Annales Mathematicae Silesianae 7. Katowice 1993, 57–61

Prace Naukowe Uniwersytetu Śląskiego nr 1399

A DISCRETE FORM OF JORDAN CURVE THEOREM

WOJCIECH SURÓWKA

Abstract. The paper includes purely combinatorial proof of a theorem that implies the following theorem, stated by Hugo Steinhaus in [6, p.35]: consider a chessboard (rectangular, not necessarily square) with some "mined" squares on. Assume that the king, while moving in accordance with the chess rules, cannot go across the chessboard from the left edge to the right one without meeting a mined square. Then the rook can go across the chessboard from the upper edge to the lower one moving exclusively on mined squares. All proofs of the Steinhaus theorem already published (see [5] and remarks on some proofs in [7, p.211]) are incomplete, except the hexagonal variant proved by Gale in [1].

Steinhaus theorem was thought in [6, p.269] as the lemma in the proof of the Brouwer fixed point theorem for the square (cf. Šaškin [5] and Gale [1]). It can also be used as the lemma for the mountain climbing theorem of Homma [2] (see Mioduszewski [3]). In this paper the Steinhaus theorem is used in the proof of a discrete analogue of the Jordan curve theorem (see Stout [8], where different proof is stated; cf. also Rosenfeld [4]).

1. Preliminaries

Let $m\times n = \{0,1,\ldots,m-1\}\times\{0,1,\ldots,n-1\}$ denote the lattice of natural numbers, where $m,n \geq 1$. By the *i-row*, where $0 \leq i \leq n-1$, we shall mean the set $\{0,1,\ldots,m-1\} \times \{i\}$; by the *j-column*, where $0 \leq j \leq m-1$, we shall mean the set $\{j\} \times \{0,1,\ldots,n-1\}$. The *boundary* $\mathrm{Bd}(m \times n)$ of the lattice is the union of all terminal rows and columns.

Two elements of the lattice will be called *8-adjacent*, if they are different and their coordinates differ at most by 1. Two elements of the lattice will be called *4-adjacent*, if they differ on only one coordinate by 1. A sequence $a_1, a_2, \ldots, a_k$ of elements of the lattice will be called a *path from A to B*, where $A, B \subset m \times n$, if $a_1 \in A, a_k \in B$, and a_j and a_{j+1} are adjacent for

Received December 16, 1991.

AMS (1991) subject classification: Primary 52C99. Secondary 54F50, 55M20.

Supported by Polish scientific grant RP.I.10

each j. There are two kinds of paths, the *8-path*, if the adjacement means 8-adjacement, and the *4-path*, if the adjacement means 4-adjacement. A path P from A to B will be called *minimal*, if any proper subset of the set of elements of P is not a path from A to B. We can identify a minimal path with the set of its elements.

For both 8-paths and 4-paths the following property holds: $(*)$ *assume that the first element of a path P belongs to the i-column of the lattice and the last element of P belongs to the j-column, where $i < j$. Then in each 8-column, where $i \leq k \leq j$, lies at least one element of P.* The analogous property holds for rows.

A subset A of the lattice will be called *connected*, if any two elements of A can be joined by a path in A. Depending on what kind the path is, the set A will be called either *4-connected* or *8-connected*. Let B be a subset of the lattice. Each maximal connected subset of B will be called a *component* of B, and similarly we have *4-components* and *8-components*.

Any function $f\colon m \times n \to \{\text{white}, \text{black}\}$ will be called a *colouring* of the lattice. We can say of *black path* in a certain colouring, if each element of that path is black in this colouring; a *white path* is a path with white elements.

Let a and b be two different elements of the boundary of the lattice. The set $\mathrm{Bd}(m \times n) \setminus \{a, b\}$ has at most two 4-components. While moving clockwise on the boundary from a to b, we meet the elements of only one of these components; the union of that component and the set $\{a, b\}$ we shall call an *arc* $\widehat{ab}$. By an arc $\widehat{aa}$ we shall mean the set $\{a\}$.

The quadruple of given elements on the boundary of the lattice a, b, c, d will be called *ordered*, if $c, d \notin \widehat{ab}$ and $d \notin \widehat{bc}$ (this corresponds to the clockwise ordering of a, b, c, d).

