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# SOME FORMULE INVOLVING HERMITE POLYNOMIAL, CONFLUENT HYPERGEOMETRIC FUNCTION AND MEIJER'S G-FUNCTION

**Abstract.** In this paper, we evaluate two integrals involving Hermite polynomial and confluent hypergeometric function, and employ one of them to evaluate an integral involving Meijer's $G$-function. We use the integral involving Meijer's $G$-function to evaluate a double integral involving Meijer's $G$-function. We further employ the integral involving Meijer's $G$-function to establish one one-dimensional Fourier-Hermite expansion and one two-dimensional Fourier-Hermite expansion. We also obtain a solution of a heat conduction problem.

**1. Introduction.** The object of this paper is to evaluate two integrals involving Hermite polynomial, confluent hypergeometric function and Meijer's $G$-function, and utilize one of them to evaluate a double integral involving Hermite polynomials, the confluent hypergeometric functions and Meijer's $G$-function. The object of this paper is also to employ the integral involving Meijer's $G$-function to establish one one-dimensional Fourier-Hermite expansion and one two-dimensional Fourier-Hermite expansion for the products of the confluent hypergeometric functions and Meijer's $G$-function. We also obtain a solution of the heat conduction problem due to Bhonsle [3], when initial temperature is given as a product of the confluent hypergeometric function and Meijer's $G$-function.

On specializing the parameters, the $G$-function may be reduced to almost all elementary functions and special functions appearing in applied mathematics, engineering and physical sciences [5, pp. 215—222]. There are also some interesting particular cases of the confluent hypergeometric functions [5, pp. 264—269]. Therefore, the results given in this paper are of a general and applied nature, and hence may encompass several cases of interest.

The following integral is required to evaluate the integral involving Hermite polynomial and the confluent hypergeometric function

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*Manuscript received January 7, 1991.*

AMS (1991) subject classification: Primary 33C15, Secondary 35Q99.

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$$\text{(1.1)} \quad \int_{-\infty}^{\infty} e^{-x^2} x^{2\sigma} H_n(x)\,dx = \frac{2^n \Gamma\left(\sigma+\frac{1}{2}\right)\Gamma(\sigma+1)}{\Gamma\left(\sigma+1-\frac{n}{2}\right)}, \quad 2\sigma = 2k+n, \quad k = 0,1,2,\ldots$$

which follows from [2, p. 10] and [5, p. 4, (11)].

The following integral is required to evaluate the integral involving the $G$-function.

$$\text{(1.2)} \qquad I = \int_{-\infty}^{\infty} e^{-x^2} x^{2\sigma} H_n(x)\, M\left(\frac{n}{2}+1; \frac{3}{2}; x^2\right) dx$$

$$= \frac{2^{n-1}\sqrt{\pi}\,\Gamma\left(\sigma+\frac{1}{2}\right)\Gamma(\sigma+1)\Gamma\left(\frac{3+\sigma}{2}\right)\Gamma\left(-\frac{\sigma+n}{2}\right)}{\Gamma\left(\frac{1-n}{2}\right)\Gamma(\sigma+2)\Gamma\left(\frac{2-\sigma}{2}\right)\Gamma\left(\frac{\sigma-n+1}{2}\right)}, \; 2\sigma = 2k+n, \; k = 0,1,2,\ldots$$

In the integrand, expressing $M\left(\frac{n}{2}+1; \frac{3}{2}; x^2\right) = \sum_{s=0}^{\infty} \frac{\left(\frac{n}{2}+1\right)_s x^{2s}}{\left(\frac{3}{2}\right)_s s!}$, interchanging the order of integration and summation, evaluating the inner-integral with the help of (1.1) and using [5, p. 182, (1)], we obtain

$$I = \frac{2^n \Gamma\left(\sigma+\frac{1}{2}\right)\Gamma(\sigma+1)}{\Gamma\left(\sigma+1-\frac{n}{2}\right)} \, {}_3F_2 \left[\begin{matrix} \sigma+1, & \sigma+\frac{1}{2}, & \frac{n}{2}+1; & 1 \\ \frac{3}{2}, & & \sigma+1-\frac{n}{2} & \end{matrix}\right]$$

Summing the well-poised ${}_3F_2$ of unit argument with the help of Dixon's theorem [5, p. 189, (5)], the right-hand side of (1.2) is obtained.

