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HULL–CONCAVE SET–VALUED FUNCTIONS

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Abstract. A set–valued function F is called hull–concave if

$$F(tx+(1-t)y)\subset \operatorname{co}(tF(x)+(1-t)F(y))$$

for all x,y from the domain of F and all $t\in[0,1]$. It is shown that if a hull–concave set–valued function F is defined on an open convex subset D of $\mathbb{R}^n$ and for every $x\in D$ the set $\operatorname{cl}F(x)$ is convex and bounded, then F is continuous on D. Some other properties of hull–concave set–valued functions are also given.

1. Introduction. The aim of this paper is to present, some results on hull–concave set–valued functions. The concept of hull–concave set–valued functions was introduced by A. V. Fiacco and J. Kyparisis in their work [3] devoted to general parametric optimization problems. Such functions are a natural generalization of concave set–valued functions. In the case of single valued functions hull–concavity means affinity.

In Section 2 we give some basic properties and a characterization of hull-concave set-valued functions with compact values in $\mathbb{R}^n$.

Section 3 is devoted to the problem of continuity. We prove that if a hull-concave set–valued function is defined on an open convex subset of $\mathbb{R}^n$ and the closures of its values are convex and bounded subsets of a topological vector space, then it is continuous. We also show that hull–midconcave set-valued functions defined on a topological vector space (not necessarily finite dimensional) and bounded on a set with a non–empty interior are continuous. The first result is a generalization of the well known fact stating that affine functions defined on $\mathbb{R}^n$ are continuous; the second one is an analogue of the classical Bernstein–Doetsch theorem for midconvex functions. The

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theorems presented here generalize some earlier results of K. Nikodem [5] obtained for concave and midconcave set-valued functions (cf. also [1] and [6]). Similar results for hull-convex set-valued functions were obtained by A. Fiacca and F. Papalini [2]. However, the method used in this paper is new and independent of [2].

2. Let X and Y be real vector spaces, D be a convex subset of X and $\mathrm{n}(Y)$ be the family of all non-empty subsets of Y. Given a set $A \subset Y$ we denote by $\mathrm{co}(A)$ the convex hull of A. A set-valued function $F : D \to \mathrm{n}(Y)$ is said to be:

– *concave* if

$$F(tx + (1-t)y) \subset tF(x) + (1-t)F(y), \quad x, y \in D, \quad t \in [0,1]; \tag{1}$$

– *hull-concave* if

$$F(tx + (1-t)y) \subset \mathrm{co}(tF(x) + (1-t)F(y)), \quad x, y \in D, \quad t \in [0,1]; \tag{2}$$

– *quasiconcave* if for every convex set $A \subset Y$ the upper inverse image $F^+(A) = \{x \in D : \ F(x) \subset A\}$ is convex.

We say that F is *midconcave* (*hull-midconcave*) if it satisfies condition (1) (condition (2)) with $t = 1/2$.

Observe first that a set-valued function $F : D \to \mathrm{n}(Y)$ is hull-concave if and only if the set-valued function $\mathrm{co}F$ defined by $\mathrm{co}F(x) = \mathrm{co}(F(x))$, $x \in D$, is concave (cf. [3, p. 110]). This follows immediately from the fact that $\mathrm{co}(A + B) = \mathrm{co}(A) + \mathrm{co}(B)$ for arbitrary sets A and B. In particular, if all values of F are convex, then F is hull-concave if and only if it is concave.

Proposition 1. *Every concave set-valued function is hull-concave and every hull-concave set-valued function is quasiconcave.*

Proof. The first statement is obvious; the second follows from the fact that a set-valued function $F : D \to \mathrm{n}(Y)$ is quasiconcave if and only if $F(tx + (1-t)y) \subset \ \mathrm{co}\ (F(x) \cup F(y))$ for all $x, y \in D$ and $t \in [0,1]$ (cf.[6, Theorem 2.8]). □

Given set-valued functions F and G we denote by $F+G$, $F \cup G$ and $F \cap G$ the set-valued functions defined by $(F+G)(x) = F(x)+G(x)$, $(F \cup G)(x) = F(x) \cup G(x)$ and $(F \cap G)(x) = F(x) \cap G(x)$, respectively.

Proposition 2. *If set-valued functions $F, G : D \to \mathrm{n}(Y)$ are hull--concave, then $F + G$ and $F \cup G$ are hull-concave.*

PROOF. Let $x, y \in D$ and $t \in [0,1]$. By assumption we get

$$\begin{aligned}(F+G)(tx+(1-t)y) &\subset \operatorname{co}(tF(x)+(1-t)F(y)) \\ &\quad + \operatorname{co}(tG(x)+(1-t)G(y)) \\ &= \operatorname{co}(t(F(x)+G(x))+(1-t)(F(y)+G(y))).\end{aligned}$$

Similarly,

$$\begin{aligned}(F\cup G)(tx+(1-t)y) &\subset \operatorname{co}(tF(x)+(1-t)F(y)) \\ &\quad \cup \operatorname{co}(tG(x)+(1-t)G(y)) \\ &\subset \operatorname{co}(t(F(x)\cup G(x))+(1-t)(F(y)\cup G(y))).\end{aligned}$$

□

REMARK 1. The set–valued function $F \cap G$ need not be hull–concave even if F and G are concave. For instance, the set–valued functions $F, G : [0,1] \to$ n($\mathbb{R}$) defined by $F(x) = [0, x]$, $G(x) = [0, 1-x]$, $x \in [0,1]$, are concave but $F \cap G$ is not hull–concave.

