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BOUNDARY VALUE PROBLEMS FOR ONE-PARAMETER SECOND-ORDER DIFFERENTIAL EQUATIONS

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Abstract. The paper establishes sufficient conditions for the existence of solutions of a one-parameter differential equation $x'' = f(t, x, x', \lambda)$ satisfying some of the following boundary conditions:

$$\gamma(x) = 0, \quad x'(a) = x'(b) = 0,$$

$$x'(a) = x'(b) = 0, \quad x(c) - x(d) = 0$$

and

$$x'(a) = x'(t_o) = x'(b) = 0.$$

Here γ is a functional. The application is given for a class of one-parameter functional boundary value problems.

1. Introduction

Consider the differential equation

$$x'' = f(t, x, x', \lambda) \tag{1}$$

depending on the parameter λ. Here $f \in C^0(\langle a, b\rangle \times \mathbb{R}^2 \times \langle A, B\rangle)$, $-\infty < a < b < \infty$, $-\infty < A < B < \infty$.

Let X be the Banach spare of C^0–functions on $\langle a, b\rangle$ with the norm $\|x\| = \max\{|x(t)|;\ a \le t \le b\}$. Let $\gamma : X \to \mathbb{R}$ be a continuous increasing (i.e. $x, y \in X$, $x(t) < y(t)$ for $t \in \langle a, b\rangle \Rightarrow \gamma(x) < \gamma(y)$) functional, $\gamma(0) = 0$.

Let $a \le c < d \le b$, $a < t_0 < b$. Consider the functional boundary condition

$$\gamma(x) = 0, \quad x'(a) = x'(b) = 0 \tag{2}$$

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and the boundary conditions

$$x'(a) = x'(b) = 0, \qquad x(c) - x(d) = 0, \tag{3}$$

$$x'(a) = x'(t_o) = x'(b) = 0. \tag{4}$$

The considered problem is to determine sufficient conditions on f guaranteeing that it is possible to choose the parameter λ such that the boundary value problem (BVP for short) (1), (2) or (1), (3) or (1), (4) has a solution. The uniqueness of solutions of BVP (1), (2) is discussed too. We observe that BVPs (1), (3) and (1), (4) are at a resonance. The proofs of results make use of the coincidence degree theory (see Theorem 2.2. [8]) and the maximum principle arguments. An application of our result for BVP (1), (2) with $\gamma(x) = x(a)$ is given for the functional differential equation

$$x'' = F(t, x, x_t, x', x_t', \lambda)$$

depending on the parameter λ.

Sufficient conditions for the two-parameter differential equation $x'' + (q(t,\lambda,\mu) + r(t))x = 0$ having a nontrivial solution $x(t)$ satisfying $x(a) = x(t_0) = x(b) = \theta$ are stated in [1] and [2]. Using a surjective mapping in $\mathbb{R}^n$, BVP (1), $x'(0) = A$, $x(1) = B$, $x(2) = C$ was studied in [6]. Some boundary value problems for differential and functional differential equations depending on the parameter were considered in [3]–[5] using the Schauder linearization technique and the Schauder fixed point theorem. We observe that the boundary value problem $x'' - q(t)x = h(t, x_t, \lambda)$, $x(t_1) = x(t_2) = x(t_3) = 0$ $(-\infty < t_1 < t_2 < t_3 < \infty)$, was studied in [7].

2. Lemmas

The function $x \in C^2(\langle a, b\rangle)$ is said to be a solution of BVP (1), (2) or (1), (3) or (1), (4) if there exists a $\lambda_0 \in \langle A, B\rangle$ such that x is a solution of (1) for $\lambda = \lambda_0$ satisfying (2) or (3) or (4), respectively.

