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ON ITERATION GROUPS OF SINGULARITY–FREE HOMEOMORPHISMS OF THE PLANE

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Abstract. Let D be a simply connected region on the plane. We prove that a continuous iteration group of homeomorphisms $\{f^t : t \in \mathbb{R}\}$ defined on D is of the form

$$f^t(x) = \varphi^{-1}(\varphi(x) + te_1) \qquad \text{for} \quad x \in D, \quad t \in \mathbb{R},$$

where $e_1 = (1,0)$ and φ is a homeomorphism mapping D onto $\mathbb{R}$, if and only if f^1 is a singularity-free homeomorphism, i.e. $f^1 =: f$ has the property that for every Jordan domain $B \subset D$ there exists an integer n_0 such that $B \cap f^n[B] = \emptyset$ for $|n| > n_0$, $n \in \mathbb{Z}$.

Let D be a topological space. A family of homeomorphisms $\{f^t : t \in \mathbb{R}\}$ defined on D is said to be a continuous iteration group if $f^t : D \to D$ for $t \in \mathbb{R}$, $f^t \circ f^s = f^{t+s}$ for $t, s \in \mathbb{R}$ and for every $x \in D$ the mapping $t \mapsto f^t(x)$ is continuous.

Let us note that $f^0(x) = x$ for $x \in D$ and f^t maps D onto itself.

REMARK 1. *If $\{f^t : t \in \mathbb{R}\}$ is a continuous iteration group defined on a topological space D homeomorphic with $\mathbb{R}^n$, then the mapping $(t,x) \mapsto f^t(x)$ is continuous in $\mathbb{R} \times D$.*

PROOF. Let $p \in S^n$, where S^n is an n-dimensional sphere. Then the set $S^n \setminus \{p\}$ is homeomorphic with $\mathbb{R}^n$ (see e.g. [6, p. 40]). Let α and β be

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homeomorphisms mapping D onto $\mathbb{R}^n$ and $\mathbb{R}^n$ onto $S^n \setminus \{p\}$, respectively. Put $\gamma := \beta \circ \alpha$ and

$$F^t(z) := \begin{cases} (\gamma \circ f^t \circ \gamma^{-1})(z), & z \in S^n \setminus \{p\}; \\ p, & z = p. \end{cases}$$

It is easy to see that $\{F^t : t \in \mathbb{R}\}$ is a continuous iteration group on S^n. In [13] it has been shown that every continuous iteration group defined on a compact metric space is continuous with respect to both variables. Hence the mapping $(t, z) \mapsto F^t(z)$ is continuous in $\mathbb{R} \times S^n$ and consequently the mapping $(t, x) \mapsto f^t(x)$ is continuous in $\mathbb{R} \times D$. □

Let us introduce the following

DEFINITION 1. A continuous iteration group $\{f^t : t \in \mathbb{R}\}$ defined on a topological space D is said to be *translative* if there exists a homeomorphism $\varphi : D \overset{onto}{\to} \mathbb{R}^n$ such that

$$f^t(x) = \varphi^{-1}(\varphi(x) + te_1) \qquad \text{for} \quad x \in D, \quad t \in \mathbb{R}, \tag{1}$$

where $e_1 = (1, 0, \dots, 0)$.

Let us note that every iteration group given by the formula $f^t(x) := \psi^{-1}(\psi(x) + ta)$ for $x \in D$ and $t \in \mathbb{R}$, where $a \in \mathbb{R}^n \setminus \{(0, \dots, 0)\}$ and ψ is a homeomorphism mapping D onto $\mathbb{R}^n$, is translative.

In the sequel we shall be concerned with iteration groups defined on a simply connected region of the plane, i.e. on a plane region which is homeomorphic to $\mathbb{R}^2$ (see [10, p. 262]).

It is known (see [8, p. 197]) that for every homeomorphism f of a simply connected region $D \subset \mathbb{R}^2$ into $\mathbb{R}^2$ there exists exactly one $d_f \in \{-1, 1\}$ such that

$$\mathrm{Ind}_\Gamma(x) = d_f \cdot \mathrm{Ind}_{f[\Gamma]}(f(x))$$

for every Jordan curve $\Gamma \subset D$ and every $x \in D \setminus \Gamma$ (for the definition of Ind_Γ see e.g. [5, p. 247]). We shall say that a *homeomorphism f preserves orientation* if $d_f = 1$.