On following the above procedure the following integral can be established easily

$$\text{(1.3)} \qquad \int_{-\infty}^{\infty} e^{-x^2} x^{2\sigma} H_n(x)\, M\left(\frac{n+1}{2}; \frac{1}{2}; x^2\right) dx$$

$$= \frac{2^n\sqrt{\pi}\,\Gamma\left(\sigma+\frac{1}{2}\right)\Gamma(\sigma+1)\Gamma\left(\frac{5+2\sigma}{4}\right)\left(-\frac{2n+2\sigma+1}{4}\right)}{\Gamma\left(-\frac{n}{2}\right)\Gamma\left(\frac{3}{2}+\sigma\right)\Gamma\left(\frac{1-2\sigma}{4}\right)\Gamma\left(\frac{3-2n+2\sigma}{4}\right)}, \; 2\sigma = 2k+n, \; k = 0,1,2,\ldots$$

NOTE 1. The integrals (1.1), (1.2) and (1.3) appear to be of fundamental importance for evaluating some integrals involving generalized hypergeometric functions of one and several complex variables.

NOTE 2. The multiple-integrals analogous to (1.1), (1.2) and (1.3) may be derived easily with the help of (1.1), (1.2) and (1.3) respectively.

In what follows for safe of brevity $a_p$ stands for $a_1, \dots, a_p$, $\lambda$ and $\mu$ are positive integers and the symbol $\Delta(\lambda, \alpha)$ represents the set of parameters $\frac{\alpha}{\lambda}, \frac{\alpha+1}{\lambda}, \dots, \frac{\alpha+\lambda-1}{\lambda}$.

**2. Integral.** The integral to be evaluated is

$$\text{(2.1)} \qquad \int_{-\infty}^{\infty} e^{-x^2} x^{2\sigma} H_m(x) M\left(\frac{m}{2}+1; \frac{3}{2}; x^2\right) G_{p,q}^{u,v}\left[zx^{2\lambda}\left|\begin{matrix} a_p \\ b_q \end{matrix}\right.\right] dx$$

$$= \frac{2^{m+\sigma-\lambda-3/2}\lambda^{\sigma+m/2-3/2}}{\pi^{\lambda-1}\Gamma\left(\frac{1-m}{2}\right)} G_{p+6\lambda, q+4\lambda}^{u+\lambda, v+5\lambda}\left[z(2\lambda)^{2\lambda}\left|\begin{matrix} \Delta\left(2\lambda, \frac{1}{2}-\sigma\right), \Delta(2\lambda, -\sigma), \\ \Delta\left(\lambda, -\frac{1+\sigma}{2}\right), a_p, \\ \Delta\left(\lambda, 1-\frac{\sigma}{2}\right); \\ \Delta\left(\lambda, -\frac{\sigma+m}{2}\right), b_q, \\ \Delta(2\lambda, -\sigma-1), \Delta\left(\lambda, \frac{1-\sigma+m}{2}\right) \end{matrix}\right.\right]$$

where $2(u+v) > p+q$, $|\arg z| < \left(u+v-\frac{1}{2}p-\frac{1}{2}q\right)\pi$, $2\sigma = 2k+m$, $k = 0, 1, 2, \dots$

Expressing the $G$-function in the integrand as a Mellin-Barnes type integral [5, p. 207, (1)], interchanging the order of integrations, evaluating the inner-integral with the help of (1.2), and using multiplication formula for gamma-function [5, p. 4, (11)] and [5, p. 207, (1)], the value of the integral is obtained.

**3. Double integral.** The integral to be evaluated is

$$\text{(3.1)} \qquad \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} x^{2\sigma} y^{2\rho} H_m(x) H_n(y)$$

$$\times M\left(\frac{m}{2}+1; \frac{3}{2}; x^2\right) M\left(\frac{n}{2}+1; \frac{3}{2}; y\right) G_{p,q}^{u,v}\left[zx^{2\lambda}y^{2\mu}\left|\begin{matrix} a_p \\ b_q \end{matrix}\right.\right] dx\, dy$$

$$= \frac{2^{m+n+\sigma+\rho-\lambda-\mu-3}\lambda^{\sigma+m/2-3/2}\mu^{\rho+n/2-3/2}}{\pi^{\lambda+\mu-2}\Gamma\left(\frac{1-m}{2}\right)\Gamma\left(\frac{1-n}{2}\right)}$$

