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ON A BOUNDARY VALUE PROBLEM

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Abstract. The equation $x''(t) = f(t, x(\alpha(t)), x'(\beta(t)))$ for $t \in [a, b]$, where the functions α, β deviated argument of type $[a, b] \longrightarrow [a, b]$ is considered.

A sufficient condition for existence of the end b of the interval $[a, b]$, such that there exists the solution x of the above equation on $[a, b]$ fulfilling the boundary value conditions $x(a) = A$, $x(b) = B$ and $||x'(a)|| = v > 0$, where the constants a, v and vectors A, B are given, is proved.

Let $D := [a, b]$ be an interval and $d := b - a$ denote length of this interval. Let the symbol $\|\cdot\|$ denote a norm in the space $\mathbb{R}^n$.

Consider a system of ordinary differential equations of the second order with a deviating argument of the form

$$x''(t) = f(t, x(\alpha(t)), x'(\beta(t))), \qquad t \in D. \tag{1}$$

Let us denote $D_1 := D \times \mathbb{R}^n \times \mathbb{R}^n$. We assume that the function $f : D_1 \to \mathbb{R}^n$ is a continuous real function and fulfils the Lipschitz condition of the form

$$\| f(t, x, y) - f(t, \bar{x}, \bar{y}) \| \leq p \| x - \bar{x} \| + q \| y - \bar{y} \| \quad \text{for} \quad t \in D, \tag{2}$$

where $p, q \geq 0$ are constants. The function f is bounded on the domain D_1, i.e.

$$\| f(\cdot, \cdot, \cdot) \| \leq K. \tag{3}$$

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$\alpha, \beta : D \to D$, functions of deviation of the argument, are continuous and $a \le \alpha(t) \le t$, $a \le \beta(t) \le t$.

We consider boundary value conditions for the system (1)

$$x(a) = A, \qquad x(b) = B \tag{4}$$

and

$$\| x'(a) \| = v, \tag{5}$$

where vectors A, $B \in \mathbb{R}^n$ and the constant $v > 0$ are given. The right end b of the interval D is unknown.

In this paper the existence of the right end b of the interval D and the solution x of the problem (1), (4), (5) on D will be proved.

In particular case, for equation

$$x''(t) = g(x(t)), \qquad t \in D$$

the similar problem was consider in the paper [3].

We will prove the following theorem:

THEOREM. *Let the function f satisfy assumptions (2) and (3). Let us assume that*

$$v > \begin{cases} h(d_1) & \text{for} \quad d_1 < d_2 \\ h(d_2) & \text{for} \quad d_1 \ge d_2, \end{cases}$$

where

$$h(d) := \frac{1}{d} \| B - A \| + \frac{K}{2} \cdot d, \qquad d > 0$$

and

$$d_1 := \sqrt{\frac{2\|B - A\|}{K}},$$

$$d_2 := \frac{\sqrt{q^2 + 2p} - q}{p}$$

and

$$\frac{1}{2} d^2 p + dq < 1. \tag{7}$$

If the vectors A and B satisfy the relation

$$A \neq B \tag{8}$$

then there exists the interval D and the solution x of the problem (1), (4), (5) on D.

PROOF. From the results of the papers [1], [2] and the assumptions (2) and (7) we obtain existence and uniqueness of the solution of the problem (1), (4) on D, where $b > a$ and b is a parameter. From uniqueness of the solution it follows that the formula of the solution may be presented in the form

$$x(t) = \int_a^t \left[\int_a^s f(z, x(\alpha(z)), x'(\beta(z)))dz \right] ds + M_b \cdot (t - a) + A, \tag{9}$$

for $t \in D$. From (9) it follows for $t = b$ that the vector $M_b \in \mathbb{R}^n$ is defined by formula

$$M_b = \frac{1}{d} \left\{ (B - A) - \int_a^b \left[\int_a^s f(z, x(\alpha(z)), x'(\beta(z)))dz \right] \right\}. \tag{10}$$

Differentiating each side of the equation (9) with respect to the variable t we obtain

$$x'(t) = \int_a^t f(z, x(\alpha(z)), x'(\beta(z)))dz + M_b, \qquad t \in D. \tag{11}$$

Using (10), the triangle inequality, the assumption (3) and properties of integrals we obtain

$$\begin{aligned} \| M_b \| &\leq \frac{1}{d} \left\{ \| B - A \| + \int_a^b \left[\int_a^s \| f(z, x(\alpha(z)), x'(\beta(z))) \| \, dz \right] ds \right\} \\ &\leq \frac{1}{d} \left\{ \| B - A \| + \int_a^b \left[\int_a^s K dz \right] ds \right\} = h(d) \end{aligned}$$

i.e.

$$\|M_b\| \leq h(d) \qquad \text{for} \quad d > 0. \tag{12}$$

From the definition (10) there exists

$$\lim_{d \leftrightarrow 0+} \| M_b \| = +\infty. \tag{13}$$

It follows from the definition of the function h that

$$\lim_{d \to 0+} h(d) = +\infty$$

and formulas (12), (13) are not in contradiction with themselves.

From continuity of the function M_b for $d > 0$, from (12) and (13), under (8) the norm $\| M_b \|$ is greather then $\min\limits_{d>0} h(d)$. But for satisfying the inequality (7) the argument d must fulfill the inequality $d \leq d_2$. Let us consider two possible cases:

(1^o) $d_1 < d_2$.

From the definition of the function h

$$\min_{0 \leq d \leq d_2} h(d) = h(d_1).$$

(2^o) $d_1 \geq d_2$.

Then

$$\min_{0 \leq d \leq d_2} h(d) = h(d_2).$$

From uniqueness of the solution x and from (5) and (11) it follows that the equality

$$\| M_b \| = v \tag{14}$$

holds. The existence of b follows from (13) and continuity of $\| M_b \|$. Then for v, satisfying the inequality (6) there exist M_b defined by (10) and solution x of the form (9) of the problem (1), (4), (5).

This is the end of the proof. □

Remark 1. Theorem is not true without the assumption (8).

PROOF. For the problem

$$x''(t) = 0, \qquad x(a) = x(b) = 0, \qquad t \in D.$$

there exists the unique constant solution $x(t) = 0$, $t \in D$. For all $v > 0$ the condition (5) is not fulfilled. $\square$

REMARK 2. The analogous theorem is true for the equation (1) with more than two deviations of the argument.

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