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A FUNCTIONAL EQUATION ARISING FROM AN ASYMPTOTIC FORMULA FOR ITERATES

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Abstract. We consider the real solutions of the functional equation

$$(*) \qquad \varphi^m(x) = \frac{1}{m}\varphi(m\,x),\ \varphi(0) = 0,$$

where $m \in \mathbb{N}$ and φ^m denotes the m-th iterate of the unknown function φ. We will handle this functional equation for a fixed m, but also for all naturals m, and give a representation of all C^2-solutions (even weaker, see Theorem 2.1) of $(*)$, but also treat the case of other solutions of this equation. In the introduction we will show the origin of this equation.

1. Introduction. Suppose the function $f : D \to \mathbb{R}$, where D is an open real interval containing 0 and $f(0) = 0$. Suppose further, that for x of a neighbourhood U of zero, $\lim_{n\to\infty} nf^n(x/n)$ exists uniformly on U. We denote this limit function with $\varphi(x)$, hence

$$\varphi(x) = \lim_{n\to\infty} nf^n(x/n) \qquad \text{with} \quad \varphi(0) = 0.$$

For arbitrary natural k we have

$$\lim_{n\to\infty} nf^{kn}(x/n) = \frac{1}{k}\lim_{n\to\infty} nkf^{kn}\left(\frac{kx}{kn}\right) = \frac{1}{k}\varphi(kx),$$

for at least all x with $kx \in U$. From this we get the asymptotic formula for the kn-th iterates of the function f:

$$f^{kn}(x/n) = \frac{1}{k}\,\varphi(kx) + o\left(\frac{1}{n}\right)$$

for $n \to \infty$ and x with $kx \in U$.

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In the paper [5] it is shown that if f is of class C^2 and $f'(0) = 1$, then such an asymptotic formula for f exists (cf. also [4] and [1] - [3]).

Now we will derive a functional equation, characterizing the function φ. For naturals k and m and sufficiently small x we conclude from $f^{(k+m)n}(x) = f^{kn} \circ f^{mn}(x)$:

$$nf^{(k+m)n}\left(\frac{x}{n}\right) = nf^{kn}\left(nf^{mn}\left(\frac{x}{n}\right)\frac{1}{n}\right).$$

Taking the limit for $n \to \infty$ we get, due to the continuity of f, uniformity of convergence, and $\lim_{n\to\infty} nf^{mn}(x/n) = \frac{1}{m}\varphi(mx)$ the following functional equation.

$$\text{(1.1)} \qquad \frac{1}{k+m}\varphi((k+m)x) = \frac{1}{k}\varphi\left(\frac{k}{m}\varphi(mx)\right), \qquad \varphi(0) = 0.$$

LEMMA 1.1. *The functional equation* (1.1), *for all* $k, m \in \mathbf{N}$, *is equivalent to:*

$$\text{(1.2)} \qquad \varphi^m(x) = \frac{1}{m}\varphi(mx), \qquad \varphi(0) = 0,$$

for all $m \in \mathbf{N}$.

PROOF. (1.1) $\to$ (1.2): For $m = k = 1$ equation (1.1) yields

$$\frac{1}{2}\varphi(2 \cdot x) = \varphi^2(x).$$

We proceed by induction. If (1.2) holds for m then from (1.1) with $k = 1$ follows

$$\frac{1}{m+1}\varphi((1+m)x) = \varphi\left(\frac{1}{m}\varphi(mx)\right) = \varphi \circ \varphi^m(x) = \varphi^{m+1}(x).$$

(1.2) $\to$ (1.1):

$$\frac{1}{k+m}\varphi((k+m)x) = \varphi^{k+m}(x) = \varphi^k \circ \varphi^m(x) = \frac{1}{k}\varphi\left(\frac{k}{m}\varphi(mx)\right).$$

□

REMARK 1.1. From the proof of the above lemma one can see that the following three functional equations are equivalent.

(i) (1.1) for all $k, m \in \mathbb{N}$,
(ii) (1.1) with $k = 1$ and for all $m \in \mathbb{N}$,
(iii) (1.2) for all $m \in \mathbb{N}$.

We shall confine ourselves to the study of equation (1.2). We shall adopt the following definition. Let $\varphi : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function and let $U \subseteq \mathbb{R}$ be a set. We say that φ *is a solution of* (1.2) *on* U if

(a) $0 \in U \subseteq \frac{1}{m} D \cap \bigcap_{i=0}^{m-1} \left(\varphi^i\right)^{-1}(D)$;
(b) (1.2) holds for every $x \in U$.

