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ON THE NUMBER OF SOLUTIONS OF THE NEUMANN PROBLEM FOR THE ORDINARY SECOND ORDER DIFFERENTIAL EQUATION

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Abstract. We have found conditions for the nonlinearity f which are sufficient for the existence of at least two solutions to the Neumann problem for the equation $u'' + f(t, u, u') = s$.

1. Introduction

Consider the second order differential equation

$$u'' + f(t, u, u') = s, \tag{1.1s}$$

where $s \in \mathbb{R}$ is a parameter, $I = [a, b] \subset \mathbb{R}$ and $f \in C(I \times \mathbb{R}^2)$. We seek results concerning the number of solutions to (1.1s), satisfying the Neumann conditions

$$u'(a) = 0, \quad u'(b) = 0. \tag{1.2}$$

Our method of proofs makes use of a relation between strict upper and lower solutions and the coincidence topological degree and is close to that of [1]. The number of solutions (2, 1 or 0) of (1.1s), (1.2) is a function of parameter s. Such multiplicity results of Ambrosetti–Prodi type are obtained in [1] and [4] for periodic and four–point problems, respectively, provided f satisfies the Berstein–Nagumo growth conditions. However they were proved under the assumption that for fixed $s_1 \in \mathbb{R}$ the set of all solutions to $\{$(1.1s), $s \leq s_1\}$, satisfying the boundary conditions, is bounded above. In contrast

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to that, our results are proved under assumptions imposed on f directly. Moreover no growth conditions (like Bernstein-Nagumo) are required here (see (3.1),(3.2)).

Other multiplicity results (one nonnegative and one nonpositive solution) for Neumann problem

$$u'' = f(t,u), \quad u'(0) = u'(1) = 0 \tag{1.3}$$

have been proved by M. N. Nkashama and J. Santanilla in [2] for a Carathéodory function f bounded below by a Lebesgue integrable function and fulfilling e.g. conditions:

$\varliminf_{|u|\to\infty} f(t,u) \geq 0$ for a.e. $t \in [0,1]$ with strict inequality on a subset of positive measure,

$f(t,u) \leq \alpha_+ u$ for a.e. $t \in [0,1]$ and all $u \geq 0$,

$f(t,u) \leq -\alpha_-^2 u$ for a.e. $t \in [0,1]$ and all $u \leq 0$,

where $\alpha_+ \in (0,\infty)$, $\alpha_- \in (0, \frac{\pi}{2})$.

We can see that the theorems of [2] cannot be used for functions f rapidly growing in their second variable.

Now, let us remind that functions $\sigma_1, \sigma_2 \in C^2(I)$ are called lower and upper solutions for (1.1s), (1.2), respectively, if they fulfil (1.2) and

$$(\sigma_i'' + f(t,\sigma_i,\sigma'_i) - s)(-1)^i \leq 0 \quad \text{for each} \quad t \in I, \quad i = 1,2. \tag{1.3}$$

The lower and upper solutions are said to be strict, if the inequalities in (1.3) are strict for all $t \in I$.

For $r_1 \in (0,+\infty)$ we shall write

$$D(-r_1) = \{x \in C^2(I): \ x(t) > -r_1 \quad \text{for each} \quad t \in I\},$$
$$D(r_1) = \{x \in C^2(I): \ x(t) < r_1 \quad \text{for each} \quad t \in I\}.$$

2. Lemmas

Let us consider the auxiliary equation

$$u'' = g(t,u,u'), \tag{2.1}$$

where $g \in C(I \times \mathbb{R}^2)$.

LEMMA 1. *Let σ_1 be a lower solution and σ_2 an upper solution to (2.1), (1.2) with $\sigma_1(t) \le \sigma_2(t)$ for each $t \in I$. Further, let there exist $k \in (0,\infty)$ such that for each $t \in I$, $x, y \in \mathbb{R}$, where $\sigma_1(t) \le x \le \sigma_2(t)$, the inequality*

$$|g(t,x,y)| \le k \tag{2.2}$$

is fulfilled.

