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ASYMPTOTIC STABILITY OF MARKOV OPERATORS CORRESPONDING TO THE DYNAMICAL SYSTEMS WITH MULTIPLICATIVE PERTURBATIONS

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Abstract. We consider discret time dynamical systems with multiplicative perturbations. We give a sufficient condition for the asymptotic stability of Markov operators on measures generated by dynamical systems with multiplicative perturbations.

Introduction. In this paper we consider a stochastically perturbed discrete time dynamical system of the form $x_{n+1} = S(x_n)\xi_n$, $n = 0, 1, 2, \ldots$, where S is a given Borel measurable transformation, and ξ_n are random variables. The trajectories of our system are sequences of random variables x_n with values in $\mathbb{R}^d$. Systems of this type has been examined recently by K. Horbacz ([1], [2]). She considered the case when ξ_n are continuously distributed with a common density g. In this case x_n are also continuously distributed. K. Horbacz gave a sufficient condition for the convergence of the densities of x_n to a unique stationary density.

We study the same problem without assumption that the common distribution of ξ_n is continuous. In our case x_n are in general random vectors without density. Our aim is to found sufficient conditions for the weak convergence of the distributions of x_n to a stationary measure. The Proof of the main result is based on a theorem of A. Lasota and J.A.Yorke [5] concerning Markov operator on measures.

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Our paper is divided into two sections. Section 1 contains some notations and definitions. The main result is formulated in Section 2.

1. Formulation of the problem. Consider a stochastically perturbed discrete time dynamical system of the form

$$x_{n+1} = S(x_n)\xi_n \quad \text{for} \quad n = 0, 1, 2, \dots \tag{1.0}$$

where S is a Borel measurable transformation of $\mathbb{R}^d$ into itself, and ξ_n are independent random variables with values in $\mathbb{R}_+$.

We assume the following conditions:

(i) The random variables $\xi_0, \xi_1, \dots$ are independent and have the same nontrivial distributions G i.e. G is not concetrated on a single point.

(ii) S is a function which satisfies the Lipschitz condition:

$$|S(x) - S(z)| \le L|x - z| \quad \text{for} \quad x, z \in \mathbb{R}^d$$

where the symbol $|\cdot|$ denotes a norm in $\mathbb{R}^d$.

(iii) There is $\alpha_0 \in (0, 1)$ such that

$$L^{\alpha_0} \int_{\mathbb{R}_+} y^{\alpha_0} G(dy) < 1.$$

(iv) The vector x_0, and variables ξ_i are independent for $i = 0, 1, 2, \dots$.

According to (1.0) the random vector x_n is function of x_0 and $\xi_0, \xi_1, \dots, \xi_{n-1}$. From this and from condition (iv) it follows that x_n and ξ_n are independent. Using this fact we will derive a recurrence formula for the measures

$$\mu_n(A) = \text{Prob}\,(x_n \in A), \qquad A \in B(\mathbb{R}^d). \tag{1.1}$$

Let consider now a bounded Borel measurable function $h : \mathbb{R}^d \to \mathbb{R}$. The expectation $E(z_{n+1})$ of the random vector $z_{n+1} = h(x_{n+1})$, (where $n \ge 0$) is given by

$$E(z_{n+1}) = E(h(x_{n+1})) = \int_{\mathbb{R}^d} h(x)\mu_{n+1}(dx). \tag{1.2}$$

Since $z_{n+1} = h(S(x_n)\xi_n)$this implies

$$E(z_{n+1}) = E(h(S(x_n)\xi_n)) = \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}_+} h(S(x)y)G(dy) \right] \mu_n(dx). \tag{1.3}$$

Comparing (1.2) and (1.3) and setting $h = 1_A$ we obtain:

$$\mu_{n+1}(A) = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}_+} 1_A(S(x)y)G(dy) \right) \mu_n(dx) \quad \text{or} \quad \mu_{n+1}(A) = P\mu_n(A),$$

where

(1.4)
$$P\mu(A) = \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} 1_A(S(x)y)\mu(dx)G(dy).$$

The operator P given by formula (1.4) maps the space M_1, of all probabilistic measures on $\mathbb{R}^d$ into itself and is called the *Markov operator corresponding to the dynamical system* (1.0).
The equation (1.4) can be rewritten in the form

(1.5)
$$P\mu(A) = \int_{\mathbb{R}^d} U1_A\mu(dx)$$

where $U : C_0(\mathbb{R}^d) \to C(\mathbb{R}^d)$ is the operator adjoint to the Markov operator P. By $C_0(\mathbb{R}^d)$ is denoted the space of all real valued continuous functions with compact support, and by $C(\mathbb{R}^d)$ the space of all continuous functions.

