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SELECTIONS OF BIADDITIVE SET-VALUED FUNCTIONS

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Abstract. In this paper we prove that there exists a biadditive selection f of a biadditive set-valued function F and a continuous selection when F is lower semicontinuous.

We begin with some notations and definitions. Let $\mathrm{n}(Y)$ denote the set of all nonempty subsets of a nonempty set Y. If Y is a normed space then $\mathrm{cc}(Y)$ denotes the set of all compact and convex elements of $\mathrm{n}(Y)$.

DEFINITION 1. Let X, Y, Z be real vector spaces. We say that a set-valued function $F : X \to n(Z)$ (abbreviated to "*s.v. function*") in the sequel is *additive* iff

$$F(x+y) = F(x) + F(y) \quad \text{for} \quad x, y \in X.$$

A s.v. function $F : X \times Y \to n(Z)$ is called *biadditive* iff F is additive with respect to each variable.

DEFINITION 2. The point x_0 of a subset C of real vector space X is called an *algebraic interior point* of C (we write $x_0 \in \mathrm{core}C$) iff for each $x \in X$ there is a real positive ε such that

$$tx + (1-t)x_0 \in C \quad \text{for} \quad |t| \leq \varepsilon.$$

DEFINITION 3. We say that a point $x_0 \in C$, $C \subseteq X$ is an *extreme point* of C iff there are no two different points $x, y \in C$ and no number $t \in (0,1)$ such that

$$x_0 = tx + (1-t)y.$$

The set of all extreme points of C is denoted by $\mathrm{Ext}C$.

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DEFINITION 4. A set $C \subseteq X$ is said to be a *convex cone* iff $C + C \subseteq C$ and $tC \subseteq C$ for all $t \in (0, \infty)$.

K. Nikodem in the paper [4] proved the following theorem.

THEOREM. *Let X, Y be real vector spaces and C be a convex cone in X. Assume that $F : C \to$ n(Y) is an additive s.v. function, $x_0 \in$ core$C$ and $p \in$ Ext$F(x_0)$. Then there exists exactly one additive selection $f : C \to Y$ of F such that $f(x_0) = p$. In addition,*

$$f(x) \in \text{Ext}F(x) \quad \textit{for} \quad x \in C.$$

The following lemma (Nikodem [4]) will be useful for us.

LEMMA. *Let B and C be subsets of a real vector space. If $p \in$ Ext$(B+C)$, then there exists exactly one point $b \in B$ and exactly one point $c \in C$ such that $b+c = p$. Moreover, $b \in$ Ext$B$ and $c \in$ Ext$C$, i.e. Ext$(B+C) \subseteq$ Ext$B+$ Ext$C$.*

Now, we shall formulate a theorem, analogue to Nikodem's Theorem.

THEOREM 1. *Let X, Y, Z be real vector spaces, C, D be convex cones in X, Y, respectively, and $F : C \times D \to$ n(Z) be a biadditive s.v. function. Moreover, let $x_0 \in$ core$C$, $y_0 \in$ core$D$ and $p \in$ Ext$F(x_0, y_0)$. Then there exists exactly one biadditive selection $f : C \times D \to Z$ of F such that $f(x_0, y_0) = p$.*

PROOF. Let $U := C \cap (x_0 - C)$. If $u \in U$ then $x_0 - u \in U$. Fix any element $a \in U$. Since $p \in$ Ext$F(x_0, y_0) =$ Ext $\{F(a, y_0) + F(x_0 - a, y_0)\}$, there exist, by Nikodem's lemma, a unique point $p_a \in$ Ext$F(a, y_0)$ and a unique point $p_{x_0-a} \in$ Ext$F(x_0 - a, y_0)$ such that

$$p = p_a + p_{x_0-a}. \tag{1.1}$$

For the additive s.v. function $F(a, \cdot) : D \to$ n(Z), $y_0 \in$ core$D$ and the point $p_a \in$ Ext$F(a, y_0)$, the assumptions of Nikodem's Theorem hold. So there exists exactly one additive selection $f_a : D \to Z$ of $F(a, \cdot)$ such that

$$f_a(y_0) = p_a.$$

It holds for any $a \in U$. Now, let us define a function $g_0 : U \times D \to Z$ as follows:

$$g_0(a, y) := f_a(y) \qquad \text{for} \quad (a, y) \in U \times D.$$

It is easy to check that g_0 is properly defined and

$$g_0(a,y) = f_a(y) \in F(a,y) \qquad \text{for} \quad (a,y) \in U \times D.$$

Moreover,

$$g_0(a, x+y) = f_a(x) + f_a(y) = g_0(a,x) + g_0(a,y) \qquad \text{for} \quad a \in U, \quad x, y \in D.$$

