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ONE–ONE AND ONE–ONE ONTO CHOICE FUNCTIONS

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Abstract. The existence of one–one and one–one onto choice functions on a family of subsets of a given nonempty set is studied.

Let $\mathcal{T}$ be a nonempty family of nonempty sets. The aim of this paper is to investigate the supplemental conditions imposed on $\mathcal{T}$ and the elements of $\mathcal{T}$ which, together with the Axiom of Choice (or one of its equivalents) imply the existence of an one-one choice function $\psi: \mathcal{T} \to \bigcup \mathcal{T}$, or stronger, the existence of an one-one choice function from $\mathcal{T}$ onto $\bigcup \mathcal{T}$.

It will be assumed throughout all the paper that the Axiom of Choice (and consequently each of its equivalents) proves true. Moreover, a given sequence (also a transfinite sequence) of sets or elements of a set will be often identified with its set of values.

THEOREM 1. *Let $\mathbb{Y} = \{Y_x : x \in X\}$ be a family of nonempty subsets of some infinite set X. If card $Y_x =$ card X, $x \in X$, then there exists an one–one choice function on $\mathbb{Y}$.*

PROOF. It is sufficient to prove that there exist pairwise disjoint sets $Z_x \subseteq Y_x$, $x \in X$, such that card $Z_x =$ card $X, x \in X$. To do this let us denote by $<$ a minimal order on the set X, i.e. a well order on X satisfying the following condition:

$$(\forall x \in X)(I(x) := \{y :\ y \in X \quad \text{and} \quad y < x\} \Rightarrow \text{ card } I(x) < \text{ card} X).$$

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Let 0 be the minimal element of the ordered set $(X, <)$. By transfinite induction we construct an one–one sequence $\{y_{u,v} : u, v \in X\}$ of elements of X such that

$$Z_u := \{y_{u,v} : v \in X\} \subseteq Y_u$$

for every $u \in X$.

Let $y_{0,0}$ be the minimal element of Y_0. Assume now that for some $w \in X$, the one–one sequence $S_w = \{y_{u,v} : u, v \in I(w)\}$ has been already defined. Moreover, assume that $\{y_{u,v} : v \in I(w)\} \subset Y_u$, for every $u \in I(w)$. By transfinite induction with respect to $u \in I(w)$ we will construct the elements $y_{u,w}$, with $u \in I(w)$. First note that

$$\text{card } S_w = \text{ card } (I(w) \times I(w)) = \text{ card } I(w) < \text{ card } X$$

whenever $I(w)$ is infinite, and therefore $Y_0 \setminus S_w \neq \emptyset$. Let $y_{0,w}$ denote the minimal element of the set $Y_0 \setminus S_w$. If for some $t < w$ the elements $y_{u,w}$ with $u < t$ are already defined then

$$\text{card } (S_w \cup \{y_{u,w} : u < t\}) \leq \text{ card } I(w) < \text{ card} X.$$

Hence $Y_t \setminus (S_w \cup \{y_{u,w} : u < t\}) \neq \emptyset$ and we define $y_{t,w}$ to be the minimal element of this set. Thus, all elements $y_{u,w}$ with $u \in I(w)$ have been already defined.

In a similar way as above we can define the elements $y_{w,u} \in Y_w$, $u \in I(w+1)$. Consequently, we may assume that the one–one sequence $\{y_{u,v} : u, v \in I(w+1)\}$ such that $\{y_{u,v} : v \in I(w+1)\} \subset Y_u$ for every $u \in I(w+1)$ is constructed.

Finally, by the principle of transfinite induction, the elements $y_{u,v}$ are defined for all $u, v \in X$. □

COROLLARY 1.1. *Let $\mathcal{T}$ be a nonempty family of subsets of some infinite set X. Suppose that the following two conditions are fulfilled:*
(1) card $\mathcal{T} \leq$ card X,
(2) $(\forall Y)$ $(Y \in \mathcal{T} \Rightarrow$ card $(X \setminus Y) <$ card $X)$.
Then there exists an one–one choice function on $\mathcal{T}$.

