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ON A FUNCTIONAL EQUATION

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Abstract. The functional equation

$$\varphi(x+y)+\psi(x)+\psi(y)=\psi(x+y)+\varphi(x)+\varphi(y)$$

is studied for set-valued functions.

All topological spaces are assumed to satisfy Hausdorff separation axiom. Throughout the paper the symbols $\mathbf{Q}$ and $\mathbf{N}$ denotes the sets of all rational numbers and positive integers, respectively.

Let X be a linear space over the field $\mathbf{Q}$ and S be a $\mathbf{Q}$-*convex cone with zero* in X, i.e. if $x \in S$ and $\lambda \in [0,\infty)\cap\mathbf{Q}$, then $\lambda x \in S$ and if $x, y \in S$, then $x+y \in S$.

Recall that $0 \in \operatorname{core} U$, where $U \subset X$, if for every $x \in X$ there is $\varepsilon > 0$ such that $\lambda x \in U$ for all $\lambda \in \mathbf{Q}\cap(-\varepsilon,\varepsilon)$.

Let $U \subset X$ be a $\mathbf{Q}$-convex set such that $0 \in \operatorname{core} U$. By $\mathrm{c}(Y)$ we denote the family of all non-empty compact subsets of a real topological vector space Y and by $\mathrm{cc}(Y)$ we denote the family of all convex members of $\mathrm{c}(Y)$.

Consider a set-valued function (abbreviated to s.v. function in the sequel) $F: U\cap S \to \mathrm{c}(Y)$ such that

$$F\left(\frac{x+y}{2}\right)=\frac{F(x)+F(y)}{2} \tag{1}$$

for all $x, y \in U\cap S$. These functions will be called *Jensen* ones. It is known that values $F(x)$ of F belong to the family $\mathrm{cc}(Y)$ (see [2, Remark 3.1]).

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Let "$\sim$" denote the Rådström's equivalence relation between pairs of members of cc(Y) defined by the formula

$$(A,B)\sim(C,D)\Leftrightarrow A+D=B+C.$$

For any pair (A,B), $[A,B]$ denotes its equivalence class. All equivalence classes form a real linear space $\tilde{Y}$ with addition defined by the rule

$$[A,B]+[C,D]=[A+C,B+D],$$

and scalar multiplication

$$\lambda[A,B]=[\lambda A,\lambda B]$$

for $\lambda\geq 0$ and

$$\lambda[A,B]=[-\lambda B,-\lambda A]$$

for $\lambda<0$, (cf. [3]).

Now, let $F:U\cap S\to$ c(Y) be a solution of Jensen equation (1) and let

$$f_0(x)=[F(x),F(0)] \tag{2}$$

for $x\in U\cap S$. Then

$$f_0\left(\frac{x}{2}\right)=\left[F\left(\frac{x+0}{2}\right),F(0)\right]=\left[\frac{F(x)+F(0)}{2},F(0)\right]$$

$$=\frac{1}{2}[F(x)+F(0),2F(0)]=\frac{1}{2}[F(x),F(0)]=\frac{1}{2}f_0(x)$$

for $x\in U\cap S$. Hence, one has

$$\begin{aligned}f_0(x+y)=&2f_0\left(\frac{x+y}{2}\right)=2\left[F\left(\frac{x+y}{2}\right),F(0)\right]=2\left[\frac{F(x)+F(y)}{2},F(0)\right]\\&=[F(x)+F(y),F(0)+F(0)]\\&=[F(x),F(0)]+[F(y),F(0)]=f_0(x)+f_0(y)\end{aligned}$$

for all $x,y\in U\cap S$ such that $x+y\in U\cap S$. Similarly as in [1] the function f_0 can be extended to an additive function f on the whole S. The function $f:S\to\tilde{Y}$ has to have a representation

$$f(x)=[\varphi(x),\psi(x)] \tag{3}$$

where $\varphi : S \to \mathrm{cc}(Y), \psi : S \to \mathrm{cc}(Y)$. The additivity of f yields

$$\varphi(x+y) + \psi(x) + \psi(y) = \psi(x+y) + \varphi(x) + \varphi(y) \tag{4}$$

for all $x, y \in S$. The main goal of the note is to study equation (4).

We will need the following lemmas.