The next theorem characterizes hull–cancave set–valued functions with compact values in $\mathbb{R}^n$. We denote by c($\mathbb{R}^n$) the family of all compact non--empty subsets of $\mathbb{R}^n$, and by cc($\mathbb{R}^n$) the family of all convex compact non--empty subsets of $\mathbb{R}^n$. The set of all extreme points of A is denoted by Ext$A$.

THEOREM 1. *A set–valued function $F : D \to$ c($\mathbb{R}^n$) is hull–concave if and only if there exists a concave set–valued function $G : D \to$ cc($\mathbb{R}^n$) such that*

$$\text{(3)} \qquad \operatorname{Ext} G(x) \subset F(x) \subset G(x), \qquad x \in D.$$

PROOF. Assume that F is hull–concave and put $G = \operatorname{co}F$. Then G is concave and $F(x) \subset G(x)$, $x \in D$. Moreover, Ext $G(x) \subset F(x)$ because extreme points of the convex hull of a set belong to this set (cf.[4, Theorem 11.2.2]).

Now, assume that F satisfies (3) with a concave set–valued function G. Then, using the fact that co(Ext$A$) = A for every compact convex set $A \subset \mathbb{R}^n$ (cf.[4,Theorem 11.2.1]), we get

$$\begin{aligned}F(tx+(1-t)y) &\subset G(tx+(1-t)y) \subset tG(x)+(1-t)G(y) \\ &= t\operatorname{co}(\operatorname{Ext} G(x)) + (1-t)\operatorname{co}(\operatorname{Ext} G(y)) \\ &\subset t\operatorname{co}(F(x)) + (1-t)\operatorname{co}(F(y)) \\ &= \operatorname{co}(tF(x)+(1-t)F(y)).\end{aligned}$$

This shows that F is hull–concave. □

REMARK 2. The above theorem not only characterizes hull–concave set–valued functions but also gives a simple method of construction of such functions. For example, if $f : D \to \mathbb{R}$ is concave, $g : D \to \mathbb{R}$ is convex and $f(x) \le g(x)$, $x \in D$, then the set–valued function $G : D \to \mathrm{cc}(\mathbb{R})$ defined by $G(x) = [f(x), g(x)]$, $x \in D$, is concave and $\mathrm{Ext}\, G(x) = \{f(x),\ g(x)\}$. Therefore every set–valued function $F : D \to \mathrm{c}(\mathbb{R})$ such that

$$\{f(x), g(x)\} \subset F(x) \subset [f(x),\ g(x)], \quad x \in D,$$

is hull–concave.

3. In this section X and Y denote topological vector spaces (satisfying the T_0 separation axiom). Recall that a set–valued function $F : X \to \mathrm{n}(Y)$ is called *upper semicontinuous* (*usc*) at a point x_0 (*lower semicontinuous* (*lsc*) at x_0) if for every neighbourhood W of zero in Y there exists a neighbourhood U of zero in X such that

$$F(x) \subset F(x_0) + W \qquad (F(x_0) \subset F(x) + W) \qquad \text{for every} \quad x \in x_0 + U.$$

F is *continuous* at a point if it is usc and lsc at this point.

Let $\mathrm{b}(Y)$ denote the family of all bounded (in topological sense) and non–empty subsets of Y. It is known that every concave set–valued function $F : D \to \mathrm{b}(Y)$, where D is an open convex subset of $\mathbb{R}^n$, is continuous ([5, Corollary 2]; cf. also [1,Theorem 5.5] and [6, Theorem 4.7]). For hull–concave set–valued functions analogous result (without any additional assumptions) is not true. For instance, the set–valued function $F : \mathbb{R} \to \mathrm{c}(\mathbb{R})$ defined by

$$F(x) = \begin{cases} [0,1], & x \in \mathbb{Q}, \\ \{0,1\}, & x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$$

is hull–concave (by Theorem 1) but it is not continuous at any point. However, we have the following result.

THEOREM 2. *Let D be an open convex subset of $\mathbb{R}^n$ and Y be a topological vector space. If a set–valued function $F : D \to \mathrm{b}(Y)$ is hull–concave and for every $x \in D$ the set $\mathrm{cl}F(x)$ is convex, then F is continuous on D.*

In the proof of this theorem we use the following two lemmas.