2. Steinhaus chessboard theorem

CHESSBOARD THEOREM. *Let a, b, c, d be the ordered quadruple of elements on the boundary of the lattice endowed with the colouring S. The existence of a black 8-path from $\widehat{ab}$ to $\widehat{cd}$ is equivalent to the non-existence of a white 4-path from $\widehat{bc}$ to $\widehat{da}$.*

The Steinhaus theorem is the chessboard theorem in which a, b, c, d are the corner elements. We shall prove the chessboard theorem indirectly, by reducing it to some special case:

REDUCED CHESSBOARD THEOREM. *Let a and b be different black elements on the boundary of the lattice endowed with the colouring S. The existence of a black 8-path from a to b is equivalent to the non-existence of a white 4-path from $\widehat{ab}$ to $\widehat{ba}$.*

To infer the chessboard theorem from the reduced chessboard theorem it is enough to enlarge the lattice $m \times n$ to the lattice $(m+2) \times (n+2)$ by adding the frame around $m \times n$, colour the frame elements 4-adjacent to the arcs $\widehat{ab}$ and $\widehat{cd}$ black and the rest white, and extend a 8-path from $\widehat{ab}$ to $\widehat{cd}$ to the frame.

Proof of the reduced chessboard theorem. For $n = 2$ the theorem is the direct consequence of the property $(*)$ and the fact that $\mathrm{Bd}(m \times 2) = m \times 2$. We shall prove that *the existence of a black 8-path from a to b implies the non-existence of a white 4-path from $\widehat{ab}$ to $\widehat{ba}$.* Restrict the colouring S to $m \times (n-1)$. A black 8-path from a to b in S determines black 8-paths in $S|_{m\times(n-1)}$ such that arcs corresponding to them are subsets of $\widehat{ab}$ or $\widehat{ba}$, and both $\widehat{ab}$ and $\widehat{ba}$ are sums of these arcs and arcs that are black. Thus the existence of a 4-path from $\widehat{ab}$ to $\widehat{ba}$ in S contradicts the inductive hypothesis for some black 8-path in $m \times (n-1)$.

Now we shall prove that *the non-existence of a white 4-path from $\widehat{ab}$ to $\widehat{ba}$ implies the existence of a black 8-path from a to b.* Let us consider the special case: suppose that a and b belong to the 0-row and that the first coordinate of a is less than the first coordinate of b. Denote by D the subset of the $(n-2)$-row containing each element x such that a white 4-path from x to $\widehat{ba}$ exists. We can define the colouring T of the lattice $m \times (n-1)$ equal to S on $m \times (n-1) \setminus D$ and black on D. Thus we get the non-existence of a white 4-path from $\widehat{ab}$ to $\widehat{ba}$ in T. From the inductive hypothesis it follows the existence of the minimal black 8-path P from a to b in T. If P and D are disjoint, then P is the black path in S too. Suppose that P and D are not disjoint, and denote by c and d the elements of $(n-1)$-row not belonging to D and bounding D from the left and from the right, and by e the first element of P belonging to D. We shall construct the path from a to b in S of three segments: from a to c, from c to d and from d to b.

To construct the path from a to c (and, similarly, from d to b) we have to construct the path Q from e to c. The existence of Q follows from the inductive hypothesis applied to the elements c and e of the lattice $m \times (n-1)$ with black e. The assertions can be obtained from the first implication of the proving theorem. Then concatenate the initial segment of P from a to e with Q and remove e.

The existence of the path from c to d can be easily proved by induction on number of components of D. For one component the needed path can be composed of c, then all elements of $(n-1)$-row 4-connected to D and then d. For more components the existence of the path connecting the components' bounding elements follows from the inductive hypothesis.

Now suppose that a belongs to the $(n-1)$-row, and b to the $(m-1)$-column. Add the m-column to the lattice and colour the elements from

$(m, n-1)$ up to the one 4-adjacent to b black and rest of them white. From the proof of the first case we get the existence of black path from a to the element of the $(m-1)$-column with second coordinate greater than the one of b. Second applying of the first case gives us the path from this element to b. The case when a belongs to the 0-row and b belongs to the $(n-1)$-row can be proved similarly, by adding n-row to the lattice. □

3. Jordan Curve Theorem

Firstly we shall prove the

LEMMA. *Let $Q \subset m \times n$ be a minimal 8-path from a to b, where a and b are different and not 4-adjacent elements of* $\mathrm{Bd}(m \times n)$. *Assume that a and b are the only elements on the boundary belonging to Q. The complement of Q in $m \times n$ has exactly two 4-components such that each element of Q is 4-adjacent to both these components.*