Condition (a) is equivalent to say that

(a') $0 \in U$ and with $x \in U$ also $x, \varphi(x), ..., \varphi^{m-1}(x) \in D$ and $mx \in D$.

In the sequel we shall use the following lemmas.

LEMMA 1.2. *If $\varphi : D \to \mathbb{R}$ is a solution of (1.2) in a set U for a fixed m then, for every natural k, the k-th iterate $\varphi^k : D_k = \bigcap_{j=0}^{k-1} \left(\varphi^j\right)^{-1}(D) \to \mathbb{R}$ is a solution of (1.2) on any set V such that*

$$0 \in V \subseteq \bigcap_{s=0}^{k-1} \left(\varphi^{sm}\right)^{-1}(U).$$

PROOF. Fix a $j \in \{0, \ldots, k-1\}$ and take $x \in V$. We have $\varphi^{jm}(x) \in U$ whence $m\varphi^{jm}(x) \in D$. On the other hand $\varphi^{sm}(x) \in U$ for every $s \in \{0, \ldots, j-1\}$ whence using (1.2) we get by an easy induction

$$D \ni m\varphi^{jm}(x) = m\varphi^m\left(\varphi^{(j-1)m}(x)\right) = \varphi\left(m\varphi^{(j-1)m}(x)\right) = \ldots = \varphi^j(mx).$$

It follows that $mx \in \left(\varphi^j\right)^{-1}(D)$ and consequently, $V \subseteq \frac{1}{m} \bigcap_{j=0}^{k-1} \left(\varphi^j\right)^{-1}(D)$ $= \frac{1}{m} D_k$.

Now fix an $i \in \{0, \ldots, m-1\}$ and a $j \in \{0, \ldots, k-1\}$. Then $ki + j \in \{0, \ldots, km-1\}$ and we can write $ki + j = ms + r$ for some $s \in \{0, \ldots, k-1\}$ and $r \in \{0, \ldots, m-1\}$. For every $x \in V$ we have

$$\varphi^{ki+j}(x) = \varphi^{ms+r}(x) = \varphi^r\left(\varphi^{ms}(x)\right) \in \varphi^r(U) \subseteq D.$$

Hence it follows that

$$V \subseteq \bigcap_{i=0}^{m-1} \left(\varphi^{ki}\right)^{-1} \left(\bigcap_{j=0}^{k-1} \left(\varphi^j\right)^{-1}(D)\right) = \bigcap_{i=0}^{m-1} \left(\varphi^{ki}\right)^{-1}(D_k)$$

which proves that V satisfies (a). To show (b) it is enough to prove by simple induction that for every $x \in V$

$$\varphi^{km}(x) = \varphi^{(k-1)m}\left(\varphi^m(x)\right) = \varphi^{(k-1)m}\left(\frac{1}{m}\varphi(mx)\right) = \ldots = \frac{1}{m}\varphi^k(mx).$$

□

LEMMA 1.3. *Let $\varphi : D \to \mathbb{R}$ be a solution of (1.2) for a fixed integer m greater than 1 on U. Then for every $k \in \mathbb{N}$ the function $\varphi^{m^k} : D_{m^k} = \bigcap_{j=0}^{m^k-1} \left(\varphi^j\right)^{-1}(D) \to \mathbb{R}$ satisfies*

$$(1.3) \qquad \varphi^{m^k}(x) = \frac{1}{m^k}\varphi(m^k x), \qquad \varphi(0) = 0$$

for every $x \in W_k(U) = \bigcap_{s=0}^{k-1} \frac{1}{m^s} \bigcap_{j=0}^{m^{k-s-1}-1} \left(\varphi^{jm}\right)^{-1}(U)$.

PROOF. For $k = 1$ we have $W_1(U) = U$ and (1.3) is simply (1.2). Suppose that the assertion holds for a $k \in \mathbb{N}$ and let $x \in W_{k+1}(U)$. Then for every $j \in \{0, \ldots, m^{k+1} - 1\}$ we have $j = pm + r$ where $p \in \{0, \ldots, m^k - 1\}$ and $r \in \{0, \ldots, m - 1\}$. Thus for every $j \in \{0, \ldots, m^{k+1} - 1\}$

$$\varphi^j(x) = \varphi^r\left(\varphi^{pm}(x)\right) \in \varphi^r(U) \subseteq D,$$

whence $\varphi^{m^{k+1}}(x)$ is well defined for $x \in W_{k+1}(U)$. Further we have by Lemma 1.2 (with k replaced by m^k) and induction

$$\varphi^{m^{k+1}}(x) = \left(\varphi^{m^k}\right)^m(x) = \frac{1}{m}\varphi^{m^k}(mx) = \frac{1}{m^{k+1}}\varphi\left(m^{k+1}x\right).$$

□

The following lemma will be given without the obvious proof.