In the following we shall assume there exist constants K, L, $K < L$, such that f satisfies the following assumptions:

(H_1) $f(t, K, 0, B) < 0 < f(t, L, 0, A)$ for $t \in \langle a, b\rangle$;

(H_2) $|f(t, x, y, \lambda)| \le p(|y|)$ for $(t, x, y, \lambda) \in \langle a, b\rangle \times \langle K, L\rangle \times \mathbb{R} \times \langle A, B\rangle$, where $p : \langle 0, \infty) \to (0, \infty)$ is a nondecreasing function such that

$$\int_0^T \frac{s\,ds}{p(s)} > L - K$$

with a positive constant T;

(H$_3$) $f(t,.,y,\lambda)$ is increasing on $\langle K,L\rangle$ for each fixed $(t,y,\lambda)\in\langle a,b\rangle\times\langle -T,T\rangle\times\langle A,B\rangle$;

(H$_4$) $f(t,x,y,.)$ is increasing on $\langle A,B\rangle$ for each fixed $(t,x,y)\in\langle a,b\rangle\times\langle K,L\rangle\times\langle -T,T\rangle$.

LEMMA 1. *Let assumptions (H$_1$)–(H$_4$) be fulfilled with constants $K<L$ and let $\mu\in\langle A,B\rangle$. Then equation (1) for $\lambda=\mu$ has a unique solution $x(t)$ satisfying*

$$x'(a)=x'(b)=0 \tag{5}$$

and

$$K<x(t)<L \quad \text{for} \quad t\in\langle a,b\rangle, \quad \|x'\|<T. \tag{6}$$

PROOF. With respect to assumptions (H$_1$)–(H$_4$) the existence of a solution $x(t)$ of (1) for $\lambda=\mu$ satisfying (5) and (6) follows from Theorem 2.2 [8] and its proof. This theorem is proved using the results of coincidence degree theory. To show the uniqueness we assume on the contrary that x_1 and x_2 are two different solutions of (1) for $\lambda=\mu$ satisfying (5) and (6) with $x=x_j$ $(j=1,2)$. Set $w=x_1-x_2$ and let $w(a)\geq 0$. If w has a negative local minimum at a point $t_1\in(a,b)$, then $w(t_1)<0$, $w'(t_1)=0$, $w''(t_1)\geq 0$ which contradicts (cf. (H$_3$)) $w''(t_1)=f(t_1,x_1(t_1),x'_1(t_1),\mu)-f(t_1,x_2(t_1),x'_1(t_1),\mu)<0$. We can similarly check that w has not a positive local maximum at an inner point of $\langle a,b\rangle$. If $w(a)>0$, then $w''(a)=f(a,x_1(a),0,\mu)-f(a,x_2(a),0,\mu)>0$, hence $w(t)>0$, $w'(t)>0$ on an interval (a,t_2) $(\subset(a,b))$ while $w'(t_2)=0$, and consequently $w''_2(t)\leq 0$ which contradicts $w''(t_2)=f(t_2,x_1(t_2),x'_1(t_2),\mu)-f(t_2,x_2(t_2),x'_1(t_2),\mu)>0$. Let $w(a)=0$. Since w is either nondecreasing or nonincreasing on $\langle a,b\rangle$ and $w(b)\neq 0$ we may withought loss of generality assume w is nondecreasing on $\langle a,b\rangle$, $w(b)>0$. Then $w''(b)=f(b,x_1(b),0,\mu)-f(b,x_2(b),0,\mu)>0$ which is impossible. Hence the lemma is proved. □

REMARK 1. Let assumptions (H$_1$)–(H$_4$) be fulfilled with constants $K<L$. For each $\lambda\in\langle A,B\rangle$ we will denote by $x(t,\lambda)$ the unique solution of (1) satisfying (5) and (6). The existence and uniqueness of this solution is ensured by Lemma 1.

LEMMA 2. *Let assumptions (H$_1$)–(H$_4$) be fulfilled with constants $K<L$ and let $A\leq\lambda_1<\lambda_1\leq B$. Then*

$$(L>)\ x(t,\lambda_1)>x(t,\lambda_2)\ (>K) \quad \text{for} \quad t\in\langle a,b\rangle.$$

PROOF. Set $u_j(t) = x(t,\lambda_j)$ for $t \in \langle a,b\rangle$, $j = 1,2$, and let $w = u_1 - u_2$.