An alternative (and more compact) proof of Corollary 1.1. will be given below.

PROOF OF COROLLARY 1.1. Let f be an arbitrary one–one function from $X \times X$ onto X. Define $A_x = f(\{x\} \times X)$, $x \in X$. Then the family $\{A_x : x \in X\}$ is a partition of X and card $A_x =$ card X, $x \in X$. Hence, combining (1) with the equality $X \setminus Y = \bigcup_{x \in X} (A_x \setminus Y)$, $Y \in \mathcal{T}$, we can assert that the families $B_Y := \{A_x : x \in X \text{ and } A_x \subset Y\}$, $Y \in \mathcal{T}$, are nonempty and, more precisely, that card $B_Y =$ card X for every $Y \in \mathcal{T}$.

Let F be a choice function on the family $\{B_Y : Y \in \mathcal{T}\}$ and let ϱ be the equivalence relation on $\mathcal{T}$ defined as follows: $Y \varrho Z \Leftrightarrow F(B_Y) = F(B_Z)$. We see that for every $\mathcal{E} \in \mathcal{T}/\varrho$ the following inclusion is fulfilled:

$$F(B_Y) \subset \bigcap \{Z : Z \in \mathcal{E}\}, \quad Y \in \mathcal{E}.$$

Fix an choice function G on $\mathcal{T}/\varrho$. Then we have

$$(\forall \mathcal{D} \in \mathcal{T}/\varrho)\ (\forall \mathcal{E} \in \mathcal{T}/\varrho)\ (\mathcal{D} \neq \mathcal{E} \Rightarrow F(B_{G(\mathcal{D})}) \cap F(B_{G(\mathcal{E})}) = \emptyset).$$

Applying (2) we get card $\mathcal{E} \leq$ card $F(B_{G(\mathcal{E})})$ for every $\mathcal{E} \in \mathcal{T}/\varrho$. Therefore for every $\mathcal{E} \in \mathcal{T}/\varrho$ there exists an one–one function $g_{\mathcal{E}} : \mathcal{E} \to F(B_{G(\mathcal{E})})$. We are now in a position to define an one–one choice function $g : \mathcal{T} \to X$, by $g(Y) = g_{\mathcal{E}}(Y)$ for every $Y \in \mathcal{E}$ and $\mathcal{E} \in \mathcal{T}/\varrho$. □

THEOREM 2. *Let $\{X_\beta\}_{\beta<\alpha}$ be a transfinite sequence of sets satisfying the following two conditions:*
(1) $(\forall \beta)$ $(\beta < \alpha \Rightarrow$ *card* $X_\beta >$ *card* $\beta)$,
(2) $(\forall \beta, \delta)$ $(\beta < \delta < \alpha \Rightarrow X_\beta \subseteq X_\delta)$.
Then there exists an one-one choice function on $\{X_\beta\}_{\beta<\alpha}$.

PROOF. By the transfinite induction, we deduce that there exists an ordinal number $\widetilde{\alpha} \leq \alpha$ and an increasing function $f : \mathcal{Z}(\widetilde{\alpha}) \to \mathcal{Z}(\alpha)$, where $\mathcal{Z}(\beta)$ denotes the set of all ordinal numbers smaller then β for every ordinal number β, such that
(3) the sets $\mathcal{Z}(\widetilde{\alpha})$ and $f(\mathcal{Z}(\widetilde{\alpha}))$ are similar, $f(0) = 0$ and $f(\widetilde{\alpha}) = \alpha$,
(4) for every isolated ordinal $\beta < \widetilde{\alpha}$, $\beta = \gamma + 1$, we have

$$\text{card}\left(X_{f(\beta)} \setminus X_{f(\gamma)}\right) = \text{card}\ X_{f(\beta)}$$

and

$$(\forall \delta)\left(\delta \in \mathcal{Z}(\alpha) \text{ and } f(\gamma) < \delta < f(\beta) \Rightarrow \text{ card}\left(X_{f(\beta)} \setminus X_\delta\right) < \text{card} X_{f(\beta)}\right),$$