LEMMA 1.4. *If $\varphi : D \to \mathbb{R}$ is a solution of (1.2) on U then*

$$\varphi^m\left(\frac{y}{m}\right) = \frac{1}{m}\varphi(y)$$

for every $y \in mU$.

2. Twice differentiable solutions of the functional equation

$$(2.1) \qquad \varphi^m(x) = \frac{1}{m}\varphi(mx), \qquad \varphi(0) = 0.$$

Proposition 2.1. *Let $\varphi : D \to \mathbf{R}$ be a solution of (2.1) for a fixed $m \geq 2$ on a set U containing a neighbourhood of 0. If φ is differentiable at 0 then $\varphi'(0) \in \{0, 1\}$ in the case where m is even and $\varphi'(0) \in \{0, 1, -1\}$ in the case where m is odd. If $\varphi'(0) = 0$ and U is an interval then*

$$\varphi(x) = 0 \quad \textit{for all} \quad x \in mU.$$

Proof. From $\varphi^m(x) = \frac{1}{m}\varphi(mx)$ follows

$$\varphi'(0)^m = \varphi'(0),$$

hence $\varphi'(0) = 0$ or $\varphi'(0)$ is a $(m-1)$–th real unit root. Let now be supposed $\varphi'(0) = 0$. Then there exists an interval V such that $0 \in V \subseteq U$ and $\varphi(V) \subsetneq V$. Hence φ^n is defined in V for all $n \in \mathbf{N}$. Fix an $\varepsilon \in (0, 1)$. There exists a $\delta > 0$ such that

$$|\varphi(x)| < \varepsilon|x|, \quad x \in (-\delta, \delta) \cap D.$$

An easy induction shows that

$$(2.2) \quad |\varphi^n(x)| = |\varphi\left(\varphi^{n-1}(x)\right)| \leq \varepsilon|\varphi^{n-1}(x)| \leq \varepsilon^n|x|, \qquad x \in (-\delta, \delta) \cap D.$$

Since $V \subseteq U$, φ satisfies (2.1) in V. Fix a $y \in V \setminus \{0\}$ and choose $k \in \mathbf{N}$ so that $\frac{y}{m^k} \in (-\delta, \delta) \cap V$. V is an interval containing 0 and therefore

$$\frac{y}{m^k} = \frac{1}{m^{k-s}}\frac{y}{m^s} \in \frac{1}{m^s}V \qquad \text{for all} \quad s \in \{0, \dots, k-1\}$$

whence $\frac{y}{m^k} \in W_k(V)$ (cf. Lemma 1.3). It follows from Lemma 1.3 and (2.2) that

$$\left|\frac{\varphi(y)}{y}\right| = \left|\frac{m^k}{y}\varphi^{m^k}\left(\frac{y}{m^k}\right)\right| \leq \varepsilon^{m^k} < \varepsilon.$$

Since $\varepsilon \in (0, 1)$ was chosen arbitrarily we get $\varphi(y) = 0$, $y \in V$. Now let $z \in U \cap mV$. Then $z = mx$ for some $x \in V$ and hence $\varphi(z) = \varphi(mx) = m\varphi^m(x) = 0$. By induction φ vanishes in $U \cap m^nV$, $n \in \mathbf{N}$, whence φ vanishes in $U = \bigcup_{n \in \mathbf{N}_0} U \cap m^nV$. From (2.1) we infer that φ vanishes in mU as well. □

The most important class of solutions of our considered functional equation is given in the following.

Theorem 2.1. *Let $\varphi : D \to \mathbb{R}$ be a real solution of equation (2.1) for a fixed natural $m > 1$ on an open interval U containing 0. Suppose that φ is continuous on U and two times differentiable at 0. If $\varphi'(0) = 1$ then*

$$\varphi(x) = \frac{x}{1 - b \cdot x} \quad \text{with} \quad b = \frac{1}{2}\frac{d^2\varphi}{dx^2}(0), \tag{2.3}$$

for $x \in W$, where

$$W = \begin{cases} mU \cap \left(-\infty, b^{-1}\right) & \text{if } b > 0, \\ mU & \text{if } b = 0, \\ mU \cap \left(b^{-1}, \infty\right) & \text{if } b < 0. \end{cases}$$

Conversely, the function $\varphi : \mathbb{R} \setminus \{b^{-1}\} \to \mathbb{R}$ given by (2.3) is a solution of (2.1) on $\mathbb{R} \setminus \{m^{-1}b^{-1}, (m-1)^{-1}b^{-1}, \ldots, b^{-1}\}$ for all $m \in \mathbb{N}$.