(i) Let $w(a) \le 0$. Then $w''(a) = f(a,u_1(a),0,\lambda_1) - f(a,u_2(a),0,\lambda_2) < 0$, hence $w(t) < 0$, $w'(t) < 0$ for $t \in (a,\xi)$, $w'(\xi) = 0$ with a $\xi \in (a,b\rangle$. Consequently $w''(\xi) \ge 0$ which contradicts $w''(\xi) = f(\xi,u_1(\xi),u'_1(\xi),\lambda_1) - f(\xi,u_2(\xi),u'_1(\xi),\lambda_2) < 0$.

(ii) Let $w(t) > 0$ for $t \in \langle a,\nu)$, $w(\nu) = 0$ with a $\nu \in (a,b\rangle$. If $w'(\nu) = 0$, then $w''(\nu) = f(\nu,u_1(\nu),u'_1(\nu),\lambda_1) - f(\nu,u_1(\nu),u'_1(\nu),\lambda_2) < 0$, a contradiction. If $w'(\nu) < 0$, then $w(t) < 0$, $w'(t) < 0$ on an interval (ν,τ) ($\subset (\nu,b\rangle$) while $w'(\tau) = 0$, which (see the case (i)) is impossible. Consequently $w(t) > 0$ on $\langle a,b\rangle$. □

LEMMA 3. *Let assumptions* (H_1)-(H_4) *be satisfied with constants* $K < L$. *If* $\{\lambda_n\} \subset \langle A,B\rangle$ *is a convergent sequence,* $\lim_{n\to\infty} \lambda_n = \mu$, *then*

$$\lim_{n\to\infty} x^{(i)}(t,\lambda_n) = x^{(i)}(t,\mu), \qquad i = 0,1, \tag{7}$$

uniformly on $\langle a,b\rangle$.

PROOF. Let $\{\lambda_n\} \subset \langle A,B\rangle$ be a convergent sequence, $\lim_{n\to\infty} \lambda_n = \mu$. Since $\|x(t,\lambda_n)\| < \max\{|K|,|L|\}$ $(:= V)$, $\|x'(t,\lambda_n)\| < T$ for $n \in \mathbb{N}$ (by Lemma 1), we have $\|x''(t,\lambda_n)\| \le S$ for $n \in \mathbb{N}$ with $S = \max\{|f(t,x,y,\lambda)|; (t,x,y,\lambda) \in \langle a,b\rangle \times \langle -V,V\rangle \times \langle -T,T\rangle \times \langle A,B\rangle\}$. Let $\{\overline{\lambda}_n\}$ be a subsequence of $\{\lambda_n\}$. Using the Arzela–Ascoli theorem we can select a subsequence $\{x(t,\overline{\lambda}_{k_n})\}$ of $\{x(t,\overline{\lambda}_n)\}$ such that $\{x^{(i)}(t,\overline{\lambda}_{k_n})\}$ uniformly convergent on $\langle a,b\rangle$ for $i = 0,1$. Setting $u(t) = \lim_{n\to\infty} x(t,\overline{\lambda}_{k_n})$ for $t \in \langle a,b\rangle$, then $u'(a) = u'(b) = 0$, and taking the limit in the equalities

$$x'(t,\overline{\lambda}_{k_n}) = \int_a^t f(s,x(s,\overline{\lambda}_{k_n}),x'(s,\overline{\lambda}_{k_n}),\overline{\lambda}_{k_n})ds, \qquad t \in \langle a,b\rangle, \qquad n \in \mathbb{N}$$

as $n \to \infty$, we get

$$u'(t) = \int_a^t f(s,u(s),u'(s),\mu)ds, \qquad t \in \langle a,b\rangle.$$

Therefore u is a solution of (1) for $\lambda = \mu$ satisfying (5) and (6) with $x = u$, and consequently $u(t) = x(t,\mu)$ by Lemma 1. Thus $\lim_{n\to\infty} x^{(i)}(t,\lambda_n) = x^{(i)}(t,\mu)$ uniformly on $\langle a,b\rangle$, $i = 0,1$. □