(5) for every limit ordinal number $\beta < \widetilde{\alpha}$ two following conditions are satisfied:

$$(\forall \gamma)\ \left(\gamma < \widetilde{\alpha} \text{ and } \gamma < \beta \Rightarrow \text{ card } \left(X_{f(\beta)} \setminus X_{f(\gamma)}\right) = \text{ card } X_{f(\beta)}\right)$$

and if $\delta \in \mathcal{Z}(f(\beta))$ and $\delta > f(\gamma)$ for every $\gamma < \beta$, then

$$\text{card}\ \left(X_{f(\beta)} \setminus X_\delta\right) < \text{card}\ X_{f(\beta)}.$$

An one–one choice function $g : \{X_\beta\}_{\beta<\alpha} \to \bigcup_{\beta<\alpha} X_\beta$ will be constructed below by the transfinite induction with respect on $\gamma \in \mathcal{Z}(\widetilde{\alpha})$.

By (1), an one-one choice function g on all sets X_β, with β finite, could be easily defined. In the sequel, g could be defined on all finite sets X_β. Assume now that g is an one-one choice function defined on all X_β with $\beta \leq f(\gamma)$ for some $\gamma < \tilde{\gamma}$, where $\tilde{\gamma} \in \mathcal{Z}(\tilde{\alpha})$ is arbitrary fixed. There is no loss of generality in assuming that the cardinality of $f(\tilde{\gamma})$ is infinite. If $\tilde{\gamma}$ is an isolated ordinal number then $\tilde{\gamma} = \gamma + 1$ for some $\gamma < \tilde{\alpha}$. By (1) we obtain

$$\text{card } X_{f(\gamma)} > \text{ card } Im(g),$$

where $Im(g)$ denotes the image of g. Hence, by (4), we deduce that

$$\begin{aligned}(\forall\delta)\,(\delta \in \mathcal{Z}(\alpha) \quad \text{and} \quad & f(\gamma) < \delta < f(\tilde{\gamma}) \Rightarrow \\ & \text{card}\left[(X_{f(\tilde{\gamma})} \setminus \text{ Im}(g)) \setminus (X_\delta \setminus \text{ Im}(g))\right] \\ & < \text{ card}\left[X_{f(\tilde{\gamma})} \setminus \text{ Im}(g)]\right).\end{aligned}$$

Therefore, if the set $(X_{f(\tilde{\gamma})} \setminus \text{Im}(g))$ is infinite then, in virtue of Corollary 1.1, there exists an one-one choice function on the family $\{X_\delta \setminus \text{ Im}(g) : f(\gamma) < \delta < f(\tilde{\gamma})\}$. Thus the definition of g can be extended on the sets X_δ with $\delta \in \mathcal{Z}(\alpha)$ and $f(\gamma) < \delta < f(\tilde{\gamma})$. In the case when the set $(X_{f(\tilde{\gamma})} \setminus \text{ Im}(g))$ is finite then by the equality

$$\text{card}(X_{f(\tilde{\gamma})} \setminus \text{ Im}(g)) = \text{ card } X_{f(\tilde{\gamma})}$$

we obtain that the set $X_{f(\tilde{\gamma})}$ is also finite, which was excluded above. If $\tilde{\gamma}$ is a limit ordinal number then applying the condition (5) we can extand the definition of g on the sets X_δ with $\delta > f(\gamma)$ for every $\gamma < \tilde{\gamma}$ and $\delta \leq f(\tilde{\gamma})$. □

LEMMA 3. *Let* $\{Y_x : x \in (0,1]\}$ *be a family of sets such that card* $Y_x \geq \mathfrak{c}$, $x \in (0,1]$ *and* $Y_x \subseteq Y_z$ *for* $x, z \in (0,1]$, $x \leq z$. *Then there exists an one-one choice function*

$$\psi : \{Y_x : x \in (0,1]\} \to Y_1.$$

PROOF. First assume that card $\bigcap_{0<x\leq 1} Y_x \geq \mathfrak{c}$. Let g be an one-one function from the interval (0,1] into the set $\bigcap_{0<x\leq 1} Y_x$. Then the mapping $Y_x \mapsto g(x)$, $x \in (0,1]$, is the desired choice function.