Then problem (2.1), (1.2) has a solution u satisfying

$$\sigma_1(t) \le u(t) \le \sigma_2(t) \qquad \text{for each} \quad t \in I. \tag{2.3}$$

PROOF. This known fact can be proved for example in the same way as in [3]. □

LEMMA 2. *Suppose $s \in \mathbb{R}$. Let σ_1 be a lower solution and σ_2 an upper solution to (1.1s), (1.2) with $\sigma_1(t) \le \sigma_2(t)$ for each $t \in I$.*

Further, let there exist $m \in \mathbb{R}$ such that

$$m \le f(t,x,y) \quad \text{for each} \quad t \in I, \quad x, y \in \mathbb{R}, \quad \text{where} \quad \sigma_1(t) \le \sigma_2(t). \tag{2.4}$$

Then problem (1.1s), (1.2) has a solution u fulfilling (2.3).

PROOF. Let us choose $s \in \mathbb{R}$ and suppose that u is a solution of (1.1s), (1.2) satisfying (2.3). We shall find an a priori estimate for u'. From (1.1s), (2.4) it follows $u''(t) + m \le s$ for each $t \in I$. Using (1.2) and integrating the last inequality on (a,t), $t \in I$, we get $u'(t) \le |s-m|(b-a)$ on I. Similarly, by integration on (t,b), $t \in I$, we have $u'(t) \ge -|s-m|(b-a)$ on I. So if we put $\varrho = |s-m|(b-a) + \max\{|\sigma_1(t)| + |\sigma_2(t)| : \ t \in I\}$, we have

$$\max\{|u'(t)| : \ t \in I\} \le p \tag{2.5}$$

and

$$\max\{f(t,x,y) : \ t \in I, \ \sigma_1(t) \le x \le \sigma_2(t), \ -\varrho \le y \le \varrho\} = M \in \mathbb{R}.$$

So, we can define a function g

$$g(t,x,y) = \begin{cases} f(t,x,y) & \text{for} \quad t \in I, \quad x \in \mathbb{R}, \quad y \in [-\varrho,\varrho] \\ f(t,x,\varrho.\ \text{sign}\ y) & \text{for} \quad t \in I, \quad x \in \mathbb{R}, \quad |y| > \varrho, \end{cases}$$

which fulfils the condition of Lemma 1 for $k = \max\{|m| + |s|, \ |M| + |s|\}$ and the same upper and lower solutions and hence problem (2.1), (1.2) has a solution u fulfilling (2.3). Then u' satisfies (2.5) and according to (2.6) u is a solution to (1.ls), (1.2) as well. □

3. Multiplicity results

Using Lemma 2 and the coincidence degree theory we get multiplicity results of Ambrosetti–Prodi type.

THEOREM 1. *Let $f \in C(I \times \mathbb{R}^2)$ and there exist $r_1 \in (0, \infty)$, $m, s_1 \in \mathbb{R}$ such that the inequalities*

$$(3.1) \qquad f(t, -r_1, 0) > s_1 > f(t, 0, 0) \quad \text{for each} \quad t \in I,$$

$$(3.2) \qquad m \leq f(t, x, y) \quad \text{for each} \quad t \in I, \quad x \in (-r_1, \infty), \quad y \in \mathbb{R}$$

are satisfied. Then there exists $s_0 \in [m, s_1)$ such that
(a) for $s < s_0$, problem (1.1s), (1.2) has no solution in $\overline{D(-r_1)}$,
(b) for $s = s_0$, problem (1.1s), (1.2) has at least one solution in $\overline{D(-r_1)}$,
(c) for $s \in (s_0, s_1]$, problem (1.ls), (1.2) has at least two solutions in $D(-r_1)$.

PROOF. Put

$$(3.3) \qquad h(t, x, y) = \begin{cases} f(t, x, y) & \text{for} \quad x \geq -r_1 \\ f(t, -r_1, y) & \text{for} \quad x < -r_1 \end{cases}$$

and for $s \in \mathbb{R}$ consider the equation

$$(3.4s) \qquad u'' + h(t, u, u') = s.$$

Proving Theorem 1 we shall need several auxiliary propositions.