$$\times G^{u+\lambda+\mu,\, v+5\lambda+5\mu}_{p+6\lambda+6\mu,\, q+4\lambda+4\mu}\left[ z(2\lambda)^{2\lambda}(2\mu)^{2\mu} \left| \begin{matrix} \Delta\left(2\lambda, \frac{1}{2}-\sigma\right), \Delta\left(2\mu, \frac{1}{2}-\rho\right), \\ \Delta(2\lambda, -\sigma), \Delta(2\mu, -\rho), a_p, \\ \Delta\left(\lambda, 1-\frac{\sigma}{2}\right), \Delta\left(\mu, 1-\frac{\rho}{2}\right); \\ \Delta\left(\lambda, -\frac{\sigma+m}{2}\right), \Delta\left(\mu, -\frac{\rho+n}{2}\right), \\ b_q, \Delta(2\lambda, -\sigma-1), \Delta(2\mu, -\rho-1), \\ \Delta\left(\lambda, \frac{1-\sigma+m}{2}\right), \Delta\left(\mu, \frac{1-\rho+n}{2}\right) \end{matrix} \right. \right],$$

where $2(u+v) > p+q$, $|\arg z| < \left(u+v-\frac{1}{2}p-\frac{1}{2}q\right)\pi$, $2\sigma = 2k+n$, $2\rho = 2k+m$, $k = 0, 1, 2, \ldots$

To establish (3.1), evaluating the $x$-integral with the help of (2.1), then evaluating the resulting $y$-integral with the help of (2.1), the value of (3.1) is obtained.

NOTE 3. The multiple-integral analogous to (3.1) can be established easily on applying the above technique repeatedly.

**4. One-dimensional Fourieur-Hermite expansion.** The one-dimensional Fourier-Hermite expansion to be established is

$$x^{2\sigma} M\left(\frac{m}{2}+1; \frac{3}{2}; x^2\right) G^{u,v}_{p,q}\left[ zx^{2\lambda} \left| \begin{matrix} a_p \\ b_q \end{matrix} \right. \right] \tag{4.1}$$

$$= \frac{2^{\sigma-\lambda-3/2}\lambda^{\sigma-3/2}}{\pi^{\lambda-1/2}} \sum_{s=0}^{\infty} \frac{\lambda^{s/2}}{s!\,\Gamma\left(\frac{1-s}{2}\right)} G^{u+\lambda,\, v+5\lambda}_{p+6\lambda,\, q+4\lambda}\left[ z(2\lambda)^{2\lambda} \left| \begin{matrix} \Delta\left(2\lambda, \frac{1}{2}-\sigma\right), \Delta(2\lambda, -\sigma), \\ \Delta\left(\lambda, -\frac{\sigma+1}{2}\right), a_p, \\ \Delta\left(\lambda, 1-\frac{\sigma}{2}\right); \\ \Delta\left(\lambda, -\frac{\sigma+s}{2}\right), b_q \\ \Delta(2\lambda, -\sigma-1), \Delta\left(\lambda, \frac{1-\sigma+s}{2}\right) \end{matrix} \right. \right]$$

$$\times H_s(x),$$

where $2(u+v) > p+q$, $|\arg z| < \left(u+v-\frac{1}{2}p-\frac{1}{2}q\right)\pi$, $2\sigma = 2k+n$, $k = 0, 1, 2, \ldots$

Let

$$f(x) = x^{2\sigma} M\left(\frac{m}{2}+1; \frac{3}{2}; x^2\right) G_{p,q}^{u,v}\left[zx^{2\lambda}\middle| \begin{matrix} a_p \\ b_q \end{matrix}\right] = \sum_{s=0}^{\infty} C_s H_s(x). \tag{4.2}$$

Equation (4.2) is valid, since $f(x)$ is continuous and of bounded variation in the open interval $(-\infty, \infty)$.