Proof. The last statement is obvious. Let us prove the first part of the assertion.

i) For a fixed $\rho \in \mathbb{R}$ consider the function $\varphi_\rho : \mathbb{R} \setminus \{\rho^{-1}\} \to \mathbb{R}$ defined by

$$\varphi_\rho(x) = \frac{x}{1 - \rho\, x}$$

and note that φ_ρ has the expansion

$$\varphi_\rho(x) = x + \rho\, x^2 + \rho^2\, x^3 + \cdots = \sum_{i=1}^{\infty} \rho^{i-1}\, x^i$$

in the interval $(-\rho^{-1}, \rho^{-1})$.

We know that the m-th iterate of φ_ρ has the form

$$\varphi_\rho^m(x) = \frac{x}{1 - \rho\, m\, x}.$$

From this one can see that each φ_ρ is a solution of (2.1) on $\mathbb{R}\setminus\{m^{-1}\rho^{-1}, \rho^{-1}\}$ and $\frac{d^2\varphi_\rho}{dx^2}(0) = 2\,\rho$. Moreover, observe that φ_ρ is strictly increasing in each component of $\mathbb{R} \setminus \{\rho^{-1}\}$ (or in $\mathbb{R}$, if $\rho = 0$).

ii) Suppose now φ to be a solution of (2.1) on U which is continuous, twice differentiable at 0 and $\varphi'(0) = 1$. For this φ we get the Taylor formula

$$\varphi(x) = x + bx^2 + o(x^2), \ x \to 0,$$

which holds for every $x \in D$. Choose arbitrary $a, c \in \mathbb{R}$ so that $a < b < c$. Since

$$\frac{\varphi_c(x) - \varphi(x)}{x^2} = c - b + o(1), \; x \to 0,$$

and

$$\frac{\varphi(x) - \varphi_a(x)}{x^2} = b - a + o(1), \; x \to 0,$$

we have

$$\varphi_a(y) < \varphi(y) < \varphi_c(y) \tag{2.4}$$

for every $y \neq 0$ from an open interval $V \subset U$, containing 0. We shall continue the proof in the case where $b > 0$ because in the remaining ones the argument is very similar. We can assume that $0 < a < c$ and we will show that (2.4) holds in $mU \cap (-\infty, c^{-1})$ (except for $y = 0$). Indeed, put

$$x = \sup\{z > 0 : \quad (2.4) \text{ holds for every } \quad y \in (0, z)\}$$

and suppose that $x < \sup mU \cap (-\infty, c^{-1})$. By continuity of φ_a, φ and φ_c we get

$$\varphi_a(x) = \varphi(x) \qquad \text{or} \qquad \varphi(x) = \varphi_c(x). \tag{2.5}$$

Note that we have

$$\varphi_c^j(x/m) = \frac{x}{m - cjx} < x, \qquad j = 0, \dots, m-1,$$

because $x < c^{-1}$. By the definition of x and because of monotonicity of φ_a, φ_c we infer that

$$\varphi_a^{m-1}(x/m) < \varphi^{m-1}(x/m) < \varphi_c^{m-1}(x/m).$$

Hence by (2.4), (2.1) and the monotonicity of φ_a and φ_c we get

$$\begin{aligned} \varphi_a(x) = m\varphi_a^m(x/m) &< m\varphi_a\left(\varphi^{m-1}(x/m)\right) < m\varphi^m(x/m) = \varphi(x) \\ &< m\varphi_c\left(\varphi^{m-1}(x/m)\right) < m\varphi_c^m(x/m) = \varphi_c(x) \end{aligned}$$

which contradicts (2.5) and proves that (2.4) holds on $mU \cap (0, c^{-1})$.

Now put

$$v = \inf\{w < 0 : \quad (2.4) \text{ holds for every } \quad y \in (w, 0)\}$$

and suppose that $v > \inf mU$. Then we infer that (2.4) does not hold for $y = v$. We have $v < \varphi_a(v)$ and

$$y < \varphi_a(y) < \varphi(y) < \varphi_c(y) < 0$$

for every $y \in mU \cap (v, 0)$. Hence and by monotonicity of φ_a, φ_c, we have

$$v < \varphi_a(v) = m\varphi_a^m(v/m) < m\varphi^m(v/m) = \varphi(v) < m\varphi_c^m(v/m) = \varphi_c(v)$$

which means that (2.4) holds for $y = v$ as well, contrary to our supposition.