LEMMA 4. *Let assumptions (H_1)-(H_4) be fulfilled with constants* $K < 0 < L$. *If assumption*
(H_5) $f(t,0,0,A).f(t,0,0,B) < 0$ *for* $t \in \langle a,b\rangle$
is satisfied, then

$$x(t,B) < 0 < x(t,A) \quad \text{for} \quad t \in \langle a,b\rangle. \tag{8}$$

PROOF. Let (H_5) be satisfied and set $u(t) = x(t,A)$, $t \in \langle a,b\rangle$. If $u(a) \le 0$, then $u''(a) = f(a,u(a),0,A) \le f(a,0,0,A) < 0$ hence $u(t) < 0$, $u'(t) < 0$ on an interval (a,ε) $(\subset (a,b\rangle)$ while $u'(\varepsilon) = 0$ which contradicts $u''(\varepsilon) = f(\varepsilon,u(\varepsilon),0,A) < f(\varepsilon,0,0,A) < 0$, and consequently $u(a) > 0$. If there exists a $\xi \in (a,b\rangle$ such that $u(t) > 0$ on $\langle a,\xi)$ while $u(\xi) = 0$, then $u'(\xi) < 0$ since in the case of $u'(\xi) = 0$ we have $u''(\xi) = f(\xi,0,0,A) < 0$, a contradiction. Thus $u(t) < 0$ on an interval (ξ,ν) $(\subset (\xi,b\rangle)$ while $u'(\nu) = 0$. This is impossible because of $u''(\nu) = f(\nu,u(\nu),0,A) < f(\nu,0,0,A) < 0$. Therefore $u(t) > 0$ on $\langle a,b\rangle$. The proof of $x(t,B) < 0$ on $\langle a,b\rangle$ is evidently analogous to the proof of $x(t,A) > 0$ on $\langle a,b\rangle$ and therefore it is omitted. □

LEMMA 5. *Let assumptions (H_1)-(H_5) be fulfilled with constants* $K < 0 < L$. *Assume moreover that the following assumptions are satisfied:*
(H_6) *If* $f(t_1,a_1,0,A) \ge 0$ *for some* $t_1 \in \langle a,b)$ *and* $0 < a_1 < L$, *then* $f(t,a_1,y,A) \ge 0$ *for all* $(t,y) \in \langle t_1,b\rangle \times \langle 0,T\rangle$;
(H_7) *If* $f(t_1,b_1,0,B) \le 0$ *for some* $t \in \langle a,b)$ *and* $K < b_1 < 0$, *then* $f(t,b_1,y,B) \le 0$ *for all* $(t,y) \in \langle t_1,b\rangle \times \langle -T,0\rangle$.
Then

$$x'(t,A) \le 0, \quad x'(t,B) \ge 0 \quad \text{for} \quad t \in \langle a,b\rangle.$$

PROOF. Let assumptions $(H_1) = (H_7)$ be satisfied. Set $u_1(t) = x(t,A)$ $u_2(t) = x(t,B)$ for $t \in \langle a,b\rangle$ and $c_j = \min\{u_j(t);\ a \le t \le b\}$, $d_j = \max\{u_j(t);\ a \le t \le b\}$, $j = 1,2$. Then $K < c_2 \le d_2 < 0 < c_1 \le d_1 < L$ by Lemma 4. Assume $c_1 < d_1$ and $u'_1(t_1) > 0$ for a $t_1 \in (a,b)$. Then there exists a $\tau \in \langle a,t_1)$ such that $u_1(\tau) \ge c_1 > 0$, $u_1{}'(\tau) = 0$, $u_1'(t) > 0$ on $(\tau,t_1\rangle$ and $u''_1(\tau) \ge 0$. Hence $(u''_1(\tau) =)$ $f(\tau,u_1(\tau),0,A) \ge 0$ and $f(t,u_1(\tau),y,A) \ge 0$ for $(t,y) \in \langle \tau,b\rangle \times \langle 0,T\rangle$ by (H_6). Consequently $u''_1(t) \ge 0$ on $\langle \tau,b\rangle$ and $u'_1(t) \ge u'_1(t_1) > 0$ for $t \in \langle t_1,b\rangle$ which contradicts $u'_1(b) = 0$. This proves $u'_1(t) \le 0$ on $\langle a,b\rangle$. Using assumption (H_7) we can verify $u'_2(t_1) < 0$ for a $t_1 \in (a,b)$ is impossible and therefore $u'_2(t) \ge 0$ for $t \in \langle a,b\rangle$. □

