ANDRZEJ KASPERSKI*

APPROXIMATION OF ELEMENTS OF THE SPACES X^1_φ AND X_φ BY NONLINEAR, SINGULAR KERNELS

Abstract. Let l^φ be a Musielak-Orlicz sequence space. Let X^1_φ and X_φ be the modular spaces of multifunctions generated by l^φ. Let $K_{w,j}: \mathbf{R} \to \mathbf{R}$ for $j = 0, 1, 2, \ldots$, $w \in W$, where W is an abstract set of indices. Assuming certain singularity assumption on the nonlinear kernel $K_{w,j}$ and setting $T_w(F) = (T_w(F)(i))_{i=0}^{\infty}$ with $(T_w(F))(i) = \{\sum_{j=0}^{i} K_{w,j}(f(j)): f(j) \in F(j)\}$, convergence theorems $T_w(F) \xrightarrow[\varphi, \mathscr{W}]{} F$ in X^1_φ and $T_w(F) \xrightarrow[d, \varphi, \mathscr{W}]{} F$ in X_φ are obtained.

1. Introduction. In [3] a general approximation theorem in modular space was obtained and applied to translation operators and linear integral operators in Musielak-Orlicz space L^φ of periodic functions as well as in Musielak-Orlicz space l^φ of sequences. The application in L^φ was extended in [4] to some nonlinear integral operators and in [1] and [2] to some operators in the space X^1_φ of multifunctions generated by L^φ. In [6] an extension of the results of [3] to the case of approximation by some nonlinear operators in the Musielak-Orlicz space l^φ of sequences was obtained. The aim of this note is to obtain an extension of the result of [6] to the case of approximation by some nonlinear operators in the spaces X^1_φ and X_φ generated by l^φ.

Let N be the set of all nonnegative integers. Let l^φ be the Musielak-Orlicz sequence space generated by a modular $\rho(x) = \sum_{i=0}^{\infty} \varphi_i(|x(i)|)$, $x = (x(i))$, with φ-functions φ_i, i.e. $\varphi_i: \mathbf{R}_+ \to \mathbf{R}_+$ and it is a nondecreasing continuous function such that $\varphi_i(u) = 0$ iff $u = 0$ and $\varphi_i(u) \to \infty$ as $u \to \infty$ for every $i \in \mathrm{N}$. Let

$$X = \{F: \mathrm{N} \to 2^{\mathbf{R}}: F(i) \text{ is compact nonempty set for all } i \in \mathrm{N}\}.$$

Let $\underline{f}(F)(i) = \min_{x \in F(i)} x$, $\bar{f}(F)(i) = \max_{x \in F(i)} x$ for all $F \in X$ and all $i \in \mathrm{N}$. Let

$$X^1_\varphi = \{F \in X: F(i) = [a(i), b(i)] \text{ for all } i \in \mathrm{N} \text{ and } a, b \in l^\varphi\}.$$

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* Instytut Matematyki, Politechnika Śląska, ul. Zwycięstwa 42, 44-100 Gliwice.

Let W be an abstract nonempty set of indices and let $\mathscr{W}$ be a filter of subsets of W.

DEFINITION 1. A *function* $g\colon W\to \mathbf{R}$ *tends to zero with respect to* $\mathscr{W}$, written $g(w)\xrightarrow{\mathscr{W}}0$, if for every $\varepsilon>0$ there is a set $W\in\mathscr{W}$ such that $|g(w)|<\varepsilon$ for all $w\in W$.