Finally, fix an $x \in mU \cap (-\infty, b^{-1})$. Then, if $c > b$ is close enough to b, we get $x \in mU \cap (-\infty, c^{-1})$ and hence

$$\varphi_a(x) < \varphi(x) < \varphi_c(x)$$

for every $a < b$ and every $c > b$, close enough to b. Letting $a \to b, c \to b$ we see that $\varphi(x) = \varphi_b(x)$ which ends the proof. □

The above result has a local character but we cannot expect a global statement as is shown by the following.

EXAMPLE 2.1. The function $\varphi : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ given by

$$\varphi(x) = \begin{cases} \dfrac{x}{1-x} & \text{if } x < 1, \\ 0 & \text{if } x > 1, \end{cases}$$

is a C^∞ solution of (2.1) on $\mathbb{R} \setminus \{1, 2^{-1}, \dots, m^{-1}\}$ for every natural $m > 1$.

To complete in some sense the results on two times differentiable solutions of (2.1) we can state the following.

THEOREM 2.2. *Let $\varphi : D \to \mathbb{R}$ be a real solution of equation (2.1) for a fixed odd natural $m > 1$ on an open interval U containing 0. Suppose further that φ is continuous on U and two times differentiable at 0. If $\varphi'(0) = -1$ then*

$$\varphi(x) = -x \qquad \text{for all } x \in mU. \tag{2.7}$$

The function φ given by (2.7) is a solution of (2.1) on $\mathbb{R}$ for all odd $m \in \mathbb{N}$.

PROOF. According to Lemma 1.2, φ^2 is a solution of (2.1) on $V = U \cap (\varphi^m)^{-1}(U)$, φ^2 is twice differentiable at 0 and $(\varphi^2)'(0) = 1$. Thus in view of Theorem 2.1

$$\varphi^2(x) = \frac{x}{1-bx} = x + bx^2 + o(x^2), \quad x \to 0, \tag{2.8}$$

for some $b \in \mathbb{R}$. On the other hand we have for some $a \in \mathbb{R}$

$$\varphi(x) = -x + ax^2 + o(x^2), \quad x \to 0,$$

whence

$$\varphi^2(x) = x + o(x^2), \quad x \to 0. \tag{2.9}$$

Comparing (2.8) and (2.9) we get $b = 0$, or $\varphi^2 = x$ for $x \in V$. Since m is odd and φ satisfies (2.1) on V we have

$$\varphi(x) = \varphi^m(x) = \frac{1}{m}\varphi(mx), \quad x \in V,$$

whence

$$\varphi(y) = m\varphi\left(\frac{y}{m}\right) \tag{2.10}$$

for $y \in mV$. We may assume that V is an interval whence $\frac{y}{m^n} \in mV$, $n \in \mathbf{N}$. Hence we get from (2.10) by induction

$$\frac{\varphi(y)}{y} = \frac{m^n}{y}\varphi\left(\frac{y}{m^n}\right), \qquad y \in mV \setminus \{0\}, \quad n \in \mathbf{N}.$$

Letting $n \to \infty$ we obtain

$$\frac{\varphi(y)}{y} = \varphi'(0) = -1,$$

whence $\varphi(y) = -y$, $y \in V$, follows. Similarly as in the proof of Proposition 2.1 we argue to get $\varphi(x) = -x$ for all $x \in U$. From (2.1) we infer that $\varphi(x) = -x$ for all $x \in mU$. □

3. Other solutions of the functional equation

$$\varphi^m(x) = \frac{1}{m}\varphi(m\,x), \quad \varphi(0) = 0. \tag{3.1}$$

We shall be concerned now with some other solutions of (3.1). It turns out that in lower classes of regularity there exist solutions different from those obtained in Section 2. We are going to describe some of them. Let us start with an easy example of a C^1 solution of (3.1) which is not twice differentiable at 0. First let us state a result without proof.

LEMMA 3.1. *Let $D_+ \subseteq \mathbb{R}_+ = [0,\infty)$ and $D_- \subseteq \mathbb{R}_- = (-\infty, 0]$ and suppose that $\varphi_+ : D_+ \to \mathbb{R}_+$ and $\varphi_- : D_- \to \mathbb{R}_-$ are solutions of (3.1) on U_+ and U_-, respectively. Then $\varphi : D = D_+ \cup D_- \to \mathbb{R}$, defined by*

$$\varphi(x) = \begin{cases} \varphi_+(x) & \text{if } x \in D_+, \\ \varphi_-(x) & \text{if } x \in D_-, \end{cases}$$

is a solution of (3.1) on $U = U_+ \cup U_-$.