Multiplying both sides of (4.2) by $e^{-x^2}H_m(x)$ and integrating with respect to $x$ from $-\infty$ to $\infty$, we get

$$\int_{-\infty}^{\infty} e^{-x^2} x^{2\sigma} H_m(x) M\left(\frac{m}{2}+1; \frac{3}{2}; x^2\right) G_{p,q}^{u,v}\left[zx^{2\lambda}\middle| \begin{matrix} a_p \\ b_q \end{matrix}\right] dx$$

$$= \sum_{s=0}^{\infty} C_s \int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_s(x)\, dx.$$

Now using (2.1) and the orthogonality property of Hermite polynomials [7, p. 837, 7.374 (1)], we have

$$C_m = \frac{2^{\sigma-\lambda-3/2}\lambda^{\sigma+m/2-3/2}}{m!\,\pi^{\lambda-1/2}\,\Gamma\left(\dfrac{1-m}{2}\right)} \tag{4.3}$$

$$\times G_{p+6\lambda,q+4\lambda}^{u+\lambda,v+5\lambda}\left[z(2\lambda)^{2\lambda}\middle| \begin{matrix} \Delta\left(2\lambda, \frac{1}{2}-\sigma\right), \Delta(2\lambda, -\sigma), \\ \Delta\left(\lambda, -\frac{1+\sigma}{2}\right), a_p, \\ \Delta\left(\lambda, 1-\frac{\sigma}{2}\right); \\ \Delta\left(\lambda, -\frac{\sigma+m}{2}\right), b_q, \\ \Delta(2\lambda, -\sigma-1), \Delta\left(\lambda, \frac{1-\sigma+m}{2}\right) \end{matrix}\right].$$

From (4.2) and (4.3), formula (4.1) follows.

**5. Two-dimensional Fourier-Hermite expansion.** The two-dimensional Fourier-Hermite expansion to be established is

$$x^{2\sigma} y^{2\rho} M\left(\frac{m}{2}+1; \frac{3}{2}; x^2\right) M\left(\frac{n}{2}+1; \frac{3}{2}; y^2\right) G_{p,q}^{u,v}\left[zx^{2\lambda}y^{2\mu}\middle| \begin{matrix} a_p \\ b_q \end{matrix}\right] \tag{5.1}$$

$$= \frac{2^{\sigma+\rho-\lambda-\mu-3}\lambda^{\sigma-3/2}\mu^{\rho-3/2}}{\pi^{\lambda+\mu-1}} \sum_{s,t=0}^{\infty} \frac{\lambda^{s/2}\mu^{t/2}}{s!\,t!\,\Gamma\left(\dfrac{1-s}{2}\right)\Gamma\left(\dfrac{1-t}{2}\right)}$$

$$\times G^{u+\lambda+\mu,\, v+5\lambda+5\mu}_{p+6\lambda+6\mu,\, q+4\lambda+4\mu}\left[ z(2\lambda)^{2\lambda}(2\mu)^{2\mu} \middle| \begin{array}{l} \Delta\left(2\lambda, \frac{1}{2}-\sigma\right), \Delta\left(2\mu, \frac{1}{2}-\rho\right), \\ \Delta(2\lambda, -\sigma), \Delta(2\mu, -\rho), \\ \Delta\left(\lambda, -\frac{\sigma+1}{2}\right), \Delta\left(\mu, -\frac{\rho+1}{2}\right), \\ a_p, \Delta\left(\lambda, 1-\frac{\sigma}{2}\right), \Delta\left(\mu, 1-\frac{\rho}{2}\right); \\ \Delta\left(\lambda, -\frac{\sigma+s}{2}\right), \Delta\left(\mu, -\frac{\rho+t}{2}\right), \\ b_q, \Delta(2\lambda, -\sigma-1), \Delta(2\mu, -\rho-1), \\ \Delta\left(\lambda, \frac{1-\sigma+s}{2}\right), \Delta\left(\mu, \frac{1-\rho+t}{2}\right) \end{array} \right].$$

$$\times H_s(x) H_t(y),$$

where $2(u+v) > p+q$, $|\arg z| < \left(u+v-\frac{1}{2}p-\frac{1}{2}q\right)\pi$, $2\sigma = 2k+n$, $2\rho = 2k+m$, $k = 0, 1, 2, \ldots$

Let

$$x^{2\sigma} y^{2\rho} M\left(\frac{m}{2}+1; \frac{3}{2}; x^2\right) M\left(\frac{n}{2}+1; \frac{3}{2}; y^2\right) G^{u,v}_{p,q}\left[ zx^{2\lambda} y^{2\mu} \middle| \begin{matrix} a_p \\ b_q \end{matrix} \right] = \sum_{s,t=0}^{\infty} C_{s,t} H_s(x) H_t(y). \tag{5.2}$$

Multiplying both sides of (5.2) by $e^{-y^2} H_n(y)$, integrating with respect to $y$ from $-\infty$ to $\infty$, and using (2.1) and [7, p. 837, 7.374 (1)], then multiplying both sides of the resulting expression by $e^{-x^2} H_m(x)$, integrating with respect to $x$ from $-\infty$ to $\infty$ and using (2.1) and [7, p. 837, 7.3.4 (1)], we obtain the value of $C_{m,n}$. Substituting the value of $C_{s,t}$ in (5.2), formula (5.1) follows.