In the case when card $\bigcap_{0<x\leq 1} Y_x < \mathfrak{c}$, it is easy to show that the following condition is sastisfied:

(1) $\quad (\forall x \in (0,1))\,(\exists z \in (0,1))\ (z < x \quad \text{and} \quad \text{card}\,(Y_x \setminus Y_z) \geq \mathfrak{c}).$

Indeed, if there exists $x \in (0,1)$ such that for any $z \in (0,1)$ we have card $(Y_x \setminus Y_z) < \mathfrak{c}$ then

$$\bigcap_{0<z\leq x} Y_z = Y_x \setminus \left(\bigcup_{0<z\leq x} (Y_x \setminus Y_z) \right) = Y_x \setminus \left(\bigcup_{n=1}^{\infty} (Y_x \setminus Y_{x/n}) \right).$$

Hence, it follows immediately that card $\bigcap_{0<z\leq x} Y_z \geq \mathfrak{c}$, which contradicts our assumption. By (1), we can choose a decreasing sequence $\{x(n) : n \in \mathbb{N}\} \subset (0,1)$ convergent to zero such that card $(Y_{x(n)} \setminus Y_{x(n+1)}) \geq \mathfrak{c}$ for every $n \in \mathbb{N}$. Let g_n denotes an one-one function which maps the interval $(x(n),\ x(n-1)]$ into the set $Y_{x(n)} \setminus Y_{x(n+1)}$, $n \in \mathbb{N}$, where $x(0) = 1$. Then the function $g(Y_x) := g_n(x)$ for $x \in (x(n),\ x(n-1)]$, $n \in \mathbb{N}$, is the desired choice function. □

THEOREM 4. *Let $\{Y_x : x \in (0,1)\}$ be a family of subsets of the interval (0,1) and let the following condition be fulfilled. There exists a sequence $\{X_n : n \in \mathbb{N}\}$ of mutually disjoint nonempty subintervals of the unit interval such that for every $x \in (0,1)$ there exists an index $n(x) \in \mathbb{N}$ such that $X_{n(x)} \subseteq Y_x$. Then we are able to define an one-one choice function $\psi : \{Y_x : x \in (0,1)\} \to \bigcup_{n\in\mathbb{N}} X_n$ effectively.*

PROOF. It is well known that we can construct an one-one function ϕ from (0,1) onto a Sierpiński's family $\mathcal{S}$ of increasing sequences of positive integers with almost disjoint sets of values effectively. From now on we assume that the family $\{Y_x : x \in (0,1)\}$ is indexed by elements $s \in \mathcal{S}$ i.e. we have $\{Y_x : x \in (0,1)\} = \{Y_s : s \in \mathcal{S}\}$.

For every $s \in \mathcal{S}$ we denote by $k(s)$ the minimal positive integer such that $X_{k(s)} \subseteq Y_s$. Moreover, let $(a_n, b_n) = \operatorname{int} X_n$, $n \in \mathbb{N}$, and let the functions $f_n : (0,1) \to (a_n, b_n)$, $n \in \mathbb{N}$, be defined as follows $f_n(x) = (b_n - a_n)x + a_n$, $x \in (0,1)$. Then a trivial verification shows that the following mapping is the desired choice function:

$$Y_s \mapsto f_{k(s)} \left(\sum_{t=1}^{\infty} 2^{-L(s,t)} \right)$$

where $s \in \mathcal{S}$, $s = \{s_i : i \in \mathbb{N}\}$, $L(s,t) = \sum_{i=1}^{t} l_i$, $t \in \mathbb{N}$, and the sequence $l = \{l_i : i \in \mathbb{N}\}$ is defined in the following way: $l_{s_i} = 2i - 1$, $i \in \mathbb{N}$, and the restriction to the complement of the set s to $\mathbb{N}$ of l is the increasing sequence of all even positive integers. □