LEMMA 1. *Let A be a subset of a topological vector space Y. Then the following conditions are equivalent:*
1. *$\mathrm{cl}A$ is convex;*
2. *$\mathrm{co}(A) \subset \mathrm{cl}A$;*
3. *$\mathrm{co}(A) \subset A + V$ for every neighbourhood V of zero in Y.*

PROOF. Implications $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 2$ are obvious. To show that $2 \Rightarrow 1$ notice first that $\mathrm{co}(\mathrm{cl}A) \subset \mathrm{cl}\,\mathrm{co}(A)$ for every set $A \subset Y$ (this follows from the fact that $\mathrm{cl}\,\mathrm{co}(A)$ is a convex set containing $\mathrm{cl}A$). Hence, by 2, we get $\mathrm{co}\,\mathrm{cl}A \subset \mathrm{cl}A$, which means that $\mathrm{cl}A$ is convex. □

LEMMA 2. *Let $F : X \to \mathrm{n}(Y)$ be a given set-valued function. If $\mathrm{co}F$ is usc at a point x_0 and $\mathrm{cl}F(x_0)$ is convex, then F is usc at x_0. If $\mathrm{co}F$ is lsc at a point x_0 and $\mathrm{cl}F(x)$ is convex for every x in some neighbourhood of x_0, then F is lsc at x_0.*

PROOF. Assume that $\mathrm{co}F$ is usc at x_0. Fix a neighbourhood W of zero in Y and take a neighbourhood V of zero in Y such that $V + V \subset W$. By assumption there exists a neighbourhood U of zero in X such that

$$\mathrm{co}F(x) \subset \mathrm{co}F(x_0) + V \qquad \text{for all} \quad x \in x_0 + U.$$

Hence, by Lemma 1, we obtain

$$F(x) \subset \mathrm{co}F(x) \subset F(x_0) + V + V \subset F(x_0) + W, \quad x \in x_0 + U,$$

which shows that F is usc at x_0. The proof of the second statement is analogous. □

REMARK 3. It is known (and easy to check) that if Y is a locally convex topological vector space, then the continuity of $F : X \to \mathrm{n}(Y)$ at a point implies the continuity of $\mathrm{co}F$ at this point.

PROOF OF THEOREM 2. The set-valued function $\mathrm{co}F$ is concave and its values are bounded. Indeed, by Lemma 1 $\mathrm{co}F(x) \subset \mathrm{cl}F(x)$, and $\mathrm{cl}F(x)$ is bounded because $F(x)$ is bounded. Therefore, by the result of K. Nikodem ([5, Corollary 2]), $\mathrm{co}F$ is continuous on D. Hence, by Lemma 2, F is continuous on D. □

Hull-concave set-valued functions defined on an infinite-dimensional space need not be continuous even if their values are convex; hull-midconcave set-valued functions may be discontinuous even if they are defined on a real interval and their values are convex. However, the following analogue of the Bernstein-Doetsch theorem holds true. Recall that F is said to be *bounded* on a set $A \subset X$ if there exists a bounded set $B \subset Y$ such that $F(x) \subset B$ for every $x \in A$.

THEOREM 3. *Let X and Y be topological vector spaces and D be an open convex subset of X. Assume that $F : D \to \mathrm{b}(Y)$ is a hull-midconcave set-valued function and $\mathrm{cl}F(x)$ is convex for every $x \in D$. If F is bounded on a set $A \subset D$ with a non-empty interior, then it is continuous on D.*

PROOF. By assumption there exists a bounded set $B \subset Y$ such that $F(x) \subset B$ for every $x \in A$. Consider the set-valued function co$F$. Using Lemma 1 we get

$$\mathrm{co}F(x) \subset \mathrm{cl}F(x) \subset \mathrm{cl}B, \quad x \in A,$$

which means that co$F$ is bounded on A. Moreover, co$F$ is midconcave and its values are bounded and convex. Therefore co$F$ is continuous on D (cf. [5, Theorem 2]). Consequently, by Lemma 2, F is continuous on D. □

The next theorem gives another condition implying the continuity of hull-midconcave set-valued functions.

THEOREM 4. *Let X be a topological vector space, D be an open convex subset of X and Y be a locally convex topological vector space. Assume that $F : D \to \mathrm{b}(Y)$ is a hull-midconcave set-valued function and $\mathrm{cl}F(x)$ is convex for every $x \in D$. If F is usc at a point $x_0 \in D$, then it is continuous on D.*

PROOF. The set-valued function co$F$ is midconcave and its values are bounded and convex. Moreover, co$F$ is usc at x_0 (cf. Remark 3). Therefore co$F$ is continuous on D (cf. [6, Theorem 4.2, for $K = \{0\}$] or [1, Corollary I, for $K = \{0\}$]). By Lemma 2 F is continuous on D. □

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