PROOF. Define the colouring S of the lattice $m \times n$ in which the elements of Q are black and all other — white. From the reduced chessboard theorem it follows that the sets $\widehat{ab} \setminus \{a, b\}$ and $\widehat{ba} \setminus \{a, b\}$ lie in different 4-components of the complement of Q. Now we must prove that for each element c of the complement of Q there exists the white 4-path from c to the boundary of the lattice. Move from c up (i.e. with fixed first coordinate and the second one increasing) on the white elements of the lattice. The white 4-path that we move on can end at the boundary or at the path Q. So we must show that for each element q of the path Q and for each white and 4-adjacent to q element d there exists the white 4-path from d to the boundary of the lattice. Fix such an element q and define the new colouring S' equal to S with one exception: q is white in S'. Since Q is minimal, there does not exist the black 8-path from a to b in S'. The reduced chessboard theorem implies the existence of a white 4-path from $\widehat{ab}$ to $\widehat{ba}$ in S'. All white 4-adjacent to q elements of $m \times n$ can be linked to one of the parts, into which this path is dissected by q. Moreover, among 4-adjacent to q elements of the lattice there exist elements of both components of the complement of Q, what provides us with the second part of the lemma. □

A *minimal circular path* is a path $\{p_1, p_2, \ldots, p_k\}$, in which p_1 and p_k are adjacent and such that after the removal of any element from this path, the remaining elements form a minimal path from the successor to the predecessor of the removed element.

JORDAN CURVE THEOREM. *Let P be a minimal circular 8-path contained in $m \times n$ and disjoint with* $\mathrm{Bd}(m \times n)$. *The complement of P in $m \times n$ has exactly two 4-components such that each element of P is 4-adjacent to both these components.*

PROOF. We can assume that in the $(n-2)$-row there exists an element c of the path P. Since P is minimal, the predecessor and the successor of c in P belong to the $(\gamma-1)$- and the $(\gamma+1)$-column respectively, where γ is the first coordinate of c. Remove from P the element c and add to P the elements $a=(\gamma-1,n-1), b=(\gamma+1,n-1), (\gamma-1,n-2)$ and $(\gamma+1,n-2)$. We get a minimal 8-path Q from a to b, a and b being boundary elements of $m\times n$. It follows from lemma that the complement of Q has exactly two 4-components. Denote the component containing c by A and the other one by B. The set B is contained in the complement of P, and after adding to B the elements of $Q\setminus P$ and the element $(\gamma,n-1)$ we get the set B', still 4-connected and contained in the complement of P. Similarly, after removing the elements $(\gamma,n-1)$ and c from A we get the set A', still 4-connected and contained in the complement of P. Notice that $m\times n\setminus P=A'\cup B'$. It remains to show that elements of P are 4-adjacent to both A' and B'. For the elements of $P\cap Q$ this follows from the lemma. For remaining elements of P this is obvious in view of the construction of A' and B'. □

Acknowledgment. The author wants to thank dr Wojciech Dębski for valuable help.

REFERENCES

[1] D. Gale, The Game of Hex and the Brouwer Fixed-Point Theorem, Amer. Math. Monthly **86** (1979), 818-827

[2] T. Homma, A theorem on continuous functions, Kodai Math., Sem. Reports **1** (1952), 13-16

[3] J. Mioduszewski, The two climbers, Delta **101** (1982), 14-15 (in Polish)

[4] A. Rosenfeld, Digital Topology, Amer. Math. Monthly **86** (1979), 621-630

[5] U. A. Šaškin, Fixed points, Nauka, Moscow 1989 (in Russian)

[6] H. Steinhaus, Mathematical Snapshot, WSiP Warszawa 1989 (in Polish)

[7] H. Steinhaus, Problems and Essays, Mir, Moscow 1974 (in Russian)

[8] L. N. Stout, Two Discrete Forms of the Jordan Curve Theorem, Amer. Math. Monthly **95** (1988), 332-336

UNIWERSYTET ŚLĄSKI
INSTYTUT MATEMATYKI
UL. BANKOWA 14
40-007 KATOWICE

E-mail: surowka@gate.math.us.edu.pl