3. Existence and uniqueness theorems

THEOREM 1. *Let assumptions (H_1)-(H_5) be fulfilled with constants $K < 0 < L$. Then there exists a unique solution $x(t)$ of BVP (1), (2) satisfying (6).*

PROOF. Let $x(t, \lambda)$ be a unique solution af (1) satisfying (6) and $x'(a, \lambda) = x'(b, \lambda) = 0$ (see Lemma 1) and set $h(\lambda) = \gamma(x(t, \lambda))$ for $\lambda \in \langle A, B \rangle$. Then h is continuous (by Lemma 3) decreasing (by Lemma 2) on $\langle A, B \rangle$ and $h(A) > 0 > h(B)$ (by Lemma 4). Therefore there exists a unique $\mu \in (A, B)$ such that $h(\mu) = 0$. Setting $x(t) = x(t, \mu)$ for $t \in \langle a, b \rangle$, then x is a unique solution of BVP (1), (2) satisfying (6). □

THEOREM 2. *Let assumptions (H_1)-(H_7) be fulfilled with constants $K < 0 < L$. Then there exists a solution of BVP (1), (3) satisfying (6).*

PROOF. Let $x(t, \lambda)$ be as in the proof of Theorem 1 and set $k(\lambda) = x(c, \lambda) - x(d, \lambda)$ for $\lambda \in \langle A, B \rangle$. Then k is continuous on $\langle A, B \rangle$ (by Lemma 3) and $k(A) \geq 0$, $k(B) \leq 0$ (by Lemma 5). Therefore there exists a $\mu \in \langle A, B \rangle$ such that $k(\mu) = 0$. Setting $x(t) = x(t, \mu)$, then $x(t)$ is a solution of BVP (1), (3) satisfying (6). □

THEOREM 3. *Let assumptions (H_1)-(H_7) be fulfilled with constants $K < 0 < L$. Then there exists a solutions $x(t)$ of BVP (1), (4) satisfying (6).*

PROOF. Let $x(t, \lambda)$ be as in proof of Theorem 1 and set $r(\lambda) = x'(t_0, \lambda)$ for $\lambda \in \langle A, B \rangle$. Then r is continuous on $\langle A, B \rangle$ (by Lemma 3) and $r(A) \leq 0$, $r(B) \geq 0$ (by Lemma 5). Therefore there exists a $\mu \in \langle A, B \rangle$ such that $r(\mu) = 0$. Setting $x(t) = x(t, \mu)$, then $x(t)$ is a solution of BVP (1), (4) satisfying (6). □

EXAMPLE 1. Consider the differential equation

$$x'' = x^3 + \exp(tx - 2) + (x/2)|x'| + (2 - t)\lambda, \quad t \in \langle 0, 1 \rangle. \tag{9}$$

Assumptions (H_1)-(H_7) are fulfilled with the constants $K = -2$, $L = 2$, $A = -4$, $B = 3$ and $T = 21$. Consequently there exists a unique solution $x(t)$ of (9) such that

$$\gamma(x) = 0, \quad x'(0) = x'(1) = 0, \quad \|x\| < 2, \quad \|x'\| < 21 \tag{10}$$

(that is, there exists a unique $\mu \in \langle -4, 3 \rangle$ such that (9) for $\lambda = \mu$ has a solution $x(t)$ satisfying (10) and, moreover, this solution is unique). For example we can set $\gamma(x) = x(t_1)$ $(0 < t_1 < 1)$ or $\gamma(x) = \max\{x(t);\ 0 \le t \le 1\}$ or $\gamma(x) = \min\{x(t);\ 0 \le t \le 1\}$ or $\gamma(x) = \int\limits_0^1 (x(s))^{2n+1} ds$ $(n \in \mathbb{N})$.