The operator U satisfes the following equation:

(1.6)
$$Uf(x) = \int_{\mathbb{R}_+} f(S(x)y)G(dy).$$

Let us define a sequence of functions $T^n(x, y_1, \ldots, y_n)$ by setting:

$$T(x,y) = S(x)y, \quad T^n(x, y_1, \ldots, y_n) = T(T^{n-1}(x, y_1, \ldots, y_{n-1}), y_n).$$

Using this notation we obtain

(1.7)
$$U^n f(x) = \int_{\mathbb{R}_+} \cdots \int_{\mathbb{R}_+} f(T^n(x, y_1, \ldots, y_n))G(dy_1)\cdots G(dy_n)$$

and

(1.8)
$$P^n\mu(A) = \int_{\mathbb{R}^d} U^n 1_A\mu(dx).$$

We introduce the class Φ of functions $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following conditions:

1^o ϕ is continuous and $\phi(0) = 0$;
2^o ϕ is nondecreasing and concave, i.e.

$$\phi\left(\frac{t_1 + t_2}{2}\right) \geq \frac{1}{2}(\phi(t_1) + \phi(t_2)) \quad \text{for} \quad t_1, t_2 \in \mathbb{R}_+;$$

3^o $\phi(t) > 0$ for $t > 0$ and $\phi(t) \to +\infty$ when $t \to +\infty$.

We define the metric ϱ_ϕ in $\mathbb{R}^d$ by the formula:

$$\varrho_\phi(x, y) = \phi(\varrho(x, y)) \quad \text{for} \quad x, y \in \mathbb{R}^d,$$

where ϱ is Euclidean metric and in the space M_1 we define the distance between measures by:

$$\|\mu_1 - \mu_2\|_\phi = \sup_{F_\phi} \left| \int_{\mathbb{R}^d} f(x)\mu_1(dx) - \int_{\mathbb{R}^d} f(x)\mu_2(dx) \right|, \tag{1.9}$$

where F_ϕ is the set of functions such that $|f| \leq 1$ and $|f(x) - f(y)| \leq \varrho_\phi(x, y) = \phi(\varrho(x, y))$.

The space M_1 with the distance $\|\mu_1 - \mu_2\|_\phi$ is a complete metric space and

$$\lim_{n \to +\infty} \|\mu_n - \mu\|_\phi = 0 \quad \text{for} \quad \mu_n, \mu \in M_1 \tag{1.10}$$

holds if and only if the sequence $\{\mu_n\}$ is weakly convergent to μ. The sequence of measures $\{\mu_n\}$ is convergent to μ in $\|\cdot\|_\varphi$ if and only if $\{\mu_n\}$ is convergent to μ in $\|\cdot\|_{\text{id}}$, where id $(x) = x$. Indeed, the identity function id belongs to the set Φ and the metrics ϱ_{id} and ϱ_ϕ define the same topology. From now, $\|\cdot\| = \|\cdot\|_{\text{id}}$.

2. Asymptotic stability. Let P be a Markov operator; a measure $\mu \in M_1$ is called stationary or invariant if $P\mu = \mu$. A Markov operator is called *asymptotically stable* if there exists a stationary distribution μ_* such that

$$\lim_{n \to +\infty} \|P^n\mu - \mu_*\| = 0 \quad \text{for} \quad \mu \in M_1. \tag{2.1}$$

From now we consider $\mathbb{R}^d$ with metric ϱ_ϕ.

