \ldots, x_i, \ldots, x_n) - $$
$$- f(x_1/p^k, \ldots, y_i, \ldots, x_n) \| \leq \varepsilon_i,$$

whence, on multiplying by $|p^k|$ and passing to the limit as $k \to \infty$, we obtain that g is n-additive. Uniqueness follows from the fact that (as in the archimedean case) a bounded n-additive function vanishes.

The symmetry of g for symmetric f can be seen as follows. Let $\pi \in \mathbf{S}_n$ be fixed. Then the function h, $h(x_1, \ldots, x_n) := g(x_{\pi(1)}, \ldots, x_{\pi(n)})$, is n--additive. The symmetry of f then implies

$$\| f(x_1, \ldots, x_n) - h(x_1, \ldots, x_n) \| \leq \varepsilon',$$

whence by the uniqueness part of the theorem $g = h$. Since π was arbitrary this means that g is symmetric. The bound for $f - g$ can be given in the form $\varepsilon' = c_p \varepsilon$ where c_p depends only on the prime p. This follows from the proof of Remark 2. $\qquad\square$

Generalized polynomials (with $\mathbf{Z}$-l-homogeneous components) can be defined as the solutions of the first or second equation of (15). The following two theorems deal with the stability of these equations.

THEOREM 8. *Let, for fixed $n \in \mathbf{N}_0$, the function $f : M \to N$ satisfy (with certain $\varepsilon > 0$)*

$$(17) \qquad \left\| \mathop{\Delta}_{x_1, \ldots, x_{n+1}} f(y) \right\| \leq \varepsilon \quad (y, x_1, \ldots, x_{n+1} \in M).$$

Then there exist symmetric k-additive functions $F_k : M^k \to N$, $0 \leq k \leq n$, unique except for F_0, such that

$$(18) \qquad\qquad f - g \quad \text{is bounded,}$$

where

$$(19) \qquad\qquad g := \sum_{k=0}^{n} F_k \circ \delta_k.$$

PROOF. For the existence proof one may use the arguments of [1] with almost no changes. These arguments can also be found in [5, Ch. XVII]. The uniqueness result follows from the fact that the boundedness of

$$h := \sum_{k=0}^{n} H_k \circ \delta_k$$

implies that h is constant, when H_k is k-additive for all k. In our non-archimedean situation this can be seen by considering the values $h(x/p^m) - H_0 = \sum_{k \geq 1} 1/p^{mk}(H_k \circ \delta_k)(x)$ and letting tend m to ∞. $\qquad\square$

THEOREM 9. *Let, for fixed* $n \in \mathbf{N}_0$, *the function* $f : M \to N$ *satisfy (with certain* $\varepsilon > 0$)

$$(20) \qquad \| \overset{n+1}{\underset{x}{\Delta}} f(y)\| \leq \varepsilon \quad (x, y \in M).$$

Then there exist symmetric k-*additive functions* $F_k : M^k \to N$, $0 \leq k \leq n$, *unique except for* F_0, *such that*

$$(21) \qquad f - g \quad \text{is bounded,}$$

where

$$(22) \qquad g := \sum_{k=0}^{n} F_k \circ \delta_k.$$

PROOF. This is derived from the previous theorem in the same way as this was done in [5, Ch. XVII] for the archimedean case. $\qquad\square$

Finally we formulate a result which in the archimedean case is formulated as an exercise in the reference above. Details in this case are to be found in [1]. Compare also with Theorem 4.

THEOREM 10. *Let, for fixed* $n \in \mathbf{N}$, *the function* $f : M \to N$ *satisfy (with a certain* $\varepsilon > 0$)

$$(23) \qquad \| \overset{n}{\underset{x}{\Delta}} f(y) - n! f(x)\| \leq \varepsilon \quad (x, y \in M).$$

Then there exists a unique symmetric n-*additive function* $F : M^n \to N$, *such that*

$$(24) \qquad f - F \circ \delta_n \quad \text{is bounded.}$$

PROOF. (23) implies that

$$\| \overset{n+1}{\underset{x}{\Delta}} f(y)\| = \| \overset{n}{\underset{x}{\Delta}} f(x+y) - \overset{n}{\underset{x}{\Delta}} f(y)\|$$

$$\leq \| \overset{n}{\underset{x}{\Delta}} f(x+y) - n! f(x)\| + \| \overset{n}{\underset{x}{\Delta}} f(y) - n! f(x)\| \leq 2\varepsilon.$$

Thus by the previous theorem we have symmetric k-additive functions F_k, $0 \leq k \leq n$, unique up to F_0, such that for some ε'

$$(25) \qquad \|f - \sum_k F_k \circ \delta_k\| \leq \varepsilon'.$$

Put $h := f - \sum_k F_k \circ \delta_k$. Then $\|h(y)\| \leq \varepsilon'$ for all $y \in M$. The formula

$$\underset{x_1,\dots,x_n}{\Delta} f(y) = \sum_{J \subseteq n} (-1)^{n-|J|} f(y + \textstyle\sum_{j \in J} x_j)$$

mentioned in the proof of Theorem 6 then leads to

$$\| \overset{n}{\underset{x}{\Delta}} h(y)\| \leq \sum_{j=0}^{n} \left\| (-1)^{n-j} \binom{n}{j} h(y + jx) \right\| \leq 2^n 2\varepsilon' = 2^{n+1}\varepsilon'.$$

But, as $\Delta_x^n F_k \circ \delta_k = 0$ for $k < n$ and $\Delta_x^n F_n \circ \delta_n = n! F_n(x,\dots,x)$, we obtain $\Delta_x^n h(y) = \Delta_x^n f(y) - n!(F_n \circ \delta_n)(x)$. This together with (23) and the inequality above then shows that $f - F_n \circ \delta_n$ is bounded. Obviously F_n is also unique. $\qquad \square$

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