Next, there exist solutions $x_1(t)$ and $x_2(t)$ of BVPs (1), (3) and (1), (4), respectively, $\|x_j\| < 2$, $\|x'_j\| < 21$ $(j = 1, 2)$.

4. An application

Let $h > 0$ be given. Let $\mathbf{C}$ be the Banach space of C^0–functions on $\langle a-h, a\rangle$ with the norm $\|x\|_0 = \max\{|x(t)|;\ a-h \le t \le a\}$, $\mathbf{D}$ be the Banach space of C^1–functions on $\langle a-h, a\rangle$ with the norm $\|x\|_1 = \|x\|_0 + \|x'\|_0$ and let $\mathbf{D}_0 = \{x;\ x \in \mathbf{D},\ x(a) = x'(a) = 0\}$. For each $U, V, H \in \mathbb{R}$, $U < V$, $H > 0$ we define sets $\mathbf{C}_{U,V}$ and $\mathbf{C}_H$ by $\mathbf{C}_{U,V} = \{x;\ x \in \mathbf{C},\ U \le x(t) \le V$ for $t \in \langle a-h, a\rangle\}$ and $\mathbf{C}_H = \{x;\ x \in \mathbf{C},\ \|x\|_0 \le H\}$.

For any continuous function $x : \langle a-h, b\rangle \to \mathbb{R}$ and each $t \in \langle a, b\rangle$, we denote by x_t the element of $\mathbf{C}$ defined by

$$x_t(s) = x(t + s - a), \qquad s \in \langle a-h, a\rangle.$$

Consider the functional differential equation

$$x'' = F(t, x, x_t, x', x'_t, \lambda) \tag{11}$$

depending on the parameter λ. Here $F : \langle a, b\rangle \times \mathbb{R} \times \mathbf{C} \times \mathbb{R} \times \mathbf{C} \times \langle A, B\rangle \to \mathbb{R}$ is a continuous locally bounded operator.

Consider boundary condition (2) with $\gamma(x) = x(a)$ for $x \in X$, that is, the boundary condition

$$x(a) = x'(a) = x'(b) = 0. \tag{12}$$

We say that x is a solution of BVP (11), (12) with an initial value $\varphi \in \mathbf{D}_0$ at the point $t = a$ if:

a) $x \in C^1(\langle a-h, b\rangle)$ and x'' is continuous on $\langle a, b\rangle$,

b) there exists a $\lambda_0 \in \langle A, B\rangle$ such that x is a solution of (11) for $\lambda = \lambda_0$,

c) x satisfies boundary condition (12),

d) $x_a = \varphi$.

THEOREM 4. *Assume there exist constants $T > 0$, $K < 0$, $L > 0$ such that F satisfies the following assumptions:*

(S_1) $F(t, K, u, 0, v, B) < 0 < F(t, L, u, 0, v, A)$ for $(t, u, v) \in \langle a, b\rangle \times \mathbf{C}_{K,L} \times$

$\mathbf{C}_T$,

(S_2) $|F(t,x,u,y,v,\lambda) \le r(|y|)$ for $(t,x,u,y,v,\lambda) \in \langle a,b) \times \langle K,L\rangle \times \mathbf{C}_{K,L} \times \mathbb{R} \times \mathbf{C}_T \times \langle A,B\rangle$, where $r: \langle 0,\infty) \to (0,\infty)$ is a nondecreasing function such that

$$\int_0^T \frac{sds}{r(s)} > L - K,$$

(S_3) $F(t,.,u,y,v,\lambda)$ is increasing on $\langle K,L\rangle$ for each fixed $(t,u,y,v,\lambda) \in \langle a,b\rangle \times \mathbf{C}_{K,L} \times \langle -T,T\rangle \times \mathbf{C}_T \times \langle A,B\rangle$,

(S_4) $F(t,x,u,y,v,.)$ is increasing on $\langle A,B\rangle$ for each fixed $(t,x,u,y,v) \in \langle a,b\rangle \times \langle K,L\rangle \times \mathbf{C}_{K,L} \times \langle -T,T\rangle \times \mathbf{C}_T$,

(S_5) $F(t,0,u,0,v,A).F(t,0,u,0,v,B) < 0$ for $(t,u,v) \in \langle a,b\rangle \times \mathbf{C}_{K,L} \times \mathbf{C}_T$.