We introduce the following definitions:
A Markov operator P is called *nonexpansive* if

$$\|P\mu_1 - P\mu_2\|_\phi \leq \|\mu_1 - \mu_2\|_\phi \quad \text{for} \quad \mu_1, \mu_2 \in M_1.$$

A Markov operator $P : M_1 \to M_1$, satisfies the *Prochorov condition* if there exists a compact set $Y \subset \mathbb{R}^d$ and a number $\beta > 0$ such that

$$\lim_{n\to+\infty} \inf P^n\mu(Y) \geq \beta \quad \text{for} \quad \mu \in M_1. \tag{2.3}$$

From [5] it follows that, if P satisfies the Prochorov condition and P is nonexpansive then the Markov operator P has an invariant measure μ_*.
We can use the following theorem [5]:

THEOREM. *Let P be a nonexpansive Markov operator Assume that for every $\varepsilon > 0$ there is a number $\lambda > 0$ having the following property: for every $\mu_1, \mu_2 \in M_1$ there exists a Borel set A with diam $A < \varepsilon$ and an integer n_0 such that*

$$P^{n_0}\mu_i(A) \geq \lambda \quad \text{for} \quad i = 1, 2. \tag{2.4}$$

Then P satisfes the following condition

$$\lim_{n\to+\infty} \|P^n(\mu_1 - \mu_2)\| = 0 \quad \text{for} \quad \mu_1, \mu_2 \in M_1. \tag{2.5}$$

Now we proof the following auxiliary lemma:

LEMMA 1. *Assume that conditions (i), (ii), (iv) hold for equation (1.0). Suppose that the Markov operator P corresponding to the dynamical system (1.0) satisfes Prochorov condition and the following inequality holds:*

$$L\left(\int_{\mathbb{R}_+} y^\alpha G(dy)\right)^{\frac{1}{\alpha}} \leq 1 \tag{2.6}$$

for some $\alpha \in (0,1)$. Then the Markov operator defined by equation (1.4) is asymptotically stable.

PROOF. First, we prove that the operator P is nonexpansive i.e.

$$\sup_{f\in F_\phi} |\int_{\mathbb{R}_+} Uf(x)\mu_1 - \int_{\mathbb{R}_+} Uf(x)\mu_2| \leq \sup_{f\in F_\phi} |\int_{\mathbb{R}_+} f(x)\mu_1 - f(x)\mu_2|$$

for $\phi(t) = |t|^\alpha$. In order to check it we show that if $f \in F_\phi$ than $Uf \in F_\phi$. Indeed

$$\begin{aligned}|Uf(x) - Uf(z)| \leq & |\int_{\mathbb{R}_+} (f(yS(x)) - f(yS(z)))G(dy)| \\ \leq & \int_{\mathbb{R}_+} \phi(y|S(x) - S(z)|)G(dy) \\ \leq & \int_{\mathbb{R}_+} y^\alpha |S(x) - S(z)|^\alpha G(dy) \\ \leq & |S(x) - S(z)|^\alpha \int_{\mathbb{R}_+} y^\alpha G(dy) \\ \leq & L^\alpha |x - z|^\alpha \int_{\mathbb{R}_+} y^\alpha G(dy) \\ \leq & |x - z|^\alpha = \phi(|x - z|).\end{aligned}$$

Since P is nonexpansive and P satisfies Prochorov condition, the operator P has an invariant measure μ_*.

Now we show that condition (2.4) holds. Fix an $\varepsilon > 0$. Then there exists an integer m such that

$$\phi(r^m \operatorname{diam}_\varrho Y) \leq \varepsilon \tag{2.7}$$

where $0 < r < 1$, Y – compact set satisfying Prochorov condition. Notice that

$$\text{Prob } \left(\xi_n < \left(\int_{\mathbb{R}_+} y^\alpha G(dy)\right)^{\frac{1}{\alpha}}\right) > 0.$$

Thus there exists

$$c < \left(\int_{\mathbb{R}_+} y^\alpha G(dy)\right)^{\frac{1}{\alpha}} \tag{2.8}$$

such that Prob $(\xi_n \leq c) > 0$.