Now, we shall show that $g_0(a+b, x) = g_0(a,x) + g_0(b,x)$ for all $x \in D$, $a, b \in U$ such that $a+b \in U$. Since $p \in \operatorname{Ext}\{F(a,y_0) + F(x_0 - a, y_0)\}$, there exist exactly one $a_1 \in F(a,y_0)$ and exactly one $b_1 \in F(x_0 - a, y_0)$ such that $p = a_1 + b_1$. Similarly $p \in \operatorname{Ext}\{F(b,y_0) + F(x_0 - b, y_0)\}$, whence $p = a_2 + b_2$, where $a_2 \in F(b,y_0), b_2 \in F(x_0 - b, y_0)$ and $p \in \operatorname{Ext}\{F(a,y_0) + F(b,y_0) + F(x_0 - a - b, y_0)\}$ so $p = a_3 + b_3 + c_3$, where $a_3 \in F(a,y_0)$, $b_3 \in F(b,y_0)$ and $c_3 \in F(x_0 - a - b, y_0)$. We get

$$p = a_3 + (b_3 + c_3) = a_1 + b_1, \quad a_1, a_3 \in F(a,y_0), \; b_1, b_3 + c_3 \in F(x_0 - a, y_0),$$

whence, by the uniqueness of the representation (1.1), we infer that $a_3 = a_1 = p_a$. In the same way we get that $b_3 = a_2 = p_b$ and $p_{a+b} = a_3 + b_3$. That is $p_a + p_b = p_{a+b}$. This means that

$$f_a(y_0) + f_b(y_0) = f_{a+b}(y_0).$$

Since the fact that f_a is a selection of $F(a,\cdot)$ and f_b is a selection of $F(b,\cdot)$ implies that $f_a + f_b$ is a selection of $F(a+b,\cdot)$, and by the uniqueness of selection passing through the point y_0, we deduce that

$$f_{a+b}(y) = f_a(y) + f_b(y) \qquad \text{for} \quad y \in D$$

and

$$g_0(a+b, y) = f_{a+b}(y) = f_a(y) + f_b(y) = g_0(a,y) + g_0(b,y)$$

for $y \in D$, $\; a, b \in U$ such that $a + b \in U$. So, we have proved that g_0 is a biadditive selection of F on the set $U \times D$.

Now, we shall extend g_0 to a biadditive function defined on $C \times D$. Fix any point $x \in C$. Since $x_0 \in \operatorname{core} C$, there exists an $\varepsilon > 0$ such that

$$x_0 + tx \in C \qquad \text{for} \quad |t| < \varepsilon.$$

Let us take a number $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$. Then

$$-\frac{1}{n}x + x_0 \in C.$$

Consequently

$$\frac{x}{n} \in x_0 - C \qquad \text{and} \qquad \frac{x}{n} \in C.$$

It implies that $\frac{x}{n} \in U$. Put $g(x,y) := ng_0(\frac{x}{n}, y)$. This definition is correct. Indeed, if $m \in \mathbf{N}$ is such a number that $\frac{x}{m} \in U$, then $\frac{x}{nm} = (1-\frac{1}{m})\cdot 0 + \frac{1}{m}\cdot\frac{x}{n} \in x_0 - C$ as well as $\frac{x}{mn} \in C$ thus $\frac{x}{mn} \in U$ and

$$mg_0\left(\frac{x}{m}, y\right) = mng_0\left(\frac{x}{nm}, y\right) = ng_0\left(\frac{x}{n}, y\right).$$

Moreover, the function $g : C \times D \to Z$ defined above is biadditive. Indeed, let $x \in C$, $y \in C$, $n \in \mathbf{N}$ be a number so large that $\frac{x}{n}, \frac{y}{n}, \frac{x+y}{n} \in U$. Then

$$g(x+y,z) = ng_0\left(\frac{x+y}{n}, z\right) = ng_0\left(\frac{x}{n}, z\right) + ng_0\left(\frac{y}{n}, z\right) = g(x,z) + g(y,z).$$

Lastly, the function g is a selection of F. If $x \in C$, $y \in D$, $n \in \mathbf{N}$ and $\frac{x}{n} \in U$, then

$$g(x,y) = ng_0\left(\frac{x}{n}, y\right) \in nF\left(\frac{x}{n}, y\right) \subseteq F\left(\frac{x}{n}, y\right) + \ldots + F\left(\frac{x}{n}, y\right) = F(x,y).$$

To end the proof we have to show that g is a unique selection of F passing through the point $((x_0, y_0), p)$. So, assume that there exists $g_1 : C \times D \to Z$ biadditive selection of F such that $g_1(x_0, y_0) = p$. Fix any $a \in U$. Then

$$p = g_1(x_0, y_0) = g_1(a, y_0) + g_1(x_0 - a, y_0).$$

Since $g_1(a, y_0) \in F(a, y_0)$ and $g_1(x_0 - a, y_0) \in F(x_0 - a, y_0)$, by the uniqueness of representation (1.1), we have that

$$g_1(a, y_0) = p_a = f_a(y_0) = g(a, y_0).$$

Thus $g_1(a, y_0) = g(a, y_0)$ for $a \in U$. Since $g_1(a, \cdot), f_a$ are additive selections of $F(a, \cdot)$ and $g_1(a, y_0) = p_a = f_a(y_0)$, we deduce that

$$g_1(a, y) = f_a(y) = g(a, y) \qquad \text{for} \quad y \in D, \quad a \in U$$

(because the selection is unique). If $a \in C$, $n \in \mathbf{N}$ and $\frac{a}{n} \in U$ then

$$g_1(a, y) = ng_1\left(\frac{a}{n}, y\right) = ng\left(\frac{a}{n}, y\right) = g(a, y) \qquad \text{for} \quad a \in C, \quad y \in D.$$