Following [11] and [4] we introduce

DEFINITION 2. Let $D \subset \mathbb{R}^2$ be a simply connected region. Then a homeomorphism f of D onto itself such that every Jordan domain $B \subset D$ meets at most a finite number of its images $f^n[B]$, $n \in \mathbb{Z}$, is said to be a *singularity-free homeomorphism* or a *Sperner homeomorphism*, where by the Jordan domain is meant the union of a Jordan curve Γ and the bounded component of $\mathbb{R}^2 \setminus \Gamma$.

We have the following characterization of singularity-free homeomorphisms of $\mathbb{R}^2$ given by Sperner and Andrea (see [11] and [2]).

PROPOSITION 1. *Let f be a homeomorphism of $\mathbb{R}^2$ onto itself. Then the following conditions are equivalent:*

(i) *f is a singularity-free homeomorphism preserving orientation;*

(ii) *there exists a homeomorphism $\varphi : \mathbb{R}^2 \overset{onto}{\rightarrow} \mathbb{R}^2$ such that*

$$f(x) = \varphi^{-1}(\varphi(x) + e_1) \qquad \text{for} \quad x \in \mathbb{R}^2;$$

(iii) *f preserves orientation and $f^n[A] \to \infty$ as $n \to \pm\infty$ for every compact set $A \subset \mathbb{R}^2$;*

(iv) *f preserves orientation and for all $x, y \in \mathbb{R}^2$ there exists an arc Γ with endpoints x and y such that $f^n[\Gamma] \to \infty$ as $n \to \pm\infty$.*

Let us note that conditions (i) and (ii) are also equivalent for any homeomorphism f mapping a simply connected region of $\mathbb{R}^2$ onto itself (in this case φ which occurs in (ii) maps D onto $\mathbb{R}^2$).

We shall prove the following

THEOREM 1. *Let $D \subset \mathbb{R}^2$ be a simply connected region. Then a continuous iteration group $\{f^t : t \in \mathbb{R}\}$ defined on D is translative if and only if f^1 is a singularity-free homeomorphism.*

PROOF. Let $\{f^t : t \in \mathbb{R}\}$ be a continuous iteration group on D such that $f^1 =: f$ is a singularity-free homeomorphism. Let ψ be a homeomorphism mapping $\mathbb{R}^2$ onto D. Put

$$F^t := \psi^{-1} \circ f^t \circ \psi \qquad \text{for} \quad t \in \mathbb{R} \tag{2}$$

and $F := \psi^{-1} \circ f \circ \psi$. Obviously $\{F^t : t \in \mathbb{R}\}$ is a continuous iteration group on $\mathbb{R}^2$ and $F = F^1$ is a singularity–free homeomorphism.

We shall show that $\{F^t : t \in \mathbb{R}\}$ is a *dispersive* iteration group, i.e. for every pair of points $x, y \in \mathbb{R}^2$ there exist neighbourhoods U_x of x and U_y of y and a constant $T > 0$ such that

$$U_x \cap F^t[U_y] = \emptyset \qquad \text{for} \quad |t| > T, \quad t \in \mathbb{R}.$$

Let $x, y \in \mathbb{R}^2$ and let K_1 be a closed disc such that $x, y \in \operatorname{Int} K_1$. Take any neighbourhoods U_x of x and U_y of y such that $U_x \cap U_y \subset K_1$. Put

$$A := \{F^t(x) : t \in [0,1], \ \ x \in K_1\}.$$

Since the set A is compact, there exists a closed disc K_2 such that $A \subset K_2$. Obviously K_2 is a Jordan domain, and so there exists an integer N such that

$$K_2 \cap F^n[K_2] = \emptyset \qquad \text{for} \quad |n| > N, \quad n \in \mathbb{Z}, \tag{3}$$

since F is a singularity-free homeomorphism.