Let $\varphi \in D_0 \cap \mathbf{C}_{K,L}$, $\varphi' \in \mathbf{C}_T$. Then BVP (11), (12) with the initial value φ at the point $t = a$ has a solution x satisfying (6).

Proof. Let $\mathbf{Y}$ be the Banach space of C^1–functions on $\langle a-h,b\rangle$ with the norm $\|x\|_{\mathbf{Y}} = \max\{|x^{(i)}(t)|;\ t \in \langle a-h,b\rangle,\ i = 0,1\}$. Let $\varphi \in \mathbf{D}_0 \cap \mathbf{C}_{K,L}$, $\varphi' \in \mathbf{C}_T$. Set $\mathcal{K}_\varphi = \{x;\ x \in \mathbf{Y},\ x_a = \varphi,\ K \le x(t) \le L,\ |x'(t)| \le T$ for $t \in \langle a,b\rangle\}$. $\mathcal{K}_\varphi$ is a convex closed bunded subset of $\mathbf{Y}$. Let $\alpha \in \mathcal{K}_\varphi$. Then the function $f: \langle a,b\rangle \times \mathbb{R}^2 \times \langle A,B\rangle \to \mathbb{R}$ defined by $f(t,x,y,\lambda) = F(t,x,\alpha_t,y,\alpha'_t,\lambda)$ is continuous and satisfies assumptions (H_1)-(H_5). By Theorem 1 there exists a unique solution x_α of BVP (1), (12), $K < x_\alpha(t) < L$, $|x'_\alpha(t)| < T$ on $\langle a,b\rangle$. If we define $\widetilde{x}_\alpha : \langle a-h,b\rangle \to \mathbb{R}$ by

$$\widetilde{x}_{\alpha,a} = \varphi, \quad \widetilde{x}_\alpha(t) = x_\alpha(t) \quad \text{for} \quad t \in \langle a,b\rangle,$$

then $\widetilde{x}_\alpha \in \mathcal{K}_\varphi$. Setting $\mathbf{V}(\alpha) = \widetilde{x}_\alpha$ we obtain an operator $\mathbf{V}: \mathcal{K}_\varphi \to \mathcal{K}_\varphi$ and to prove of our theorem it is sufficient to show that V has a fixed point. Let $\{x_n\} \subset \mathcal{K}_\varphi$ be a convergent sequence, $\lim_{n\to\infty} x_n = x$. Set $\mathbf{V}(x_n) = z_n$, $\mathbf{V}(x) = z$. Then a sequence $\{\lambda_n\} \subset \langle A,B\rangle$ and a $\lambda_0 \in \langle A,B\rangle$ exist such that

$$z_n''(t) = F(t,z_n(t),x_{n,t},z'_n(t),x'_{n,t},\lambda_n),$$

$$z''(t) = F(t,z(t),x_t,z'(t),x'_t,\lambda_0)$$

for $t \in \langle a,b\rangle$, $n \in \mathbb{N}$, and

$$z_n(a) = z'_n(a) = z'_n(b) = 0, \quad z(a) = z'(a) = z'(b) = 0,$$

$$z_{n,a} = \varphi, \quad z_a = \varphi$$

for $n \in \mathbb{N}$. Next, we have

$$K < z_n(t) < L, \quad |z'_n(t)| < T, \quad |z''_n(t)| \le M \quad \text{rm for} \quad t \in \langle a,b\rangle, \quad n \in \mathbb{N},$$

where $M := \sup\{|F(t,x,u,y,v,\lambda)|;\ (t,x,u,y,v,\lambda) \in \langle a,b\rangle \times \langle K,L\rangle \times \mathbf{C}_{K,L} \times \langle -T,T\rangle \times \mathbf{C}_T \times \langle A,B\rangle\}$ $(< \infty)$.