Hence $g = g_1$ on the set $C \times D$. This completes the proof. □

REMARK 1. The last proof implies that

$$f(x,y) \in \text{ Ext } F(x,y) \qquad \text{for} \quad (x,y) \in C \times D,$$

whenever $F : C \times D \to \text{conv}(Z)$, where $\text{conv}(Z)$ denotes the set of nonempty convex subsets of Z. Indeed, if $x \in U$ and $y \in D$, then $g_0(x,y) \in \text{Ext}F(x,y)$. Fix $x \in C$, $y \in D$, $n \in \mathbb{N}$ such that $\frac{x}{n} \in U$. Then

$$g(x,y) = ng_0\left(\frac{x}{n}, y\right) \in n\text{Ext}F\left(\frac{x}{n}, y\right) \subseteq \text{ Ext } \left(nF\left(\frac{x}{n}, y\right)\right) \subseteq \text{ Ext}F(x,y).$$

THEOREM 2. *Let X, Y, Z be real vector spaces, and C, D convex cones in X, Y, respectively. Assume that $F : C \times D \to \text{conv}(Z)$ is a biadditive s.v. function and $x_0 \in \text{core}C, y_0 \in \text{core}D$ and $p \in \text{conv}[\text{Ext}F(x_0, y_0)]$. Then there exists a biadditive function $f : C \times D \to Z$ such that $f(x_0, y_0) = p$ and*

$$f(x,y) \in \text{ conv}[\text{Ext}F(x,y)] \qquad \textit{for} \quad (x,y) \in C \times D.$$

PROOF. The point p belongs to $\text{conv}[\text{Ext}F(x_0, y_0)]$, so there exist a number $n \in \mathbb{N}$, points $p_1, \dots, p_n \in \text{Ext}F(x_0, y_0)$ and nonnegative numbers $\lambda_1, \dots, \lambda_n$ such that $\sum_{i=1}^n \lambda_i = 1$ and $p = \sum_{i=1}^n \lambda_i p_i$. By Theorem 1, there exist biadditive functions $f_i : C \times D \to Z$ for which $f_i(x_0, y_0) = p_i$ and

$$f_i(x,y) \in \text{ Ext } F(x,y) \quad . \quad \text{for} \quad (x,y) \in C \times D, \quad i = 1, \dots, n.$$

It is easy to check that the function $f : C \times D \to Z$ given by formula

$$f(x,y) := \sum_{i=1}^{n} \lambda_i f_i(x,y) \qquad \text{for} \quad (x,y) \in C \times D$$

is biadditive, $f(x_0, y_0) = \sum_{i=1}^n \lambda_i p_i = p$ and $f(x,y) \in \text{conv}[\text{Ext}F(x,y)]$ for all $(x,y) \in C \times D$. □

DEFINITION 5. Assume that X, Y are topological vector spaces and C is an open subset of X. We say that a s.v. function $F : C \to \text{n}(Y)$ is *lower semicontinuous* (*l.s.c.*) *at* a point $x_0 \in C$ iff for any neighbourhood V of zero in Y, there exists a neighbourhood U of zero in X such that

$$F(x_0) \subseteq F(x) + V \qquad \text{for} \quad x \in x_0 + U. \tag{5.1}$$

We say that F is *upper semicontinuous* (*u.s.c.*) *at* $x_0 \in C$ iff for every neighbourhood V of zero in Y there exists a neighbourhood U of zero in X such that

$$F(x) \subseteq F(x_0) + V \qquad \text{for} \quad x \in x_0 + U. \tag{5.2}$$

F is called *continuous at* $x_0 \in C$ iff it is both l.s.c. and u.s.c. at x_0.

Theorem 3. *Let X, Y, Z be topological vector spaces and Z be locally convex, C, D open convex cones in X, Y, respectively. A s.v. function $A : C \times D \to \mathrm{cc}(Z)$ is biadditive if and only if there exist a biadditive continuous s.v. function $L : C \times D \to \mathrm{cc}(Z)$ and a biadditive function $a : C \times D \to Z$ such that*

$$A(x, y) = a(x, y) + L(x, y) \qquad \text{for} \quad (x, y) \in C \times D.$$

Proof. By Theorem 1, there exists a biadditive selection $a : C \times D \to Z$ of A. Let us define an s.v. function $L : C \times D \to \mathrm{cc}(Z)$ as follows:

$$L(x, y) := A(x, y) - a(x, y) \qquad \text{for} \quad (x, y) \in C \times D.$$

Obviously $0 \in L(x, y)$ for all $(x, y) \in C \times D$. Fix any $(x_0, y_0) \in C \times D$. Let W be a neighbourhood of zero in Z. $L(x_0, y_0)$ is bounded, so there is a positive integer $n \geq 3$ such that

$$\frac{2}{n} L(x_0, y_0) \subseteq W.$$

There exist a balanced neighbourhood U of 0 in X such that $\frac{1}{n}x_0 + u \in C$, $x_0 + u \in C$ for all $u \in U$ and a neighbourhood V of 0 in Y such that $\frac{1}{n}y_0 + v \in D, y_0 + v \in D$ for $v \in V$. Then