PROPOSITION 1. *If $s \in (-\infty, s_1]$, then any solution of (3.4s), (1.2) belongs to $D(-r_1)$.*

PROOF OF PROPOSITION 1. Let u be a solution of (3.4s), (1.2) for some $s \leq s_1$. Suppose that $\min\{u(t) : t \in I\} = u(t_0) \leq -r_1$. Then, by (1.2), $u'(t_0) = 0$, $u''(t_0) \geq 0$. On the other hand from (3.1), (3.3) it follows $u''(t_0) = s - f(t_0, -r_1, 0) < 0$, a contradiction. □

PROPOSITION 2. *There exists $s_0 \in [m, s_1)$ such that for $s < s_0$ problem (3.4s), (1.2) has no solution.*

PROOF OF PROPOSITION 2. Suppose that (3.4s), (1.2) has a solution u for some $s \in \mathbb{R}$. Then, integrating (3.4s) on (a, b), we get $m(b-a) \leq \int_a^b h(\tau, u(\tau), u'(\tau))d\tau = s(b-a)$, thus $m \leq s$, and we can take

$$(3.5) \qquad s_0 = \inf\{s \in [m, \infty) : \text{ (3.4s),(1.2) has a solution}\}.$$

Let us show that the set in (3.5) is nonempty. Put

$$s^* = \max\{h(t,0,0) : \quad t \in I\}.$$

Then 0 is an upper solution and $-r_1$ a lower solution of (3.4s*), (1.2). Thus, by Lemma 2, problem (3.4s*), (1.2) has a solution u^* with $-r_1 \leq u^*(t) \leq 0$ on I. Clearly $s_0 \leq s^* < s_1$. □

PROPOSITION 3. *For any $s \in (s_0, s_1]$ problem (3.4s), (1.2) has at least one solution.*

PROOF OF PROPOSITION 3. Let $\widetilde{s} \in (s_0, s_1)$ and u be a solution of (3.4$\widetilde{s}$), (1.2). By Proposition 1, $\widetilde{u} \in D(-r_1)$. Let us choose $\sigma \in [\widetilde{s}, s_1]$. Then $\widetilde{u}$ is an upper solution and $-r_1$ is a lower solution of (3.4σ), (1.2). Therefore, by Lemma 2, (3.4σ), (1.2) has at least one solution. Since σ is an arbitrary number of $[\widetilde{s}, s_1]$, problem (3.4s), (1.2) has a solution for any $s \in [\widetilde{s}, s_1]$, and according to (3.5) for any $s \in (s_0, s_1]$. □

From now on, let $\widetilde{s} \in (s_0, s_1)$ be arbitrary but fixed and let $\widetilde{u}$ denote a solution of (3.4$\widetilde{s}$), (1.2). Further, let us put for all $t \in I$, $x, y \in \mathbb{R}$

$$\alpha(x) = \begin{cases} -r_1 & \text{for} \quad x < -r_1 \\ x & \text{for} \quad -r_1 \leq x \leq \widetilde{u}(t) \\ \widetilde{u}(t) & \text{for} \quad x > \widetilde{u}(t) \end{cases}$$

and

$$(3.6) \qquad g(t, x, y) = f(t, \alpha(x), y) - x + \alpha(x).$$

We shall consider the equation

$$(3.7s) \qquad u'' + g(t, u, u') = s.$$

PROPOSITION 4. *For each $s \in (\widetilde{s}, s_1]$ any solution u of problem (3.7s), (1.2) satisfies*

$$-r_1 < u(t) < \widetilde{u}(t) \qquad \textit{for all} \quad t \in I.$$

PROOF OF PROPOSITION 4. Let $s \in (\tilde{s}, s_1]$ and u be a solution of (3.7s), (1.2). Suppose that for some $t \in I$ $u(t) \geq \tilde{u}(t)$. Then there exists $t_0 \in I$ such that $u(t_0) \geq \tilde{u}(t_0)$, $u'(t_0) = \tilde{u}'(t_0)$, $u''(t_0) \leq \tilde{u}''(t_0)$. But from (3.6) we can get $u''(t_0) > \tilde{u}''(t)$, which is a contradiction. The inequality $-r_1 < u$ can be proved by similar arguments. □

Now, for $s \in (-\infty, s_1]$, let us consider the class of equations

$$u'' - (1-\lambda)u + \lambda[g(t,u,u') - s] = 0, \quad \lambda \in [0,1]. \tag{3.8sλ}$$