LEMMA 5. *Let $\mathcal{H}$ be a Hilbert space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $\{\mathcal{H}_\alpha : \alpha \in A\}$ be a family of nonempty, closed and convex subsets of $\mathcal{H}$. Then, by the countable Axiom of Choice, there exists a choice function*

$$\psi : \{\mathcal{H}_\alpha : \alpha \in A\} \to \bigcup_{\alpha \in A} \mathcal{H}_\alpha .$$

Moreover, if one of the following two conditions is satisfied: either the sets $\mathcal{H}_\alpha$, $\alpha \in A$, are mutually disjoint, or $i_\alpha \neq i_\beta$ for any two $\alpha, \beta \in A$, $\alpha \neq \beta$, where $i_\alpha = \inf\{\| h \| : h \in \mathcal{H}_\alpha\}$, then we can additionally assume that ψ is one-one.

PROOF. It is sufficient to note that, in virtue of the countable Axiom of Choice, each set $\mathcal{H}_\alpha, \alpha \in A$, contains precisely one element with minimal norm (cf [3], Theorem 2.3.1). □

REMARK. Since for every $z \in \mathcal{H}$ and for every $\alpha \in A$ there exists precisely one element $z_\alpha \in \mathcal{H}_\alpha$ such that $\| z - z_\alpha \| = \text{dist}\,(z, \mathcal{H}_\alpha)$, we may suppose that there exist, in general case, many different choice functions discussed in the above Lemma.

THEOREM 6. *Let $\mathcal{T}$ be a family of subsets of some infinite set X such that*

$$(1) \qquad (\forall x)\,(x \in X \Rightarrow \text{ card}\,\{Y : Y \in \mathcal{T} \text{ and } x \in Y\} = \text{ card}\,X).$$

Then there exists a choice function ψ from $\mathcal{T}$ onto X.
Additionally, if the following two conditions are fulfilled:

$$(2) \qquad \text{card}\,\mathcal{T} = \text{ card}\,X,$$

and

$$(3) \qquad (\forall Y)\,(Y \in \mathcal{T} \Rightarrow \text{ card}\,Y = \text{ card}\,X),$$

then there exists an one-one onto choice function $\psi : \mathcal{T} \to X$.

PROOF. Suppose that δ and δ' are minimal ordinal numbers such that card $\delta =$ card X and card $\delta' =$ card $\mathcal{T}$. Let $\{x_\alpha : \alpha < \delta\}$ and $\{Y_\alpha : \alpha < \delta'\}$ be two one-one transfinite sequences containing all elements of the sets X and $\mathcal{T}$, respectively. Define, by transfinite induction, an one-one function g from $\{\alpha : \alpha < \delta\}$ into $\{\alpha : \alpha < \delta'\}$ as follows:

$$g(0) = \min\{\alpha : \alpha < \delta' \text{ and } x_0 \in Y_\alpha\},$$

$$g(\beta) = \min\{\alpha : \alpha < \delta' \text{ and } \alpha \notin \{g(\gamma) : \gamma < \beta\} \text{ and } x_\beta \in Y_\alpha\}$$

for every $\beta < \delta$. We note that by (1) this definition is correct. Obviously, then g is one-one and $x_\beta \in Y_{g(\beta)}$ for every $\beta < \delta$.