$$\begin{aligned}
L(x_0, y_0) =& L\left(\frac{n-2}{n}x_0, y_0\right) + \frac{2}{n}L(x_0, y_0) \\
\subseteq& L\left(\frac{n-2}{n}x_0, y_0\right) + L\left(\frac{1}{n}x_0 + \frac{n-1}{n}u, y_0\right) + W \\
=& L\left(\frac{n-1}{n}x_0 + \frac{n-1}{n}u, y_0\right) + W = L\left(x_0 + u, \frac{n-1}{n}y_0\right) + W \\
\subseteq& L\left(x_0 + u, \frac{n-1}{n}y_0\right) + L\left(x_0 + u, \frac{1}{n}y_0 + v\right) + W \\
=& L(x_0 + u, y_0 + v) + W,
\end{aligned}$$

where $(u, v) \in U \times V$. So, $L(x_0, y_0) \subseteq L(x, y) + W$ for $(x, y) \in (x_0, y_0) + U \times V$. Hence the function L is lower semicontinuous at (x_0, y_0) and L is l.s.c. in $C \times D$.

Since $(\frac{1}{n}x_0, \frac{1}{n}y_0) \in C \times D$ and $C \times D$ is open, there exist a balanced neighbourhood U of 0 in X and a balanced neighbourhood V of 0 in Y such that $\frac{1}{n}x_0 - u \in C, x_0 + u \in C$ for $u \in U$, $\frac{1}{n}y_0 - \frac{n+1}{n}v \in D, y_0 + v \in D$ for

$v \in V$. Let $(u, v) \in U \times V$. Then

$$\begin{aligned} L(x_0+u, y_0+v) &\subseteq L(x_0+u, y_0+v) + L\left(\frac{1}{n}x_0 - u, y_0+v\right) \\ &= L\left(\frac{n+1}{n}x_0, y_0+v\right) = L\left(x_0, \frac{n+1}{n}y_0 + \frac{n+1}{n}v\right) \\ &\subseteq L\left(x_0, \frac{n+1}{n}y_0 + \frac{n+1}{n}v\right) + L\left(x_0, \frac{1}{n}y_0 - \frac{n+1}{n}v\right) \\ &= L\left(x_0, \frac{n+2}{n}y_0\right) = L(x_0, y_0) + \frac{2}{n}L(x_0, y_0) \\ &\subseteq L(x_0, y_0) + W. \end{aligned}$$

So, $L(x_0+u, y_0+v) \subseteq L(x_0, y_0) + W$ for $(u, v) \in U \times V$. Hence L is upper semicontinuous at (x_0, y_0). By the first part of the proof L is continuous in $C \times D$. □

For the next theorem we need some Banach-Steinhaus-type theorems for a bilinear function, which are probably known, however we will give them here for convenience of readers.

DEFINITION 6. Let X, Y, Z be real normed spaces. A bilinear map $T : X \times Y \to Z$ is called *bounded* iff there exists a real number $M > 0$ such that

$$\| T(x,y) \| \leq M \| x \| \cdot \| y \| \qquad \text{for} \quad (x,y) \in X \times Y.$$

The norm of a bilinear bounded map T is defined by the formula

$$\| T \| = \sup_{\|x\| \leq 1, \|y\| \leq 1} \| T(x,y) \| .$$

A bilinear map is bounded if and only if it is continuous.

THEOREM 4. *Let X, Y be Banach spaces and Z be a normed space. Assume that bilinear maps $T_n : X \times Y \to Z$ are continuous, $n \in \mathbf{N}$. If the sequence $\{T_n(x,y)\}_{n \in \mathbf{N}}$ is bounded for all $(x,y) \in X \times Y$, then the sequence $\{\| T_n \|\}_{n \in \mathbf{N}}$ is bounded.*

PROOF. Let $A_k := \{(x,y) \in X \times Y : \| T_n(x,y) \| \leq k,\ n \in \mathbf{N}\}$, $k \in \mathbf{N}$. It is easy to verify that

$$X \times Y = \bigcup_{k \in \mathbf{N}} A_k.$$

The continuity of the maps T_n and the norm implies that sets A_k are closed, $k \in \mathbf{N}$. Since X, Y are Banach spaces, we deduce by Baire's theorem that

$X \times Y$ is the second category set; this means that there exists a number $k_0 \in \mathbf{N}$ such that A_{k_0} is not a nowhere dense set; in other words $\mathrm{Int}A_{k_0} \neq \emptyset$. so there exist real numbers $r_1 > 0, r_2 > 0$ such that

$$\mathrm{cl}K_1(x_0, r_1) \times \ \mathrm{cl}K_2(y_0, r_2) \subseteq A_{k_0}$$

(where K_1 is a ball in X, K_2 is a ball in Y). If $\| x - x_0 \| \leq r_1$ and $\| y - y_0 \| \leq r_2$, then $\| T_n(x, y) \| \leq k_0$ for all $n \in \mathbf{N}$. Fix $(x, y) \in X \times Y$ such that $x \neq 0$ and $y \neq 0$. Since $\| \left(\frac{x}{\|x\|} r_1 + x_0\right) - x_0 \| = r_1$ and $\| \left(\frac{y}{\|y\|} r_2 + y_0\right) - y_0 \| = r_2$ one has