PROPOSITION 5. *There exist $R, \varrho \in (0,\infty)$ such that for any $s \in [s_0, s_1]$ and any $\lambda \in [0,1]$ each solution u of (3.8sλ), (1.2) satisfies*

$$|u(t)| < R, \quad |u'(t)| < \varrho \qquad \textit{for all} \quad t \in I.$$

PROOF OF PROPOSITION 5. Let us denote

$$\tilde{r} = \max\{\tilde{u}(t): \ t \in I\}, \qquad \tilde{m} = \max\{f(t,x,0): \ t \in I, \ x \in [-r, \tilde{r}]\}.$$

Let us choose a real number R with

$$R > \max\{r_1 + s_1 - m, \quad \tilde{r} + \tilde{m} - s_0\}. \tag{3.9}$$

Suppose that for some $s \in [s_0, s_1]$ and $\lambda \in [0,1]$ there exists a solution u of (3.8sλ), (1.2) with $\max\{u(t) : \ t \in I\} = u(t_0) \geq R$. Then, in view of (1.2), $u'(t_0) = 0$, $u''(t_0) \leq 0$ and by (3.8sλ), (1.2), (3.9) we get $u''(t_0) = (1-\lambda)u(t_0) + \lambda[s - g(t_0, u(t_0), u'(t_0))] \geq (1-\lambda)R + \lambda[s_0 - \tilde{m} + R - \tilde{r}] > 0$, a contradiction.

Similarly, if $u(t_0) \leq -R$, we get

$$0 \leq u''(t_0) \leq -(1-\lambda)R + \lambda[s_1 - m - R + r_1] < 0,$$

a contradiction. Thus

$$|u(t)| < R \qquad \text{for all} \quad t \in I.$$

Further, $u'' = (1-\lambda)u + \lambda[s - f(t, \alpha(u), u') + u - \alpha(u)] < R + \lambda[s_1 - m + r_1]$, hence $u''(t) < K$ for all $t \in I$, where $K = R + |s_1| + |m| + r_1$. Therefore

$$|u'(t) < \varrho| \qquad \text{for all} \quad t \in I, \quad \text{where} \quad \varrho = K(b-a).$$

□

Let us put dom $L = \{u \in C^2(I): \ u'(a) = 0, \ u'(b) = 0\}$, $L : \text{dom } L \to C(I)$, $u \to u''$, $N_s : C^1(I) \to C(I)$, $u \to h(\cdot, u(\cdot), u'(\cdot)) - s$. Then problem (3.4s), (1.2) can be written in the form

$$(3.10s) \qquad (L + N_s)u = 0.$$

Let us consider two open bounded sets in $C^1(I)$:

$$\Omega = \{u \in C^1(I): \ -r_1 < u(t) < \widetilde{u}(t), \ |u'(t)| < \varrho \ \ \text{rm for all} \ \ t \in I\},$$

and

$$\Omega_1 = \{u \in C^1(I): \ |u(t)| < R, \ |u'(t)| < \varrho \ \ \text{for all} \ \ t \in I\},$$

where $\widetilde{u}$ is the above fixed solution of (3.4$\widetilde{s}$), (1.2) and R, ϱ are the constants of Proposition 5. In the same way as in [4] we can prove that $d_L(L+N_s, \Omega) = \pm 1$ and $d_L(L + N_s, \Omega_1 - \overline{\Omega}) = \mp 1$, for any $s \in (\widetilde{s}, s_1]$. This implies that for $s \in (\widetilde{s}, s_1]$ problem (3.10s) has at least one solution in Ω and at least another one in $\Omega_1 - \overline{\Omega}$. Using Proposition 1 and the fact that $\widetilde{s}$ is a fixed but arbitrary number in (s_0, s_1), we get the assertion (c) of Theorem 1. Now, using Arzelà–Ascoli Theorem and Proposition 5, we can find a solution of (3.10s_0) as a limit of a sequence of solutions u_n of (3.10s_n) for $s_n \to s_0$. Finally, the assertion (a) of Theorem 1 follows from (3.3) and Propositions 1,2. Theorem is proved. □

Replacing f by $-f$ and x by $-x$, a dual version of Theorem 1 can be given.