EXAMPLE 3.1. Fix numbers $b < 0 < c$ arbitrarily and define $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(x) = \begin{cases} \dfrac{x}{1-bx} & \text{if } x \geq 0, \\ \dfrac{x}{1-cx} & \text{if } x < 0. \end{cases}$$

Obviously, $\varphi_+ = \varphi|_{\mathbb{R}_+}$ and $\varphi_- = \varphi|_{\mathbb{R}_-}$ are solutions of (3.1) for every $m \in \mathbb{N}$ in $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively. Moreover, φ is of class C^1 in $\mathbb{R}, \varphi'(0) = 1$, but

$$\varphi_+''(0) = 2b \neq 2c = \varphi_-''(0).$$

By Lemma 3.1, φ is a solution of (3.1) in $\mathbb{R}$.

In the sequel we present a description of a family of C^1 solutions of (3.1) which are not twice differentiable at 0.

Let $\varphi : [0,\infty) \to [0,\infty)$ and denote by $\mathrm{Fix}\varphi$ the set of fixed points of φ.

LEMMA 3.2. *If φ is a continuous, nondecreasing solution of (3.1) on $[0,\infty)$ for some $m > 1$ then for every $x \in \mathrm{Fix}\varphi$ and every $k \in \mathbb{Z}$ also $m^k x \in \mathrm{Fix}\varphi$.*

PROOF. If $x_o \in \mathrm{Fix}\varphi$ then we have for $k \in \mathbb{N}$ (cf. Lemma 1.3)

$$\varphi(m^k x_o) = m^k \varphi^{m^k}(x_o) = m^k x_o.$$

Suppose now that $x_o \in \mathrm{Fix}\varphi$ and $\frac{x_o}{m} \notin \mathrm{Fix}\varphi$. Then $x_o > 0$. Put

$$y_1 := \sup \mathrm{Fix}\varphi \cap \left[0, \frac{x_o}{m}\right)$$

and

$$y_2 := \inf \mathrm{Fix}\varphi \cap \left(\frac{x_o}{m}, \infty\right).$$

Then $y_1, y_2 \in \mathrm{Fix}\varphi$ by continuity of φ and

$$0 \leq y_1 < \frac{x_o}{m} < y_2 \leq x_o.$$

Since φ has no fixed points in (y_1, y_2) and φ is nondecreasing, we have either

$$y_1 \le \varphi(x) < x < y_2 \quad \text{for} \quad x \in (y_1, y_2) \tag{3.2}$$

or

$$y_1 < x < \varphi(x) \le y_2 \quad \text{for} \quad x \in (y_1, y_2). \tag{3.3}$$

Assume that (3.2) holds. Then

$$\lim_{n\to\infty} \varphi^n(x) = y_1 \tag{3.4}$$

for every $x \in (y_1, y_2)$. On the other hand (cf. Lemma 1.2)

$$\varphi^{mr}\left(\frac{x_o}{m}\right) = [\varphi^r]^m\left(\frac{x_o}{m}\right) = \frac{1}{m}\varphi^r\left(m\frac{x_o}{m}\right) = \frac{x_o}{m}$$

for every $r \in \mathbf{N}$, which yields a contradiction to (3.4). Similarly we proceed if (3.3) holds. The lemma is proved because by induction $\frac{x_o}{m^k} \in \mathrm{Fix}\varphi$ for every $k \in \mathbf{N}$. □

COROLLARY. *Under the assumptions of Lemma 3.2, if φ satisfies*

$$\varphi^m(x) = \frac{1}{m}\varphi(mx)$$

and

$$\varphi^p(x) = \frac{1}{p}\varphi(px)$$

for two different primes m, p and all $x \in [0,\infty)$, and $\mathrm{Fix}\varphi \setminus \{0\} \neq \emptyset$ then $\varphi = \mathrm{id}_{[0,\infty)}$.