LEMMA 1 ([3, Lemma 1]). *Assume that A, B, C are subsets of Y such that $A + C \subset B + C$. If B is closed and convex and C is bounded and non-empty, then $A \subset B$.*

LEMMA 2. *There exists a base $E \subset S$ of the linear subspace $S - S$ of X over $\mathbb{Q}$.*

PROOF. The linear subspace $S - S$ of X over $\mathbb{Q}$ has a base

$$\{x_i - y_i : \quad i \in I\},$$

where $x_i, y_i \in S, \ i \in I$. Therefore

$$\mathrm{Lin}_{\mathbb{Q}}(\{x_i : \quad i \in I\} \cup \{y_i : \quad i \in I\}) = S - S.$$

There exists a minimal set

$$E \subset \{x_i : \quad i \in I\} \cup \{y_i : \quad i \in I\}$$

for which

$$\mathrm{Lin}_{\mathbb{Q}} E = S - S.$$

□

LEMMA 3. *Let $D \subset X$ be a $\mathbb{Q}$-convex set containing the origin. If $F, G, H : D \to \mathrm{cc}(Y)$ are s.v. functions such that*

$$F(x+y) = G(x) + H(y)$$

for all $x, y \in D$ for which $x + y \in D$, then F, G and H are Jensen s.v. functions on D.

The proof of the above lemma runs like that of Lemma 4 in [4] (there X is a real linear space and D is convex in X).

Suppose that functions $\varphi : S \to \mathrm{cc}(Y)$ and $\psi : S \to \mathrm{cc}(Y)$ fulfil equation (4). Putting $x = y = 0$ in (4) we obtain

$$\varphi(0) + \psi(0) + \psi(0) = \psi(0) + \varphi(0) + \varphi(0).$$

Lemma 1 allows us to get

$$\varphi(0) = \psi(0).$$

Using Lemma 1, we get by induction the equality

$$\begin{aligned}\varphi(x_1 + \ldots + x_n) + \psi(x_1) + \ldots + \psi(x_n) \\ = \psi(x_1 + \ldots + x_n) + \varphi(x_1) + \ldots + \varphi(x_n)\end{aligned} \tag{5}$$

for all $x_1, \ldots, x_n \in S$. Setting $x_1 = x_2 = \ldots = x_n = x$ for $x \in S$ we have

$$\varphi(nx) + n\psi(x) = \psi(nx) + n\varphi(x) \tag{6}$$

for each $n \in \mathbf{N}$. Replacing x in (6) by $\frac{x}{n}$ we obtain

$$\varphi(x) + n\psi\left(\frac{x}{n}\right) = \psi(x) + n\varphi\left(\frac{x}{n}\right),$$

whence

$$\frac{1}{n}\varphi(x) + \psi\left(\frac{x}{n}\right) = \frac{1}{n}\psi(x) + \varphi\left(\frac{x}{n}\right). \tag{7}$$

Similarly, setting $mx, m \in \mathbf{N}$, instead of x in (7), we get

$$\frac{1}{n}\varphi(mx) + \psi\left(\frac{m}{n}x\right) = \frac{1}{n}\psi(mx) + \varphi\left(\frac{m}{n}x\right).$$

In view of (6) this implies that

$$\begin{aligned}&\varphi\left(\frac{m}{n}x\right) + \frac{1}{n}\psi(mx) + \frac{m}{n}\psi(x) \\ &= \psi\left(\frac{m}{n}x\right) + \frac{1}{n}\varphi(mx) + \frac{m}{n}\psi(x) = \psi\left(\frac{m}{n}x\right) + \frac{1}{n}[\varphi(mx) + m\psi(x)] \\ &= \psi\left(\frac{m}{n}x\right) + \frac{1}{n}[\psi(mx) + m\varphi(x)] = \psi\left(\frac{m}{n}x\right) + \frac{1}{n}\psi(mx) + \frac{m}{n}\varphi(x);\end{aligned}$$

hence by Lemma 1

$$\varphi\left(\frac{m}{n}x\right) + \frac{m}{n}\psi(x) = \psi\left(\frac{m}{n}x\right) + \frac{m}{n}\varphi(x). \tag{8}$$