THEOREM 2. *Let $f \in C(I \times R^2)$ and there exists $r_1 \in (0, \infty)$, $m, s_1 \in R$ such that the inequalities*

$$(3.11) \qquad f(t, 0, 0) > s_1 > f(t, r_1, 0) \qquad \text{for each} \quad t \in I,$$

$$(3.12) \qquad f(t, x, y) \le m \qquad \text{for each} \quad t \in I, \ \ x \in (-\infty, r_1), \ \ y \in R$$

are satisfied.

Then there exists $s_0 \in (s_1, m]$ such that

(a) for $s > s_0$ problem (1.1s), (1.2) has no solution in $\overline{D(r_1)}$,
(b) for $s = s_0$ problem (1.1s), (1.2) has at least one solution in $\overline{D(r_1)}$,
(c) for $s \in [s_1, s_0)$ problem (1.1s), (1.2) has at least two solutions in $D(r_1)$.

The proof of Proposition 2 implies the following criterion of nonexistence.

THEOREM 3. *Let $f \in C(I \times R^2)$.*
(a) If f is bounded below, i.e.

$$\inf \{f(t,x,y): \ (t,x,y) \in IxR^2\} = m_1 \in R,$$

then for $s < m_1$ problem (1.1s), (1.2) has no solution.
(b) If f is bounded above, i.e.

$$\sup \{f(t,x,y): \ (t,x,y) \in I \times R^2\} = m_2 \in R,$$

then for $s > m_2$ problem (1.1s), (1.2) has no solution.

4. Examples

EXAMPLE 1. Let us consider the equation

$$u'' + c|u'|^n + u^{2k} + \Phi(t) = s, \tag{4.1s}$$

where $\phi \in C(I)$, $c \in [0,\infty]$, $k,m \in \mathbb{N}$, $s \in \mathbb{R}$. The function

$$f(t,x,y) = c|y|^n + x^{2k} + \phi(t)$$

satisfies the assumptions of Theorem 1 with $m = \min\{\phi(t) : \ t \in I\}$ and arbitrary $s_1 > \max\{\phi(t) : \ t \in I\}$. We can see that f also fulfils (a) of Theorem 3, where $m = m_1$. On the other hand, for $c > 0$, $n > 2$, f does not fulfil the conditions of the theorems in [1], [4], and for $c = 0$ f does not satisfy the growth conditions of [2].

EXAMPLE 2. Let us show that Theorem 1 can be applied on the equation

$$u'' + c(e^{u'} + 1) - \text{ arctg } u = s, \tag{4.2s}$$

where $c, s \in \mathbb{R}$.

Let $c \geq 0$. Then the function $f(t,x,y) = c(e^y + 1) - \text{arctg } x$ satisfies conditions (3.1), (3.2) of Theorem 1 with $m = -\frac{\pi}{2} + c$ and arbitrary $s_1 \in (2c, 2c + \frac{\pi}{2})$. Since $m = m_1$, Theorem 3 implies that for $s < m$ problem (4.2s), (1.2) has no solution.

Let $c < 0$. Then f satisfies (3.11), (3.12) of Theorem 2 with $m = \frac{\pi}{2} + c$ and $s_1 \in (2c - \frac{\pi}{2}, 2c)$. By Theorem 3, for $s > m$ our problem has no solution.

But if $c \neq 0$, we cannot use theorems of [1], [4] and if $c = 0$, theorems of [2] cannot be applied as well.

EXAMPLE 3. Consider the equation

$$u'' + c(u')^{2k} + 2 \sin u - \sin t = s, \tag{4.3s}$$

where $k \in \mathbb{N}$, $c, s \in \mathbb{R}$, $I = [0, \pi]$.

If $c \geq 0$, then the function $f(t, x, y) = cy^{2k} + 2 \sin x - \sin t$ satisfies (3.1), (3.2) with $m = -3$ and $s \in (0, 1)$. For $c < 0$, f satisfies (3.11), (3.12) with $m = 2$ and $s_1 \in (-2, -1)$.

If $c \geq 0$, $s < -3$ or $c < 0$, $s > 2$, problem (4. 3s), (1.2) has no solution.

But for $c \neq 0$ f does not fulfil the growth conditions of [1] and moreover the function $g(t, x) = 2 \sin x - \sin t$ fulfils neither conditions of [2] nor hypothese (H4) of [1].

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