Let $\{\widetilde{z}_n\}$ be a subsequence of $\{z_n\}$ and let $\{\widetilde{\lambda}_n\}$ and $\{\widetilde{x}_n\}$ be the corresponding subsequences of $\{\lambda_n\}$ and $\{x_n\}$, respectively. Going if necessary to a subsequence (cf. the Ascoli–Arzela theorem), we can assume that $\{\widetilde{z}_n\}$ and $\{\widetilde{\lambda}_n\}$ are convergent and let $\lim\limits_{n\to\infty} \widetilde{z}_n = w$, $\lim\limits_{n\to\infty} \widetilde{\lambda}_n = \mu_0$. Thus, taking the limit in the equalities

$$\widetilde{z}_n(t) = \int_a^t \int_a^\beta F(s, \widetilde{z}_n(s), \widetilde{x}_{n,s}, \widetilde{z}'_n(s), \widetilde{x}'_{n,s}, \widetilde{\lambda}_n) ds d\beta, \quad t \in \langle a,b\rangle, \quad n \in \mathbb{N},$$

as $n \to \infty$ we have

$$w(t) = \int_a^t \int_a^\beta F(s, w(s), x_s, w'(s), x'_s, \mu_0) ds d\beta, \qquad t \in \langle a,b\rangle,$$

and consequently w is a solution (on $\langle a,b\rangle$) of the equation $u'' = g(t,u,u',\lambda)$ for $\lambda = \mu_0$ satisfying (12) with $x = w$. Here $g(t,u,v,\lambda) = F(t,u,x_t,v,x'_t,\lambda)$ for $(t,u,v,\lambda) \in \langle a,b\rangle \times \mathbb{R}^2 \times \langle a,b\rangle$. By Theorem 1 a unique solution of the above BVP exists and therefore $w = z$ and $\mu_0 = \lambda_0$. This proves that $\{z_n\}$ is convergent and $\lim\limits_{n\to\infty} z_n = z$, that is, $\mathbf{V}$ is a continuous operator.

Since $\mathbf{V}\ (\mathcal{K}_\varphi) \subset \{x;\ x \in \mathcal{K}_\varphi,\ \ x'' \text{ is continuous on } \langle a,b\rangle,\ \ |x''(t)| \le M \text{ for } t \in \langle a,b\rangle\} =: \mathcal{L}$ and $\mathcal{L}$ is a compact subset of $\mathbf{Y}$, $\mathbf{V}\ (\mathcal{K}_\varphi)$ is relative compact in $\mathbf{Y}$. Therefore by the Schauder fixed point theorem there exists a fixed point of $\mathbf{V}$ in $\mathcal{K}_\varphi$.

EXAMPLE 2. Consider the functional differential equation

$$\begin{aligned} x''(t) =& x^3(t)(e^t + |x(t-1)x'(t)|) \\ &+ \frac{(\sin t)^2}{1 + |x'(t-1/2)|} + (1 + (|x(t - \frac{t}{2}))|/2)^{\frac{1}{2}})\lambda, \qquad t \in \langle 0,1\rangle. \end{aligned}$$

We see that (13) is of the form (11) with $F(t,x,u,y,v,\lambda) = x^3(e^t + |u(-1)y|) + \frac{(\sin t)^2}{1+|v(-1/2)|} + (1 + (|u(-t/2)|/2)^{\frac{1}{2}})\lambda$ and $h = 1$.

Assumptions (S_1)-(S_5) are satisfied for $a = 0$, $b = 1$, $K = -2$, $L = 2$, $A = -2$, $B = 2$ and $T = 64$. Thus for each $\varphi \in \mathbf{D}_0 \cap \mathbf{C}_{-2,2}$, $\varphi' \in \mathbf{C}_{64}$ there exists a solution x of BVP (12), (13) with the initial value φ at the point $t = 0$ satisfying

$$|x(t)| < 2, \qquad |x'(t)| < 64 \quad \text{for} \quad t \in \langle 0,1\rangle.$$

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