On account of Lemma 2, there exists a base $E \subset S$ of subspace $S - S$ over $\mathbf{Q}$. Write

$$S_0 = \left\{\sum_{i=1}^{n} \lambda_i e_i : \quad e_i \in E, \quad \lambda_i \in \mathbf{Q} \cap [0, \infty) \quad \text{for} \quad i = 1, 2, \ldots, n, \ n \in \mathbf{N}\right\}$$

and define

$$\tilde{\varphi}(y) = \sum_{i=1}^{n} \lambda_i \varphi(e_i), \ \tilde{\psi}(y) = \sum_{i=1}^{n} \lambda_i \psi(e_i)$$

whenever

$$y = \sum_{i=1}^{n} \lambda_i e_i \in S_0.$$

It is clear that $\tilde{\varphi}, \tilde{\psi} : S_0 \to \mathrm{cc}(Y)$ are additive. Each $x \in S \setminus \{0\}$ can be represented in the form

$$x = y - z \tag{9}$$

where

$$y = \sum_{i=1}^{n} \lambda_i e_i \in S_0, \quad z = \sum_{i=1}^{n} \mu_i e_i \in S_0 \tag{10}$$

with $\lambda_i, \mu_i \geq 0$ and $\lambda_i \cdot \mu_i = 0$ for $i = 1, \ldots, n$. This representation is unique. In view of (5) and (10) we have

$$\varphi(y) + \sum_{i=1}^{n} \psi(\lambda_i e_i) = \psi(y) + \sum_{i=1}^{n} \varphi(\lambda_i e_i),$$

whence

$$\begin{aligned}\varphi(y) + \sum_{i=1}^{n} \psi(\lambda_i e_i) + \tilde{\psi}(y) =& \psi(y) + \sum_{i=1}^{n} [\varphi(\lambda_i e_i) + \lambda_i \psi(e_i)] \\ =& \psi(y) + \sum_{i=1}^{n} \psi(\lambda_i e_i) + \tilde{\varphi}(y)\end{aligned}$$

by (8). Now, Lemma 1 yields

$$\psi(y) + \tilde{\varphi}(y) = \varphi(y) + \tilde{\psi}(y). \tag{11}$$

Similar consideration leads to

$$\varphi(z) + \tilde{\psi}(z) = \psi(z) + \tilde{\varphi}(z). \tag{12}$$

Conditions (4) and(9) imply the equality

$$\varphi(y) + \psi(x) + \psi(z) = \psi(y) + \varphi(x) + \varphi(z). \tag{13}$$

Now adding (11), (12) and (13) we obtain

$$\begin{aligned}\psi(y)+&\tilde{\varphi}(y) + \varphi(z) + \tilde{\psi}(z) + \varphi(y) + \psi(x) + \psi(z)\\ &=\varphi(y) + \tilde{\psi}(y) + \psi(z) + \tilde{\varphi}(z) + \psi(y) + \varphi(x) + \varphi(z)\end{aligned}$$

whence, again by Lemma 1, the equality

$$\psi(x) + \tilde{\varphi}(y) + \tilde{\psi}(z) = \varphi(x) + \tilde{\psi}(y) + \tilde{\varphi}(z) \tag{14}$$

holds.

Conversely, suppose that $S_0 \subset S$ is a subcone (over $\mathbb{Q}$) of S, $\varphi, \psi : S \to \mathrm{cc}(Y), S \subset S_0 - S_0$ and $\tilde{\varphi}, \tilde{\psi} : S_0 \to \mathrm{cc}(Y)$ are additive such that (14) is fulfilled. Taking $x = y - z \in C$, $\bar{x} = \bar{y} - \bar{z} \in C$ for $y, z, \bar{y}, \bar{z} \in S_0$, by (14) we get

$$\begin{aligned}\varphi(x + \bar{x}) + \tilde{\psi}(y + \bar{y}) + \tilde{\varphi}(z + \bar{z}) &=\psi(x + \bar{x}) + \tilde{\varphi}(y + \bar{y}) + \tilde{\psi}(z + \bar{z}),\\ \psi(x) + \tilde{\varphi}(y) + \tilde{\psi}(z) &=\varphi(x) + \tilde{\psi}(y) + \tilde{\varphi}(z),\\ \psi(\bar{x}) + \tilde{\varphi}(\bar{y}) + \tilde{\psi}(\bar{z}) &=\varphi(\bar{x}) + \tilde{\psi}(\bar{y}) + \tilde{\varphi}(\bar{z}).\end{aligned}$$

Adding up those equalities and using Lemma 1 we obtain

$$\varphi(x + \bar{x}) + \psi(x) + \psi(\bar{x}) = \psi(x + \bar{x}) + \varphi(x) + \varphi(\bar{x}).$$

□

Above considerations allow us to establish the following result.