Now let $|s| > N + 1$, $s \in \mathbb{R}$. Then we may write $s = n + r$, where $n \in \mathbb{Z}$ and $0 \le r < 1$. Hence

$$F^s[K_1] = F^{n+r}[K_1] = F^n[F^r[K_1]] \subset F^n[A] \subset F^n[K_2].$$

Thus by (3)

$$F^s[K_1] \cap K_2 = \emptyset \qquad \text{for} \quad |s| > N + 1$$

and consequently

$$F^s[U_x] \cap U_y = \emptyset \qquad \text{for} \quad |s| > N + 1,$$

since $U_x \subset K_1$ and $U_y \subset K_1 \subset A \subset K_2$.

To prove our assertion we shall use the Nemytskii and Stepanov theorem (see [3, p. 49]) which states that every dispersive iteration group on a locally compact separable metric space has a continuous section, which means in our case that there exists a set $S \subset \mathbb{R}^2$ such that for every $x \in \mathbb{R}^2$ there is a unique $\tau(x) \in \mathbb{R}$ such that $F^{\tau(x)}(x) \in S$ and the function τ is continuous.

Define

$$h(x) := (-\tau(x),\ F^{\tau(x)}(x)) \qquad \text{for} \quad x \in \mathbb{R}^2. \tag{4}$$

Note that h is a continuous bijection of $\mathbb{R}^2$ onto $\mathbb{R} \times S$ and $h^{-1}(t, y) = F^t(y)$ for $t \in \mathbb{R}$ and $y \in S$. Hence h is a homeomorphism.

Let $\Phi^t : \mathbb{R} \times S \to \mathbb{R} \times S$ be a family of the functions defined by the formula

$$\Phi^t(u, x) := (u + t, x) \qquad \text{for} \quad u, t \in \mathbb{R}, \quad x \in S.$$

We shall show that

$$h \circ F^t = \Phi^t \circ h \qquad \text{for} \quad t \in \mathbb{R}.$$

Fix an $x \in \mathbb{R}^2$ and put $y := F^{\tau(x)}(x)$. By the definition of the function τ we have

$$\tau(F^u(y)) = -u \qquad \text{for} \quad u \in \mathbb{R},$$

since $F^{-u}(F^u(y)) = y \in S$. Hence by (4)

$$h(F^u(y)) = (-\tau(F^u(y)), \quad F^{\tau(F^u(y))+u}(y)) = (u, y), \qquad u \in \mathbb{R},$$

so

$$h(F^t(x)) = h(F^t(F^{-\tau(x)}(y)) = h(F^{t-\tau(x)}(y)) = (t - \tau(x), y)$$
$$= \Phi^t(-\tau(x), y) = \Phi^t(h(F^{-\tau(x)}(y))) = \Phi^t(h(x))$$

for $t \in \mathbb{R}$, since $x = F^{-\tau(x)}(y)$. Thus

$$F^t = h^{-1} \circ \Phi^t \circ h \qquad \text{for} \quad t \in \mathbb{R}. \tag{5}$$

Since the set $\mathbb{R} \times S$ is homeomorphic to $\mathbb{R}^2$, S is homeomorphic to $\mathbb{R}$ (see [12] and [9]). Denote by α a homeomorphism from $\mathbb{R}$ onto S and define

$$H(x_1, x_2) := (x_1, \alpha(x_2)) \qquad \text{for} \quad (x_1, x_2) \in \mathbb{R}^2.$$

Obviously H is a homeomorphism of $\mathbb{R}^2$ onto $\mathbb{R} \times S$ and $H^{-1}(y_1, y_2) := (y_1, \ \alpha^{-1}(y_2))$ for $(y_1, y_2) \in \mathbb{R} \times S$.

Put

$$T^t := H^{-1} \circ \Phi^t \circ H \qquad \text{for} \quad t \in \mathbb{R}.$$

We have

$$T^t(x_1, x_2) = (H^{-1} \circ \Phi^t \circ H)(x_1, x_2) = (H^{-1} \circ \Phi^t)(x_1, \alpha(x_2))$$
$$= H^{-1}(x_1 + t, \alpha(x_2)) = (x_1 + t, x_2) = (x_1, x_2) + t(1, 0).$$

Thus

$$T^t(x) = x + te_1 \qquad \text{for} \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^2,$$

where $e_1 = (1, 0)$.