PROOF. Suppose that $0 < x_o \in \mathrm{Fix}\varphi$. From the preceding lemma we infer that

$$m^k p^r x_o \in \mathrm{Fix}\varphi$$

for every $r, k \in \mathbf{Z}$. Since the set $\{m^k p^r x_o :\ k \in \mathbf{Z}, r \in \mathbf{Z}\}$ is dense in $(0,\infty)$ we obtain our assertion by continuity of φ. □

PROPOSITION 3.1. *Let $\varphi : [0,\infty) \to [0,\infty)$ be a nondecreasing continuous solution of (3.1) and suppose that $\varphi(x_o) = x_o$ for some $x_o > 0$. Denote $I_o = [x_o, mx_o]$. The functions $\{\psi_k : k \in \mathbf{Z}\}$ defined by*

$$\psi_k(x) = m^{-k}\varphi(m^k x), \quad x \in I_o, \tag{3.5}$$

have the following properties

(i) $\psi_k(I_o) = I_o$, $k \in \mathbf{Z}$,
(ii) $\psi_k(x_o) = x_o$ *and* $\psi_k(mx_o) = mx_o$, $k \in \mathbf{Z}$,
(iii) $\psi_k^m = \psi_{k+1}$, $k \in \mathbf{Z}$,
(iv) ψ_k *are continuous*, $k \in \mathbf{Z}$,
(v) ψ_k *are nondecreasing*, $k \in \mathbf{Z}$.

PROOF. It follows from Lemma 3.2 that $\varphi(mx_o) = mx_o$ and thus $\varphi(I_o) = I_o$ because φ is continuous and nondecreasing. From (3.5) we directly get (i), (ii), (iv) and (v). From (3.1) we infer

$$\psi_k^m(x) = m^{-k}\varphi^m(m^k x) = m^{-(k+1)}\varphi(m^{k+1}x) = \psi_{k+1}(x)$$

for all $k \in \mathbb{Z}$ and $x \in I_o$. □

PROPOSITION 3.2. *Fix an $x_o > 0$ and denote $I_o = [x_o, mx_o], I_k = m^k I_o$, $k \in \mathbb{Z}$. Suppose that $\{\psi_k : k \in \mathbb{Z}\}$ is a family of mappings defined on I_o and satisfying conditions (i)-(iv) from Proposition 3.1. Then ψ_o can be uniquely extended to a continuous solution $\varphi : [0,\infty) \to [0,\infty)$ of (3.1) such that $\psi_k(x) = m^{-k}\varphi(m^k x)$ for $x \in I_o$.*

PROOF. Define $\varphi : [0,\infty) \to [0,\infty)$ by

$$\varphi(x) = \begin{cases} m^k\psi_k(m^{-k}x) & \text{if } x \in I_k,\ k \in \mathbb{Z}, \\ 0 & \text{if } x = 0. \end{cases} \tag{3.6}$$

From (ii) and (iv) it follows that

$$\lim_{x\to m^k x_o+} \varphi(x) = m^k \lim_{x\to m^k x_o+} \psi_k(m^{-k}x) = m^k x_o = \varphi(m^k x_o), \quad k \in \mathbb{Z},$$

and similarly

$$\lim_{x\to m^{k+1}x_o-} \varphi(x) = \varphi(m^{k+1}x_o), \quad k \in \mathbb{Z},$$

which proves continuity of φ in $(0,\infty)$. Now, fix $\varepsilon > 0$ and let $p \in -\mathbb{N}$ be such that $m^{p+1}x_o < \varepsilon$. Choose $x \in (0, m^p x_o]$ arbitrarily. Then $x \in I_k$ for some $k \leq p$. From (i) we get

$$0 < m^k x_o \leq \varphi(x) = m^k\psi_k(m^{-k}x) \leq m^k m x_o \leq m^{p+1}x_o < \varepsilon.$$

This shows that φ is continuous at 0. To show that (3.1) holds let $x \in (0,\infty)$.Then $x \in I_k$ for some $k \in \mathbb{Z}$ and $\varphi(x) \in I_k$ whence

$$\begin{aligned}\varphi^m(x) &= m^k\psi_k^m(m^{-k}x) = m^k\psi_{k+1}(m^{-k}x) \\ &= m^k m^{-(k+1)}\varphi(m^{k+1}m^{-k}x) = m^{-1}\varphi(mx).\end{aligned}$$

The uniqueness of extension is obvious. □

REMARK 3.1. Note that in general a function $\psi_o : I_o \to I_o$ may have several extensions to a solution of (3.1). Indeed, if ψ_o is a homeomorphism then we can define ψ_k to be m^k-th iterate of ψ_o. Then (i)-(iv) are satisfied, but ψ_k's are not uniquely determined for $k < 0$ because in general there are many homeomorphic iterative roots of a homeomorphism of an interval.

REMARK 3.2. Observe that if we assume that ψ_k are nondecreasing (increasing) then so will be φ. It follows from the formula (3.6) and the inclusion $\varphi(I_k) \subset I_k$.