Now we prove that if the conditions (2) and (3) hold true then g is onto $\{\alpha : \alpha < \delta\}$, because then $\delta' = \delta$. To this aim assume that there exists $\beta < \delta$ such that $\beta \notin g(\{\alpha : \alpha < \delta\})$. It follows from (3) that card $g(\{\alpha : \alpha < \delta$ and $x_\alpha \in Y_\beta\}) =$ card X. On the other hand, by the definition of g, the following implication holds true:

$$(\forall\alpha)\ (\alpha < \delta \text{ and } x_\alpha \in Y_\beta \Rightarrow \text{ either } g(\alpha) < \beta \text{ or } \beta \in \{g(\gamma) : \gamma < \alpha\}).$$

The condition $\beta \in \{g(\nu) : \nu < \alpha\}$ is excluded from our discussion. Therefore card $g(\{\alpha : \alpha < \delta$ and $x_\alpha \in Y_\beta\}) \leq$ card β which contradicts the inequality card $\beta <$ card X. Accordingly, g is onto $\{\alpha : \alpha < \delta\}$ as claimed. Then it follows immediately that $h(Y_{g(\beta)}) := x_\beta$, $\beta < \delta$, is the desired choice function on $\mathcal{T}$.

In the case when only the condition (1) is satisfied and the set

$$\mathcal{T}' := (\mathcal{T} \setminus \{Y_{g(\beta)} : \beta < \delta\})$$

is nonempty, then h should be extended arbitrarily onto $\mathcal{T}'$. □

COROLLARY 6.1. *Let X be an infinite set and let*

$$\mathcal{T} := \{Y : Y \subset X \quad \text{and} \quad \text{card}\,(X \setminus Y) < \text{ card}\, X\}.$$

If card $\mathcal{T}$ = card X then there exists an one-one choice function from $\mathcal{T}$ onto X.

COROLLARY 6.2. *Let $\mathcal{T}$ be the family of all open intervals of the real line. Then there exists an one-one onto choice function $f : \mathcal{T} \to \mathbb{R}$.*

Let ρ be the relation on $\mathbb{R}$ defined as follows: $(\forall x, y \in \mathbb{R})\ (x\rho y \Leftrightarrow x - y \in Q)$. The relation ρ is obviously an equivalence. We denote by $\mathbb{R}/\rho$ the corresponding family of equivalence classes. Our next result is a consequence of the existence of a choice function $\psi : \mathbb{R}/\rho \to \mathbb{R}$.

LEMMA 7. *Let $\{Q_n\}_{n=1}^{\infty}$ be a sequence of nontrivial intervals of $\mathbb{R}$ satisfying the following conditions:*
(1) $(\forall i)\ (i \in \mathbb{N}$ *and* $i < n \Rightarrow Q_i \cap Q_n \neq Q_i)$,
and
(2) for every real number x there exist infinite many intervals Q_i which contain x.
If $\mathcal{F}_n$, $n \in \mathbb{N}$, are nonempty families of subsets of $\mathbb{R}$ such that
(3) $(\forall n, m)\ (n, m \in \mathbb{N}$ *and* $n \neq m \Rightarrow \mathcal{F}_n \cap \mathcal{F}_m = \emptyset)$
and

(4) $(\forall n)\ (\forall F)\ (n \in \mathbb{N}$ *and* $F \in \mathcal{F}_n \Rightarrow Q_n \subset F)$
then there exists an one-one choice function $\psi : \bigcup_{n\in\mathbb{N}} \mathcal{F}_n \to \mathbb{R}$.

• *If, additionally, the following condition is fulfilled:*
(5) $(\forall n)\ (n \in \mathbb{N} \Rightarrow \text{card}\ \mathcal{F}_n = \mathfrak{c})$
then there exists an one-one onto choice function $\psi : \bigcup_{n\in\mathbb{N}} \mathcal{F}_n \to \mathbb{R}$.

PROOF. For every $E \in \mathbb{R}/\rho$ we denote by $\{x_n(E)\}_{n=1}^{\infty}$ an one-one sequence of all elements of the set E. More precisely, if $\{q_n\}_{n=1}^{\infty}$ is an one-one sequence of all rational numbers and $\psi : \mathbb{R}/\rho \to \mathbb{R}$ is a choice function then we could set $x_n(E) = \psi(E) + q_n$ for every $n \in \mathbb{N}$.