From the definition of T^t we get

$$\Phi^t = H \circ T^t \circ H^{-1} \qquad \text{for} \quad t \in \mathbb{R}.$$

Hence by (5)

$$F^t = h^{-1} \circ \Phi^t \circ h = h^{-1} \circ H \circ T^t \circ H^{-1} \circ h.$$

Thus by (2)

$$f^t = \psi \circ F^t \circ \psi^{-1} = \varphi^{-1} \circ T^t \circ \varphi \qquad \text{for} \quad t \in \mathbb{R},$$

where $\varphi = H^{-1} \circ h \circ \psi^{-1}$. Consequently $\{f^t : t \in \mathbb{R}\}$ is translative.

Conversely, if $\{f^t : t \in \mathbb{R}\}$ is a translative iteration group, then it is easy to verify that f^1 is a singularity–free homeomorphism. This fact may also be obtained from Proposition 1. □

From Proposition 1 and Theorem 1 we get immediately

COROLLARY 1. *Every singularity–free homeomorphism f mapping a simply connected region $D \subset \mathbb{R}^2$ which preserves orientation is embeddable in a continuous iteration group and every continuous iteration group $\{f^t : t \in \mathbb{R}\}$ such that $f^1 =: f$ is a singularity–free homeomorphism is given by the formula* (1), *where $\varphi : D \to \mathbb{R}^2$ is a homeomorphic solution of the Abel equation*

$$\varphi(f(x)) = \varphi(x) + e_1 \qquad \text{for} \quad x \in D.$$

The homeomorphic solutions of the Abel equation on the plane depend on an arbitrary function. The general construction of all such solutions has been given in [7].

Further on we shall show that every continuous iteration group $\{f^t : t \in \mathbb{R}\}$ which is a subgroup of a continuous iteration group of homeomorphisms $\{f^z : z \in \mathbb{R}^n\}$ without fixed points is translative. To this end we shall prove a more general theorem.

THEOREM 2. *Let $\{f^z : z \in \mathbb{R}^n\}$ be a family of homeomorphisms mapping a region $D \subset \mathbb{R}^n$ onto itself such that $f^{z_1} \circ f^{z_2} = f^{z_1+z_2}$ for $z_1, z_2 \in \mathbb{R}^n$ and $f^z(x) \neq x$ for $x \in D$, $z \in \mathbb{R}^n \setminus \{(0,\dots,0)\}$ and the mapping $z \mapsto f^z(x)$ is continuous. Then there exists a homeomorphism φ mapping D onto $\mathbb{R}^n$ such that*

$$f^z(x) = \varphi^{-1}(\varphi(x) + z) \qquad \text{for} \quad x \in D, \quad z \in \mathbb{R}^n.$$

PROOF. Fix an $x \in D$ and define the function h_x by the formula

$$h_x(z) := f^z(x) \qquad \text{for} \quad z \in \mathbb{R}^n.$$

The function h_x is invertible. Indeed, if $h_x(z_1) = h_x(z_2)$, then $f^{z_1}(x) = f^{z_2}(x)$ and consequently $f^{z_1-z_2}(x) = x$, so $z_1 = z_2$. By the Brouwer invariance of region theorem (see e.g. [6, p. 199]), h_x is a homeomorphism as an invertible and continuous function on $\mathbb{R}^n$ and consequently the set $\Omega_x := h_x[\mathbb{R}^n]$ is open. For all $y \in \Omega_x$ and $z \in \mathbb{R}^n$ we have

$$f^z(y) = f^z(h_x(h_x^{-1}(y))) = f^z(f^{h_x^{-1}(y)}(x)) = f^{z+h_x^{-1}(y)}(x),$$

so

$$f^z(y) = h_x(z + h_x^{-1}(y)). \tag{6}$$

Suppose $\Omega := \Omega_u \cap \Omega_v \neq \emptyset$ for some $u, v \in D$. We shall show that $\Omega_u = \Omega_v$. Since $f^z[\Omega_u] = \Omega_u$ and $f^z[\Omega_v] = \Omega_v$ for $z \in \mathbb{R}^n$, we have $f^z[\Omega] = \Omega$ for $z \in \mathbb{R}^n$.