Now we are going to show that there exist solutions of equation (3.1) which are of class C^1 but differ from those occurring in Theorem 2.1.

PROPOSITION 3.3. *Under the assumptions of Proposition 3.2, if moreover*
(vi) ψ_k *are of class* C^1 *in* $\mathrm{Int} I_o$ *and*

$$\lim_{x \to x_o+} \psi_k'(x) = \lim_{x \to m x_o -} \psi_k'(x) = 1;$$

(vii) $\lim_{k\to -\infty} \psi_k'(x) = 1$, *uniformly in* $x \in (x_o, m x_o)$;
then ψ_o *can be uniquely extended to a solution* φ *of* (3.1) *in* $[0, \infty)$ *such that* $\psi_k(x) = m^{-k}\varphi(m^k(x))$, $x \in I_o$, φ *is of class* C^1 *in* $[0,\infty)$ *and* $\varphi'(0) = 1$.

PROOF. By Proposition 3.2, φ given by (3.6) is the unique continuous extension of ψ_o such that $\psi_k(x) = m^{-k}\varphi(m^k(x))$, $x \in I_o$. It is enough to check regularity properties of φ. Obviously, φ is of class C^1 in $D := \bigcup_{k\in\mathbb{Z}} \mathrm{Int} I_k$ and

$$\varphi'(x) = \psi_k'(m^{-k}x)$$

for $x \in \mathrm{Int} I_k$ and $k \in \mathbf{Z}$. Thus by the mean value theorem

$$\frac{\varphi(x) - m^k x_o}{x - m^k x_o} = \frac{\varphi(x) - \varphi(m^k x_o)}{x - m^k x_o} = \varphi'(\xi) \tag{3.7}$$

for every $x \in \mathrm{Int} I_k$, where ξ is a point in $(m^k x_o, x)$. Now, letting $x \to m^k x_o+$ we see that $m^{-k}\xi \to x_o+$ and we get from (3.7), (3.6) and (vi)

$$\varphi'(m^k x_o+) = 1,$$

and, in a similar way,

$$\varphi'(m^{k+1} x_o-) = 1.$$

Since k was arbitrary, we infer that φ is differentiable in $(0,\infty)$ and

$$\varphi'(x) = \begin{cases} \psi_k'(m^{-k}x) & \text{if } x \in (m^k x_o, m^{k+1} x_o),\ k \in \mathbf{Z}, \\ 1 & \text{if } x \in (0,\infty) \setminus D. \end{cases} \tag{3.8}$$

From (vi) we see that φ is of class C^1 in $(0,\infty)$.

To show that φ is differentiable at 0, fix an $\varepsilon > 0$ and choose $p \leq 0$ so that (cf. (vii))

$$\sup_{y\in I_o} \mid \psi'_k(y) - 1 \mid < \varepsilon, \tag{3.9}$$

for all $k \leq p$. Let us take $x \in (0, m^{p+1}x_o)$. We have

$$\left|\frac{\varphi(x)}{x} - 1\right| = \mid \varphi'(\xi) - 1 \mid$$

for a $\xi \in (0,x)$. If $\xi = m^k x_o$ for some $k \leq p$ then $m^{-k}\xi \in (x_o, mx_o)$ and we get in view of (3.8) and (3.9):

$$\left|\frac{\varphi(x)}{x} - 1\right| = \mid \varphi'(\xi) - 1 \mid = \mid \psi'_k(m^{-k}\xi) - 1 \mid < \varepsilon.$$

Thus we have proved that φ is right differentiable at 0 and $\varphi'(0+) = 1$. Using (3.8) and (3.9) again we see that φ' is right continuous at 0. This concludes the proof. □

The following example shows that there exist functions satisfying conditions (i)-(vii).

EXAMPLE 3.2. Fix $m \in \mathbf{N}, m > 1$, put $x_o = 1, I_o = [1,m]$ and define $\psi_k : I_o \to I_o$, $k \in \mathbf{Z}$, by

$$\psi_k(x) = \begin{cases} \alpha^{-1}\left(\alpha(x) + m^k\right) & \text{if} \quad x \in (1,m), \\ x & \text{if} \quad x \in \{1,m\}, \end{cases}$$

where $\alpha : (1,m) \to (-\infty,\infty)$ is given by

$$\alpha(x) = \cot\frac{\pi\cdot(x-1)}{m-1}.$$

Then the family $\{\psi_k : k \in \mathbf{Z}\}$ satisfies (i)-(vii). □

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