Now, by induction on n, we define auxiliary sequences of indices

$$\{s(n, E)\}_{n=1}^{\infty},\ E \in \mathbb{R}/\rho.$$

First, we define $s(1, E)$ to be the smallest positive integer with the property that $x_{s(1,E)} \in Q_1$ for every $E \in \mathbb{R}/\rho$. Let for some $n \in \mathbb{N}$, $n \geq 2$, the indices $s(i, E)$, $i = 1, \ldots, n-1$, $E \in \mathbb{R}/\rho$, are already defined. Then for every $E \in \mathbb{R}/\rho$ the index $s(n, E)$ is the smallest positive integer satisfying: $s(n, E) \neq s(i, E)$ for every $i = 1, \ldots, n-1$, and $x_{s(n,E)} \in Q_n$. We note that, by (2), the following implication holds true:

$$(\forall E)\ (E \in \mathbb{R}/\rho \Rightarrow \{s(n, E) :\ n \in \mathbb{N}\} = \mathbb{N}).$$

Let f_n denotes an one-one function from $\mathcal{F}_n$ into (onto, resp., whenever the condition (5) holds true) the set $\{x_{s(n,E)} :\ E \in \mathbb{R}/\rho\}$ for every $n \in \mathbb{N}$. Then the function $f(T) := f_n(T)$ for $T \in \mathcal{F}_n$ and $n \in \mathbb{N}$, is the desired choice function. □

REMARK 7.1. Let $\{Q_n\}_{n=1}^{\infty}$ be the sequence built from all elements of the succesive finite sequences of the following intervals of $\mathbb{R}$:

$$\left((i - 3^{-1})2^{-n},\ (i + 1 + 3^{-1})2^{-n}\right),\quad -n2^n \leq i < n2^n,$$

for every $n \in \mathbb{N}$.

Then $[-n, n] \subset \bigcup_{i=k(n)}^{k(n+1)} Q_i$, for every $n \in \mathbb{N}$, where $k(n) := n(n+1)$, $n \in \mathbb{N}$. Hence the condition (2) of Lemma 7 is fulfilled. It can be readily checked that the condition (1) of Lemma 7 is also satisfied.

COROLLARY 7.1. *Note that Corollary 6.2 also follows from Lemma 7. For the proof of this fact it is sufficient to define families* $\mathcal{F}_n$, $n \in \mathbb{N}$, *in the following way. Let* $\mathcal{F}_1$ *be the set of all intervals* $(a, b) \subseteq \mathbb{R}$ *such that* $Q_1 \subseteq (a, b)$. *If for some* $n \in \mathbb{N}$, $n \geq 2$, *the pairwise disjoint sets* $\mathcal{F}_1, \ldots, \mathcal{F}_{n-1}$ *are defined, then we set* $\mathcal{F}_n$ *to be the family of all intervals* $(a, b) \subseteq \mathbb{R}$ *such that* $Q_n \subseteq (a, b) \notin \bigcup_{i=1}^{n-1} \mathcal{F}_i$. *We note that by (1) the condition (5) is satisfied.*

COROLLARY 7.2. *Let us define by $\mathcal{T}$ the family of all open intervals (a, b) of the real line such that $(b - a) \in Q$. Then there exists an one-one onto choice function $\phi : \mathcal{T} \to \mathbb{R}$. We note that then $\mathcal{T}$ is equal to the family of all intervals $I(q, \psi(E))$ with endpoints: q and $q + \psi(E)$ for $q \in Q, E \in \mathbb{R}/\rho$, where ψ and ρ are defined in the same way as in the proof of Lemma 7. Hence we may put $\phi(I(q, \psi(E))) = \frac{1}{2}q + \psi(E)$.*

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