$$\| T_n \left(\frac{x}{\| x \|} r_1 + x_0, \frac{y}{\| y \|} r_2 + y_0 \right) \| \leq k_0$$

and

$$\begin{aligned}
\|T_n(x, y)\| &= \left\| T_n \left(\frac{x}{\|x\|} r_1, y \right) \right\| \cdot \frac{\|x\|}{r_1} \\
&= \frac{\|x\|}{r_1} \left\| T_n \left(\frac{x}{\|x\|} r_1 + x_0, y \right) - T_n(x_0, y) \right\| \\
&\leq \frac{\|x\|}{r_1} \left(\left\| T_n \left(\frac{x}{\|x\|} r_1 + x_0, y \right) \right\| + \|T_n(x_0, y)\| \right) \\
&= \frac{\|x\|}{r_1} \left\{ \frac{\|y\|}{r_2} \left\| T_n \left(\frac{x}{\|x\|} r_1 + x_0, \frac{y}{\|y\|} r_2 + y_0 \right) \right. \right. \\
&\quad \left. - T_n \left(\frac{x}{\|x\|} r_1 + x_0, y_0 \right) \right\| \\
&\quad \left. + \frac{\|y\|}{r_2} \left\| T_n \left(x_0, \frac{y}{\|y\|} r_2 + y_0 \right) - T_n(x_0, y_0) \right\| \right\} \\
&\leq \frac{4k_0}{r_1 \cdot r_2} \|x\| \cdot \|y\|
\end{aligned}$$

for $(x, y) \in X \times Y$ such that $x \neq 0, y \neq 0$. Hence

$$\| T_n \| = \sup_{\|x\| = \|y\| = 1} \| T_n(x, y) \| \leq \frac{4k_0}{r_1 r_2} \quad \text{for} \quad n \in \mathbf{N}.$$

□

DEFINITION 7. A subset A of a normed space X is called *linearly dense* in X iff the set

$$\left\{ \sum_{i=1}^{n} \lambda_i a_i; \qquad a_i \in A, \quad \lambda_i \in \mathbf{R}, \quad i = 1, \ldots, n; \quad n \in \mathbf{N} \right\}$$

is dense in X.

THEOREM 5. *Let X, Y, Z be Banach spaces and A_1, A_2 be linearly dense sets in X, Y, respectively. Assume that $T_n : X \times Y \to Z$, $n \in \mathbf{N}$ is a sequence of bilinear and continuous maps. The sequence $\{T_n(x,y)\}_{n\in\mathbf{N}}$ is convergent for all $(x,y) \in X \times Y$ iff $\{T_n(x,y)\}_{n\in\mathbf{N}}$ is convergent for all $(x,y) \in A_1 \times A_2$ and the sequence $\{\| T_n \|\}_{n\in\mathbf{N}}$ is bounded.*

PROOF. If the sequence $\{T_n(x,y)\}_{n\in\mathbf{N}}$ is convergent in $X \times Y$ then it is in $A_1 \times A_2$. Since $\{T_n(x,y)\}_{n\in\mathbf{N}}$ is convergent, the sequence $\{\| T_n(x,y) \|\}_{n\in\mathbf{N}}$ is bounded for any $(x,y) \in X \times Y$. Hence, by Theorem 4, the sequence $\{\| T_n \|\}_{n\in\mathbf{N}}$ is bounded.

Now we assume that $\{T_n(x,y)\}_{n\in\mathbf{N}}$ is convergent in $A_1 \times A_2$ and $\{\| T_n \|\}_{n\in\mathbf{N}}$ is bounded by M. Fix any pair $(x_0, y_0) \in X \times Y$ and let a be an element of the set A_1. Then the map $F_n : Y \to Z$, given by the formula $F_n(y) := T_n(a,y)$ for $y \in Y$, is linear and continuous in Y. Moreover, the sequence $\{F_n(y)\}_{n\in\mathbf{N}}$ is convergent for any $y \in A_2$ and $\{\| F_n \|\}_{n\in\mathbf{N}}$ is bounded. Indeed,

$$\begin{aligned} \| F_n \| &= \sup_{\|y\|=1} \| F_n(y) \| = \sup_{\|y\|=1} \| T_n(a,y) \| \\ &\leq \sup_{\|y\|=1} \| T_n \| \| a \| \| y \| = M \cdot \| a \|, \qquad n \in \mathbf{N}. \end{aligned}$$

So, by Theorem 16.8 ([3] p.156), we get the convergence of the sequence $\{F_n(y)\}_{n\in\mathbf{N}}$ for all $y \in Y$. Hence, in particular, $\{F_n(y_0)\}_{n\in\mathbf{N}}$ is convergent. Since $a \in A_1$ is arbitrary, the sequence $\{T_n(a, y_0)\}_{n\in\mathbf{N}}$ is convergent for any $a \in A_1$.