NOTE 4. The $n$-dimensional Fourier-Hermite expansion analogous to (5.1) can be established easily on applying the above technique repeatedly.

**6. Heat conduction and Hermite polynomials.** Hermite polynomials have been utilized by Kampé de Fériet [8] in solving a heat conduction equation. He obtained four theorems which are of the nature of existence theorems. Bhonsle [3] employed Hermite polynomials in solving the partial differential equation:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - kux^2, \tag{6.1}$$

when $u(x, t)$ tends to zero for large values of $t$ and when $|x| \to \infty$.

This equation is related to the problem of heat conduction [4, p. 130]:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - h(u-u_0), \tag{6.2}$$

provided $u_0 = 0$ and $h = kx^2$.

The solution of (6.1) given by Bhonsle [3, p. 360, (2.3)] is

$$u(x,t) = \sum_{s=0}^{\infty} A_s e^{-(1+2s)t - x^2/2} H_s(x). \tag{6.3}$$

We now consider the problem of determining a function $u(x,t)$ if

$$u(x,0) = x^{2\sigma} M\left(\frac{m}{2}+1; \frac{3}{2}; x^2\right) G^{u,v}_{p,q}\left[zx^{2\lambda}\middle| \begin{matrix} a_p \\ b_q \end{matrix}\right]. \tag{6.4}$$

When $t = 0$, then by virtue of (6.3) and (6.4), we have

$$e^{-x^2} x^{2\sigma} M\left(\frac{m}{2}+1; \frac{3}{2}; x^2\right) G^{u,v}_{p,q}\left[zx^{2\lambda}\middle| \begin{matrix} a_p \\ b_q \end{matrix}\right] = \sum_{s=0}^{\infty} A_s e^{-x^2/2} H_s(x). \tag{6.5}$$

Multiplying both sides of (6.5) by $H_m(x)$ and integrating with respect to $x$ from $-\infty$ to $\infty$, using (2.1) and orthogonality property of Hermite polynomials [6, p. 289, (9) & (11)], we obtain the value of $A_m$. Substituting the value of $A_s$ in (6.3), we get

$$u(x,t) = \frac{2^{\sigma-\lambda-2}\lambda^{\sigma-3/2}}{\pi^{\lambda-1/2}} \sum_{s=0}^{\infty} \frac{2^s \lambda^{s/2}}{s!\,\Gamma\left(\frac{1-s}{2}\right)} e^{-(1+2s)t - x^2/2} \tag{6.6}$$

$$\times G^{u+\lambda, v+5\lambda}_{p+6\lambda, q+4\lambda}\left[ z(2\lambda)^{2\lambda} \middle| \begin{matrix} \Delta\left(2\lambda, \frac{1}{2}-\sigma\right), \Delta(2\lambda, -\sigma), \\ \Delta\left(\lambda, -\frac{1+\sigma}{2}\right), a_p, \\ \Delta\left(\lambda, 1-\frac{\sigma}{2}\right); \\ \Delta\left(\lambda, -\frac{s+\sigma}{2}\right), b_q, \\ \Delta(2\lambda, -\sigma-1), \Delta\left(\lambda, \frac{1-\sigma+s}{2}\right) \end{matrix} \right] H_s(x),$$

where $2(u+v) > p+q$, $|\arg z| < \left(u+v-\frac{1}{2}p-\frac{1}{2}q\right)\pi$, $2\sigma = 2k+m$, $k = 0, 1, 2, \ldots$

NOTE 5. Similarly with the help of (2.1), we can obtain a solution of the simple harmonic oscillator problem [1, pp. 172—173], which occurs in quantum mechanics.

NOTE 6. Using (1.3) instead of (1.2), we can obtain a set of results corresponding to the results given in sections 2—6.

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