Fix a $y \in \Omega$. Then by (6)

$$f^z(y) = h_u(z + h_u^{-1}(y)) \qquad \text{for} \quad z \in \mathbb{R}^n$$

and

$$f^z(y) = h_v(z + h_u^{-1}(y)) \qquad \text{for} \quad z \in \mathbb{R}^n,$$

whence

$$h_u(z + c_1) = h_v(z + c_2) \qquad \text{for} \quad z \in \mathbb{R}^n,$$

where $c_1 := h_u^{-1}(y)$ and $c_2 := h_v^{-1}(y)$. Thus

$$h_u(z) = h_v(z + c) \qquad \text{for} \quad z \in \mathbb{R}^n,$$

where $c := c_2 - c_1$, so $\Omega_u = \Omega_v$.

Since $x \in \Omega_x$ for every $x \in D$, we have $\cup_{x \in D} \Omega_x = D$. By the connectivity of D we have $\Omega_x = D$ for every $x \in D$, because for each $x \in D$ the set Ω_x is open and for all $x, y \in D$ either $\Omega_x = \Omega_y$, or $\Omega_x \cap \Omega_y = \emptyset$. Again fix an $x \in D$ and put $\varphi := h_x^{-1}$. Then by (6) we have our assertion. □

From Theorem 2 we get

Corollary 2. *Let $\{f^t : t \in \mathbb{R}\}$ and $\{g^t : t \in \mathbb{R}\}$ be continuous iteration groups defined on a region $D \subset \mathbb{R}^2$ such that*

$$f^t \circ g^t = g^t \circ f^t \qquad \text{for} \quad t \in \mathbb{R} \tag{7}$$

and

$$\text{if } f^t(x_0) = g^s(x_0) \qquad \text{for some} \quad x_0 \in D, \qquad \text{then} \quad s = t = 0. \tag{8}$$

Then there exists a homeomorphism $\varphi : D \overset{onto}{\rightarrow} \mathbb{R}^2$ such that

$$f^t(x) = \varphi^{-1}(\varphi(x) + (0, t)) \qquad \text{for} \quad x \in D, \quad t \in \mathbb{R}$$

and

$$g^t(x) = \varphi^{-1}(\varphi(x) + (t, 0)) \qquad \text{for} \quad x \in D, \quad t \in \mathbb{R}.$$

Proof. From (7) we get $f^{nu} \circ g^{mu} = g^{mu} \circ f^{nu}$ for $u \in \mathbb{R}$ and $n, m \in \mathbb{Z}$. Setting $u = \frac{s}{n}$, we have $f^s \circ g^{rs} = g^{rs} \circ f^s$ for all rationals $r = \frac{m}{n}$. From the continuity of iteration group $\{g^t : t \in \mathbb{R}\}$ we get

$$f^s \circ g^t = g^t \circ f^s \qquad \text{for} \quad t, s \in \mathbb{R}. \tag{9}$$

Define the following family of functions

$$h^{(s,t)} := g^s \circ f^t \qquad \text{for} \quad t, s \in \mathbb{R}.$$

In view of (9) we have $h^u \circ h^v = h^{u+v}$ for $u, v \in \mathbb{R}^2$ and by (8) $h^z(x) \neq x$ for $x \in D$, $z \in \mathbb{R}^2 \setminus \{(0,0)\}$. From the fact that the functions $t \mapsto h^{(s_0,t)}(x)$ and $s \mapsto h^{(s,t_0)}(x)$ are continuous for all fixed $s_0, t_0 \in \mathbb{R}$, it follows that the function $(s,t) \mapsto h^{(s,t)}(x)$ is continuous at at least one $(s_1, t_1) \in \mathbb{R}^2$ (see e.g. [1, p. 237]). Hence it is continuous on the whole plane as a composition of continuous functions, since

$$h^{(s,t)}(x) = h^{(s-s_1,t-t_1)}(h^{(s_1,t_1)}(x)) = g^{s-s_1}(f^{t-t_1}(h^{(s_1,t_1)}(x))).$$

Thus by Theorem 2 there exists a homeomorphism $\varphi : D \overset{onto}{\rightarrow} \mathbb{R}^2$ such that

$$(g^s \circ f^t)(x) = h^{(s,t)}(x) = \varphi^{-1}(\varphi(x) + (s,t)), \qquad (s,t) \in \mathbb{R}^2, \quad x \in D.$$

Putting respectively $s = 0$ and $t = 0$ we get our assertion. □

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