Let us define maps $G_n : X \to Z$ as follows:

$$G_n(x) := T_n(x, y_0) \qquad \text{for} \quad x \in X, \quad n \in \mathbf{N}.$$

G_n are linear and continuous maps and the sequence $\{G_n(x)\}_{n\in\mathbf{N}}$ is convergent for any $x \in A_1$. Moreover,

$$\| G_n \| = \sup_{\|x\|=1} \| G_n(x) \| \leq M \cdot \| y_0 \|, \quad n \in \mathbf{N}.$$

Hence, by the same theorem, the sequence $\{G_n(x)\}_n \in \mathbf{N}$ is convergent for any $x \in X_1$, in particular for $x = x_0$. Consequently $\{T_n(x_0, y_0)\}_{n\in\mathbf{N}}$ is convergent. □

THEOREM 6. *Let X, Y, Z, A_1, A_2 be just like in the last theorem. If a sequence $T_n : X \times Y \to Z$ of bilinear and continuous maps is convergent in $A_1 \times A_2$ and the sequence $\{\| T_n \|\}_{n\in\mathbf{N}}$ is bounded then the function $T : X \times Y \to Z$ given by*

$$T(x,y) := \lim_{n \to \infty} T_n(x,y) \qquad \text{for} \quad (x,y) \in X \times Y$$

is a bilinear as well as continuous map and

$$\| T \| \leq \sup_{n \in \mathbb{N}} \| T_n \| .$$

PROOF. Theorem 5 implies the convergence of the sequence $\{T_n(x,y)\}_{n\in\mathbb{N}}$ for all $(x,y) \in X \times Y$ and hence, the correctness of definition of the map T. Its bilinearity and continuity follow from the Theorem 48.4 ([1] p.139).

Let $x \in X$, $y \in Y$ and $\| x \| \leq 1, \| y \| \leq 1$. Then

$$\begin{aligned}\| T(x,y) \| &\leq \| T(x,y) - T_n(x,y) \| + \| T_n(x,y) \| \\ &\leq \| T(x,y) - T_n(x,y) \| + M \| x \| \| y \| \\ &\leq \| T(x,y) - T_n(x,y) \| + M\end{aligned}$$

for $n \in \mathbf{N}$, where $M = \sup_{n\in\mathbf{N}} \| T_n \|$. By letting $n \to \infty$, we obtain $\| T(x,y) \| \leq M$ for $(x,y) \in X \times Y$, $\| x \| \leq 1, \| y \| \leq 1$. Thus

$$\| T \| = \sup_{\|x\|\leq 1, \|y\| \leq 1} \| T(x,y) \| \leq M = \sup_{n\in\mathbb{N}} \| T_n \| .$$

□

LEMMA 1. *Let X, Y, Z be real vector spaces, C, D convex cones in X, Y, respectively. Let $f : C \times D \to Z$ be a biadditive function. Then there exists a biadditive function $\bar{f} : X \times Y \to Z$ such that $\bar{f}(x,y) = f(x,y)$ for $(x,y) \in C \times D$. If C, D are open then*

$$\bar{f}(x,y) := f(x_1,y_1) - f(x_2,y_1) - f(x_1,y_2) + f(x_2,y_2).$$

where $x = x_1 - x_2$, $y = y_1 - y_2$, $x_1, x_2 \in C, y_1, y_2 \in D$.

PROOF. If C, D are cones then $(C \times D) - (C \times D) = (C - C) \times (D - D)$ is a subspace of $X \times Y$. Let us define a function f_0 on $(C - C) \times (D - D)$ as follows:

$$f_0(x,y) := f(x_1,y_1) - f(x_2,y_1) - f(x_1,y_2) + f(x_2,y_2),$$

where $x = x_1 - x_2$, $y = y_1 - y_2$, $x_1, x_2 \in C, y_1, y_2 \in D$.

At first we shall show that the definition of f_0 is correct. Assume that $x = x_1 - x_2 = z_1 - z_2$ and $y = y_1 - y_2$ where $x_1, x_2, z_1, z_2 \in C$ and $y_1, y_2 \in D$. Then $x_1 + z_2 = z_1 + x_2$ and

$$\begin{aligned}&[f(x_1,y_1) - f(x_1,y_2) - f(x_2,y_1) + f(x_2,y_2)] \\ &\quad - [f(z_1,y_1) - f(z_1,y_2) - f(z_2,y_1) + f(z_2,y_2)] \\ &= f(x_1+z_2,y_1) + f(x_2+z_1,y_2) - f(x_2+z_1,y_1) - f(x_1+z_2,y_2) \\ &= [f(x_1+z_2,y_1) - f(x_2+z_1,y_1)] + [f(x_2+z_1,y_2) - f(x_1+z_2,y_2)] = 0.\end{aligned}$$

The case when $x = x_1 - x_2$ and $y = y_1 - y_2 = u_1 - u_2$, $(x_1, x_2 \in C, y_1, y_2, \ u_1, u_2 \in D)$ is similar.