Fix $\widetilde{y} \in [O, c]$. According to (ii) we have

$$\begin{aligned}|T(x, \widetilde{y}) - T(z, \widetilde{y})| =& |S(x)\widetilde{y} - S(z)\widetilde{y}| \\ =& \widetilde{y}|S(x) - S(z)| \leq cL|x - z|.\end{aligned}$$

Conditions (2.8) and (2.6) imply that $cL < 1$. Thus, we can set in (2.7) $r = cL$, $(0 < r < 1)$. Observe that

(2.9)
$$\begin{aligned}|T^m(x,\widetilde{y}_1,\dots,\widetilde{y}_m) - T^m(z,\widetilde{y}_1,\dots,\widetilde{y}_m)| \\ =&|T(T^{m-1}(x,\widetilde{y}_1,\dots,\widetilde{y}_{m-1}),\widetilde{y}_m) - T(T^{m-1}(z,\widetilde{y}_1,\dots,\widetilde{y}_{m-1}),\widetilde{y}_m)| \\ \le& r|T^{m-1}(x,\widetilde{y}_1,\dots,\widetilde{y}_{m-1}) - T^{m-1}(z,\widetilde{y}_1,\dots,\widetilde{y}_{m-1})| \le r^m|x-z|.\end{aligned}$$

where $(\widetilde{y}_1,\dots,\widetilde{y}_m) \in [O,c]^m$ is fixed. Condition (2.9) implies that

$$\text{diam}_\varrho(T^m(Y,\widetilde{y}_1,\dots,\widetilde{y}_m)) \le r^m \text{ diam}_\varrho Y. \tag{2.10}$$

Define

$$A = T^m(Y,\widetilde{y}_1,\dots,\widetilde{y}_m). \tag{2.11}$$

Then

$$\text{diam}_{\varrho_\phi}(A) \le \phi(\text{diam}_\varrho A) \le \phi(r^m \text{ diam}_\varrho Y) < \varepsilon. \tag{2.12}$$

According to Prochorov condition there exists $\overline{n} = \overline{n}(\mu_i)$ such that

$$P^n\mu_i(Y) \ge \beta \quad \text{for} \quad n \ge \overline{n}, \quad i = 1,2. \tag{2.13}$$

Set $n_0 = \overline{n} + m$, then

$$\begin{aligned}P^{n_0}\mu_i(A) =& \int_{\mathbb{R}^d}\int_{\mathbb{R}_+^{n_0}}\!\!\cdots\!\int 1_A(T^{n_0}(x,y_1,\dots,y_{n_0}))\mu_i(dx)G(dy_1)\dots G(dy_{n_0}) \\ \ge& \int_{\mathbb{R}^d}\int_{\mathbb{R}_+^{\overline{n}}\times[0,c]^m}\!\!\cdots\!\int 1_A(T^m(T^{\overline{n}}(x,y_{k_1},\dots,y_{k_{\overline{n}}}),y_{k_{\overline{n}+1}},\dots,y_{k_{n_0}}))\mu_i(dx) \\ &\times G(dy_{k_1})\dots G(dy_{k_{n_0}}).\end{aligned}$$

Define

$$T^{-m}_{(y_1,\dots,y_m)}(A) = \{\omega \in \mathbb{R}^d : T^m(\omega,y_1,\dots,y_m) \in A\}$$

and notice that condition

$$T^m(T^{\overline{n}}(x,y_{k_1},\dots,y_{k_{\overline{n}}}),y_{k_{\overline{n}+1}},\dots,y_{k_{n_0}}) \in A$$

gives

$$T^{\overline{n}}(x,y_{k_1},\dots,y_{k_{\overline{n}}}) \in T^{-m}_{(y_{k_{\overline{n}+1}}},\dots,y_{k_{n_0}})(A).$$

This implies:

$$\begin{aligned} &P^{n_0}\mu_i(A) \\ &\geq (G[O,c])^m \int_{\mathbb{R}^d}\int_{\mathbb{R}^{\overline{n}}_+} 1_{T^{-m}_{[0,c]^m}(A)}(T^{\overline{n}}(x,y_{k_1},\dots,y_{k_{\overline{n}}}))\mu_i(dx) \\ &\quad\times G(dy_{k_1})\cdots G(dy_{k_{\overline{n}}}), \end{aligned}$$

where

$$\begin{aligned} &T^{-m}_{[0,c]^m}(A) \\ &\quad=\{\omega\in\mathbb{R}^d \text{ such that there is } (y_1,\dots,y_m)\in[O,c]^m : \\ &\quad\quad T^m(\omega,y_1,\dots,y_m)\in A\}. \end{aligned}$$