THEOREM 1. *Let S be a $\mathbb{Q}$-convex cone containing the origin in a linear space X over $\mathbb{Q}$ and let Y be a real topological vector space Y. S.v. functions $\varphi, \psi : S \to \mathrm{cc}(Y)$ fulfil equation (4) if and only if there exists a subcone (over $\mathbb{Q}$) S_0 of S and additive s.v.functions $\tilde{\varphi} : S_0 \to \mathrm{cc}(Y)$ and $\tilde{\psi} : C_0 \to \mathrm{cc}(Y)$ such that $S \subset S_0 - S_0$ and (14) holds whenever $x = y - z \in S$, $y, z \in S_0$.*

Theorem 1 can be used to prove the following one.

THEOREM 2. *A s.v. function $F : S \cap U \to \mathrm{c}(Y)$ fulfils the Jensen functional equation if and only if all values of F are convex and there exist a subcone S_0 of S and additive s.v. functions $\tilde{\varphi}, \tilde{\psi} : S_0 \to \mathrm{cc}(Y)$ such that $S \subset S_0 - S_0$ and*

$$F(x) + \tilde{\varphi}(z) + \tilde{\psi}(y) = F(0) + \tilde{\varphi}(y) + \tilde{\psi}(z) \tag{15}$$

whenever $y, z \in S_0$ and $x = y - z \in S \cap U$.

PROOF. Suppose that $F : S \cap U \to c(Y)$ fulfils (1). F must be convex-valued. Relations (2) and (3) yield

$$F(x) + \psi(x) = F(0) + \varphi(x) \tag{16}$$

for $x \in S \cap U$, where $\varphi, \psi : S \to cc(Y)$ fulfils (4). Accoding to Theorem 1 there exists a subcone S_0 of S with $S \subset S_0 - S_0$ and additive s.v. mappings $\tilde{\varphi}, \tilde{\psi} : S_0 \to cc(Y)$ for which (14) holds. By (16) and (14) and Lemma 1 we have

$$F(x) + \tilde{\varphi}(z) + \tilde{\psi}(y) = F(0) + \tilde{\varphi}(y) + \tilde{\psi}(z),$$

which means that (15) holds.

Conversely, suppose that $F : S \cap U \to cc(Y)$. There exists a subcone S_0 of S and additive s.v. mappings $\tilde{\varphi}, \tilde{\psi} : S_0 \to cc(Y)$ for which (15) holds for $y, z \in S_0$ whenever $x = y - z \in S \cap U$. Consequently

$$F(x) + \tilde{\varphi}(z) + \tilde{\psi}(y) = F(0) + \tilde{\varphi}(y) + \tilde{\psi}(z),$$
$$F(\bar{x}) + \tilde{\varphi}(\bar{z}) + \tilde{\psi}(\bar{y}) = F(0) + \tilde{\varphi}(\bar{y}) + \tilde{\psi}(\bar{z}),$$

and

$$2F(0) + \tilde{\varphi}(y + \bar{y}) + \tilde{\psi}(z + \bar{z}) = 2F\left(\frac{x + \bar{x}}{2}\right) + \tilde{\varphi}(z + \bar{z}) + \tilde{\psi}(y + \bar{y})$$

whenever $z, \bar{z}, y, \bar{y} \in S_0$, $x = y - z$, $\bar{x} = \bar{y} - \bar{z}$. The last equality has been obtained setting in (15)

$$\frac{x + \bar{x}}{2}, \qquad \frac{y + \bar{y}}{2}, \qquad \frac{z + \bar{z}}{2}$$

instead of x, y, z, respectively. Now, it suffices to add the last three equalities and apply Lemma 1. Obtained in this way the expression

$$F(x) + F(\bar{x}) = 2F\left(\frac{x + \bar{x}}{2}\right)$$

for $x, \bar{x} \in S$ completes the proof. □

COROLLARY ([2, Theorem 5.6]). *A s.v. function $F : S \to c(Y)$ fulfils the Jensen functional equation if and only if there exists an additive set-valued function $A : S \to cc(Y)$ such that*

$$F(x) = F(0) + A(x).$$

for $x \in S$.