We shall check that f_0 is a biadditive map on $(C - C) \times (D - D)$ to Z and $f_0(x, y) = f(x, y)$ for $(x, y) \in C \times D$. Indeed, let $x, z \in C - C$ and $y \in D - D$. Then there exist $x_1, x_2, z_1, z_2 \in C$ and $y_1, y_2 \in D$ such that $x = x_1 - x_2$, $y = y_1 - y_2$, $z = z_1 - z_2$. By defintion of f_0

$$\begin{aligned} f_0(x + z, y) =& f_0((x_1 + z_1) - (x_2 + z_2), y_1 - y_2) \\ =& f(x_1 + z_1, y_1) - f(x_1 + z_1, y_2) \\ & - f(x_2 + z_2, y_1) + f(x_2 + z_2, y_2) \\ =& [f(x_1, y_1) - f(x_1, y_2) - f(x_2, y_1) + f(x_2, y_2)] \\ & + [f(z_1, y_1) - f(z_1, y_2) - f(z_2, y_1) + f(z_2, y_2)] \\ =& f_0(x, y) + f_0(z, y). \end{aligned}$$

In the same way we can prove the addivity of f_0 with respect to the second variable. Finally, we shall check that f_0 is an extension of f. Let $(x, y) \in C \times D$. Then $(x, y) = (2x, 2y) - (x, y)$ and

$$\begin{aligned} f_0(x, y) =& f(2x, 2y) - f(x, 2y) - f(2x, y) + f(x, y) \\ =& f(x, 2y) - [f(2x, y) - f(x, y)] = f(x, 2y) - f(x, y) = f(x, y). \end{aligned}$$

Let X_1 be a subspace of X, and Y_1 be a subspace of Y such that $(C - C) \oplus X_1 = X$ and $(D - D) \oplus Y_1 = Y$. So, if $(x, y) \in X \times Y$ then

$$(x, y) = (x_1 + x_2, y_1 + y_2), \text{ where } x_1 \in C - C, x_2 \in X_1, y_1 \in D - D, y_2 \in Y_2.$$

Let us define a function $\bar{f} : X \times Y \to Z$ as follows:

$$\bar{f}(x, y) = f_0(x_1, y_1).$$

It is easy to check that $\bar{f}$ is properly defined biadditive extension of f. □

REMARK 2. With respect to the above lemma we may assert in Theorem 3 that the biadditive function a is given on $X \times Y$. Similarly in the next theorem.

Now, we shall prove the following theorem, analogue to Theorem 2 from [6].

THEOREM 7. *Let X, Y be real separable Banach spaces, C, D be open, convex cones in X, Y, respectively, and let Z be a real Banach space. Assume that $F : C \times D \to \mathrm{cc}(Z)$ is a biadditive s.v. function, $x_0 \in C$, $y_0 \in D$ and*

$p \in F(x_0, y_0)$. Then there exists a biadditive selection $f : C \times D \to Z$ of F such that $f(x_0, y_0) = p$. Moreover, if F is lower semicontinuous, then f is continuous.

PROOF. Since F is compact and convex valued in Z, by the Krein-Milman Theorem ([5])

$$p \in F(x_0, y_0) = \mathrm{cl}[\mathrm{convExt} F(x_0, y_0)].$$

Then for each $n \in \mathbb{N}$ there is an element $p_n \in \mathrm{convExt} F(x_0, y_0)$ such that

$$\| p_n - p \| < \frac{1}{n}.$$

Theorem 2 guarantees the existence of biadditive functions $f_n : C \times D \to Z$ such that

$$f_n(x_0, y_0) = p_n$$

and

$$f_n(x, y) \in \mathrm{convExt}\ F(x, y) \subseteq F(x, y) \quad \text{for} \quad (x, y) \in C \times D.$$

The set $C \times D$ is an open cone in $X \times Y$ and the set $(C \times D) - (C \times D)$ is an open subspace of $X \times Y$, whence

$$(C \times D) - (C \times D) = \mathrm{lin}\ (C \times D) = (C - C) \times (D - D) = X \times Y.$$

By Lemma 1

$$\bar{f}_n(x, y) := f_n(x_1, y_1) - f_n(x_2, y_1) - f_n(x_1, y_2) + f_n(x_2, y_2),$$

where $x = x_1 - x_2$, $y = y_1 - y_2$, $x_1, x_2 \in C$, $y_1, y_2 \in D$, is a biadditive map from $X \times Y$ to Z and $\bar{f}_n(x, y) = f_n(x, y)$ for $(x, y) \in C \times D$.

Now, we assume that F is a lower semicontinuous s.v.function. For a fixed $x \in C$ a function $y \to F(x, y)$ is additive and $\mathbb{Q}_+$-homogeneous on D (see Lemma 5.1 in [4]). There exists a constant $M(x) > 0$ such that $\| F(x, y) \| \leq M(x) \| y \|$, where $\| F(x, y) \| = \sup\{\| u \|; u \in F(x, y)\}$ for $y \in D$ (see Theorem 4 in [7]). Then, for each $x \in C$, the set

$$F(x, \Sigma) = \bigcup_{y \in \Sigma} F(x, y),$$

where $\Sigma = \{y \in D;\ \| y \| \leq 1\}$ is bounded. By Smajdor's theorem from [7] there exists a constant M such that

$$\sup_{y \in \Sigma} \| F(x, y) \| \leq M \| x \| \quad \text{for} \quad x \in C.$$