From the definition of the set A it follows that:

$$Y \subset T^{-m}_{[0,c]^m}(A).$$

Conseqently

$$\begin{aligned} &P^{n_0}\mu_i(A) \\ &\geq (G[O,c])^m \int_{\mathbb{R}^d}\int_{\mathbb{R}^{\overline{n}}_+} 1_Y(T^{\overline{n}}(x,y_{k_1},\dots,y_{k_{\overline{n}}}))\mu_i(dx)G(dy_{k_1})\dots G(dy_{k_{\overline{n}}}) \\ &=(G[O,c])^m P^{\overline{n}}\mu_i(Y) \geq (G[O,c])^m\beta > 0, \quad \text{where } m \text{ is fixed.} \end{aligned}$$

If $\lambda = (G[O,c])^m\beta$, than λ satisfies conditions (2.4). Since P is nonexpansive and satisfes conditions (2.4), operator P is asymptotically stable which completes the proof. □

A continuous $V : \mathbb{R}^d \to [0,+\infty)$ is called a *Liapunov function* if

(2.14) $$\lim_{\varrho(x,x_0)\to+\infty} V(x) = +\infty$$

for some $x_0 \in \mathbb{R}^d$.

Now we present an auxiliary proposition concerning the Prochorov condition ([5]).

PROPOSITION 1. *Let P be a Markov operator and let U be a operator dual to P. Assume that there is a Liapunov function V such that*

(2.15) $$UV(x) \leq aV(x) + b \quad \text{for} \quad x \in \mathbb{R}^d$$

where a, b are nonnegative constants and $a < 1$. Then P satisfes the Prochorov condition.

From Lemma 1 and Proposition 1 we have the following.

THEOREM 1. *If conditions (i)-(iv) hold for equation (1.0), then the operator P given by equation (1.4) is asymptotically stable.*

PROOF. Setting $V(x) = |x|^{\alpha_0}$ we have

$$\begin{aligned} UV(x) = & \int_{\mathbb{R}_+} |S(x)y|^{\alpha_0} G(dy) = |S(x)|^{\alpha_0} \int_{\mathbb{R}_+} y^{\alpha_0} G(dy) \\ = & |S(x) - S(x_0) + S(x_0)|^{\alpha_0} \int_{\mathbb{R}_+} y^{\alpha_0} G(dy) \\ \leq & |S(x) - S(x_0)|^{\alpha_0} \int_{\mathbb{R}_+} y^{\alpha_0} G(dy) + |S(x_0)|^{\alpha_0} \int_{\mathbb{R}_+} y^{\alpha_0} G(dy). \end{aligned}$$

Since S satisfies Lipschitz condition (ii), it is easy to notice that following inequalities hold:

$$\begin{aligned} UV(x) \leq & L^{\alpha_0} |x - x_0|^{\alpha_0} \int_{\mathbb{R}_+} y^{\alpha_0} G(dy) + |S(x_0)|^{\alpha_0} \int_{\mathbb{R}_+} y^{\alpha_0} G(dy) \\ \leq & L^{\alpha_0} \int_{\mathbb{R}_+} y^{\alpha_0} G(dy) |x|^{\alpha_0} + L^{\alpha_0} \int_{\mathbb{R}_+} y^{\alpha_0} G(dy) |x_0|^{\alpha_0} \\ & + |S(x_0)|^{\alpha_0} \int_{\mathbb{R}_+} y^{\alpha_0} G(dy). \end{aligned}$$

Thus condition (2.15) holds with

$$a = L^{\alpha_0} \int_{\mathbb{R}_+} y^{\alpha_0} G(dy)$$

and

$$b = (L^{\alpha_0} |x_0|^{\alpha_0} + |S(x_0)|^{\alpha_0}) \int_{\mathbb{R}_+} y^{\alpha_0} G(dy).$$

Consequently Markov operator P corresponding to the dynamical system (1.0) satisfes the Prochorov condition (2.3). According to Lemma 1 the Markov operator P is asymptotically stable. The proof is completed. □

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