PROOF. Assume that $F : S \to c(Y)$ is a Jensen s.v. function. Let $S_0, \tilde{\varphi}, \tilde{\psi}$ be such as in Theorem 2 for $U = X$. Then (15) holds whenever $y, z \in S_0$ and $x = y - z \in S$. Let n be an arbitrary positive integer. Replacing y by ny, z by nz and $x = y - z$ by $nx = ny - nz$ in (15) we obtain

$$\frac{1}{n}F(nx) + \tilde{\varphi}(z) + \tilde{\psi}(y) = \frac{1}{n}F(0) + \tilde{\varphi}(y) + \tilde{\psi}(z).$$

This implies that there exists

$$A(x) = \lim_{n\to\infty} \frac{1}{n}F(nx),$$

where the limit is in the Hausdorff metric sense. The s.v. function A is additive and convex compact valued. Moreover,

$$\tilde{\varphi}(y) + \tilde{\psi}(z) = A(x) + \tilde{\varphi}(z) + \tilde{\psi}(y).$$

Adding up this equality and (15) and using Lemma 1 we obtain

$$F(x) = F(0) + A(x).$$

The converse implication is obvious. □

Now we proceed to prove a characterization of a s.v. solution of the Pexider equation.

THEOREM 3. *S.v. functions $F, G, H : S \cap U \to \mathrm{cc}(Y)$ fulfil the Pexider functional equation*

$$F(x + y) = G(x) + H(y) \tag{17}$$

if and only if

$$F(0) = G(0) + H(0) \tag{18}$$

and there exists a subcone S_0 of S with $S \subset S_0 - S_0$ and additive s.v. functions $\tilde{\varphi}, \tilde{\psi} : S_0 \to \mathrm{cc}(Y)$ such that

$$F(x) + \tilde{\varphi}(z) + \tilde{\psi}(y) = F(0) + \tilde{\varphi}(y) + \tilde{\psi}(z), \tag{19}$$

$$G(x) + \tilde{\varphi}(z) + \tilde{\psi}(y) = G(0) + \tilde{\varphi}(y) + \tilde{\psi}(z), \tag{20}$$

$$H(x) + \tilde{\varphi}(z) + \tilde{\psi}(y) = H(0) + \tilde{\varphi}(y) + \tilde{\psi}(z), \tag{21}$$

whenever $z, y \in S_0$ *and* $x = y - z \in S \cap U$.

PROOF. Suppose that $F, G, H : S \cap U \to \mathrm{cc}(Y)$ are solutions of (17). By Lemma 3 these functions have to fulfil equation (1). Now, Theorem 2 states that there exist a subcone S_0 of S and additive s.v. functions $\tilde{\varphi}, \tilde{\psi} : S_0 \to \mathrm{cc}(Y)$ such that (19) holds. Condition (18) is obvious. To prove (20) put $y = 0$ in (17). The equality

$$F(x) = G(x) + H(0),$$

conditions (18) and (19) imply that

$$G(x) + H(0) + \tilde{\varphi}(z) + \tilde{\psi}(y) = G(0) + H(0) + \tilde{\varphi}(y) + \tilde{\psi}(z).$$

Now (20) for $y, z \in S_0$ follows from Lemma 1. Equality (21) can be obtained in the same way.

Conversely, assume that $F, G, H : S \cap U \to \mathrm{cc}(Y)$ fulfil (18) and additive s.v. functions $\tilde{\varphi}, \tilde{\psi} : S_0 \to \mathrm{cc}(Y)$ fulfil (19), (20) and (21), where S_0 is a subcone of S. Then we have

$$\begin{aligned} F(x + \bar{x}) + \tilde{\varphi}(z + \bar{z}) + \tilde{\psi}(y + \bar{y}) &= F(0) + \tilde{\varphi}(y + \bar{y}) + \tilde{\psi}(z + \bar{z}), \\ G(0) + \tilde{\varphi}(y) + \tilde{\psi}(z) &= G(x) + \tilde{\varphi}(z) + \tilde{\psi}(y), \\ H(0) + \tilde{\varphi}(\bar{y}) + \tilde{\psi}(\bar{z}) &= H(\bar{x}) + \tilde{\varphi}(\bar{z}) + \tilde{\psi}(\bar{y}), \end{aligned}$$

whenever $y, \bar{y}, z, \bar{z} \in S_0$ and $x = y - z \in S \cap U$, $\bar{x} = \bar{y} - \bar{z} \in S \cap U$ with $x + \bar{x} \in U$. To get

$$F(x + \bar{x}) = G(x) + H(\bar{x})$$

for $x, \bar{x} \in S \cap U$ with $x + \bar{x} \in U$, it is enough to add up the last three equalities and to apply Lemma 1. □

K. Nikodem's theorem ([2, Theorem 5.7]) on the form of solutions of (17) can be derived from Theorem 3 (only on a $\mathbb{Q}$-convex cone) by similar considerations as in the proof of the Corollary.

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