Let us take a point $y \in D$ and let $\{r_n\}_{n\in\mathbf{N}}$ be a sequence of rational numbers such that $\lim_{n\to\infty} r_n = \| y \|$ and $\| y \| < r_n$ for $n \in \mathbf{N}$. Since $\frac{y}{r_n} \in \Sigma$, $\| F(x, \frac{y}{r_n}) \| \le M \| x \|$ for all $n \in \mathbf{N}$, $x \in C$. Hence $\| F(x,y) \| \le M r_n \| x \|$. Passing to the limit with $n \to \infty$, we get

$$\| F(x,y) \| \le M \| x \| \| y \| \quad \text{for } (x,y) \in C \times D. \tag{7.1}$$

Hence and by the relation $f_n(x,y) \in F(x,y)$ we deduce that

$$\| f_n(x,y) \| \le M \| x \| \| y \| \quad \text{for } (x,y) \in C \times D,\ n \in \mathbf{N}. \tag{7.2}$$

For every $x \in X$, the function $\bar{f}_n(x,\cdot) : Y \to Z$ is additive in Y and bounded in some neighbourhood of any point of D, so by the Mehdi theorem (Theorem 4 in [2]) $\bar{f}_n(x,\cdot)$ is continuous. Similarly, we get continuity of $\bar{f}_n(\cdot,y)$ for any $y \in Y$. Thus $\bar{f}_n$ is a bilinear and continuous map on $X \times Y$.

Now, we shall show that the sequence $\{\| \bar{f}_n \|\}_{n\in\mathbf{N}}$ is bounded. Let us fix $(x,y) \in X \times Y$ and $x_1, x_2 \in C, y_1, y_2 \in D$ such that $x = x_1 - x_2$, $y = y_1 - y_2$. Then

$$\begin{aligned}\| \bar{f}_n(x,y) \| &= \| f_n(x_1,y_1) - f_n(x_1,y_2) - f_n(x_2,y_1) + f_n(x_2,y_2) \| \\ &\le \| f_n(x_1,y_1) \| + \| f_n(x_1,y_2) \| \\ &\quad + \| f_n(x_2,y_1) \| + \| f_n(x_2,y_2) \|,\end{aligned}$$

whence and by (7.2) we get

$$\| \bar{f}_n(x,y) \| \le M(\| x_1 \| \| y_1 \| + \| x_1 \| \| y_2 \| + \| x_2 \| \| y_1 \| + \| x_2 \| \| y_2 \|).$$

Thus, by Theorem 4 the sequence $\{\| \bar{f}_n \|\}_{n\in\mathbf{N}}$ is bounded.

Let sets A and B be dense and countable in C and D, respectively. The set

$$S := A \times B = \{(x_1,y_1), (x_2,y_2), \ldots\}$$

is dense in $C \times D$ and linearly dense in $X \times Y$. We choose a subsequence $\{\bar{f}_{\lambda_n}\}_{n\in\mathbf{N}}$ of the sequence $\{\bar{f}_n\}_{n\in\mathbf{N}}$ convergent to the point (x_1,y_1). We are able to do it because $\{\bar{f}_n(x_1,y_1)\}_{n\in\mathbf{N}}$ is a sequence of elements of the compact set $F(x_1,y_1)$. Next, we choose a subsequence of $\{\bar{f}_{\lambda_n}\}_{n\in\mathbf{N}}$ convergent to (x_2,y_2), etc. Using the diagonal method we get the subsequence $\{\bar{f}_{n_k}\}_{k\in\mathbf{N}}$ of $\{\bar{f}_n\}_{n\in\mathbf{N}}$ convergent on S. The sequence $\{\bar{f}_{n_k}\}_{k\in\mathbf{N}}$ is convergent on the lineary dense in $X \times Y$ set S and the sequence $\{\| \bar{f}_{n_k} \|\}_{k\in\mathbf{N}}$ is bounded, so by Theorem 5 it converges to some bilinear and continuous map $\bar{f} : X \times Y \to Z$. For any $(x,y) \in C \times D$ we have

$$\bar{f}(x,y) \in \operatorname{cl}[\operatorname{convExt} F(x,y)] = F(x,y).$$

Therefore $f := \bar{f}|_{C\times D}$ is a selection of F on the cone $C \times D$.

If $F : C \times D \to \mathrm{cc}(Z)$ is a biadditive s.v.function, then there exist a biadditive function $a : X \times Y \to Z$ and a biadditive continuous s.v.function $L : C \times D \to \mathrm{cc}(Y)$ such that

$$F(x,y) = a(x,y) + L(x,y) \qquad \text{for} \quad (x,y) \in C \times D$$

(cf. Theorem 3 and Remark 2). By the first part of the proof there exists a bilinear and continuous function $f : X \times Y \to Z$ such that $f|_{C\times D}$ is a selection of L on the cone $C \times D$ and

$$f(x_0,y_0) = p - a(x_0,y_0).$$

Then the function $f_1 : X \times Y \to Z$ given by

$$f_1(x,y) := a(x,y) + f(x,y) \quad \text{for} \quad (x,y) \in X \times Y,$$

restricted to $C \times D$, is a biadditive selection of F satisfying the condition

$$f_1(x_0,y_0) = a(x_0,y_0) + f(x_0,y_0) = p.$$

This completes the proof. □

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