2. General Lemma.

DEFINITION 2. A family $T=(T_w)_{w\in W}$ of operators $T_w\colon X^1_\varphi\to X^1_\varphi$ will be called $\mathscr{W}$*-bounded* if there exist positive constants $k_1,\dots,k_8$ and a function $g\colon W\to\mathbf{R}_+$ such that $g(w)\xrightarrow{\mathscr{W}}0$ and for all $F,G\in X^1_\varphi$ there is a set $W_{F,G}\in\mathscr{W}$ for which

$$\rho\big(a(\underline{f}(T_w(F))-\underline{f}(T_w(G)))\big)$$
$$\leqslant k_1\rho\big(ak_2(\underline{f}(F)-\underline{f}(G))\big)+k_3\rho\big(ak_4(\underline{f}(F)-\overline{f}(G))\big)+g(w),$$
$$\rho\big(a(\overline{f}(T_w(F))-\overline{f}(T_w(G)))\big)$$
$$\leqslant k_5\rho\big(ak_6(\underline{f}(F)-\underline{f}(G))\big)+k_7\rho\big(ak_8(\overline{f}(F)-\overline{f}(G))\big)+g(w),$$

for all $w\in W_{F,G}$ and every $a>0$.

DEFINITION 3. Let $F_w\in X^1_\varphi$ for every $w\in W$ and let $F\in X^1_\varphi$. We write $F_w\xrightarrow[\varphi,\mathscr{W}]{}F$ if for every $\varepsilon>0$ and every $a>0$ there is a $W\in\mathscr{W}$ such that $\rho\big(a(\underline{f}(F_w)-\underline{f}(F))\big)<\varepsilon$ and $\rho\big(a(\overline{f}(F_w)-\overline{f}(F))\big)<\varepsilon$ for every $w\in W$.

DEFINITION 4. Let $S\subset X^1_\varphi$. We denote

$$S_{\varphi,\mathscr{W}}=\{F\in X^1_\varphi\colon F_w\xrightarrow[\varphi,\mathscr{W}]{}F \text{ for some } F_w\in S,\ w\in W\}.$$

LEMMA 1. *Let* $S\subset X^1_\varphi$ *and let* $T=(T_w)_{w\in W}$ *be* $\mathscr{W}$*-bounded. If* $T_w(F)\xrightarrow[\varphi,\mathscr{W}]{}F$ *for every* $F\in S$, *then* $T_w(F)\xrightarrow[\varphi,\mathscr{W}]{}F$ *for every* $F\in S_{\varphi,\mathscr{W}}$.

Proof. Let $a,\varepsilon>0$ be arbitrary and let $F\in S_{\varphi,\mathscr{W}}$ be given. Then there exist $G\in S$ and $W_1\in\mathscr{W}$ such that: $\rho\big(3ak_2(\underline{f}(F)-\underline{f}(G))\big)<\varepsilon/6k_1$, $\rho\big(3ak_4(\overline{f}(F)-\overline{f}(G))\big)<\varepsilon/6k_3$, $\rho\big(3a(\underline{f}(T_w(G))-\underline{f}(G))\big)<\varepsilon/6$, $\rho\big(3a(\overline{f}(T_w(G))-\overline{f}(G))\big)<\varepsilon/6$, $\rho\big(3a(\underline{f}(F)-\underline{f}(G))\big)<\varepsilon/6$, $\rho\big(3a(\overline{f}(F)-\overline{f}(G))\big)<\varepsilon/6$, $g(w)<\varepsilon/6$ for every $w\in\mathscr{W}_1$, where we may assume $k_1,k_3\geqslant 1$. Let $W_{F,G}$ be chosen for $(T_w)_{w\in W}$ and F,G according to the definition of $\mathscr{W}$-boundedness. Then we have

$$\rho\big(a(\underline{f}(T_w(F))-\underline{f}(F))\big)\leqslant\rho\big(3a(\underline{f}(T_w(F))-\underline{f}(T_w(G))\big)$$
$$+\rho\big(3a(\underline{f}(T_w(G))-\underline{f}(G))\big)+\rho\big(3a(\underline{f}(F)-\underline{f}(G))\big)$$
$$\leqslant k_1\rho\big(3ak_2(\underline{f}(F)-\underline{f}(G))\big)+k_3\rho\big(3ak_4(\overline{f}(F)-\overline{f}(G))\big)$$
$$+\rho\big(3a(\underline{f}(T_w(G))-\underline{f}(G))\big)+\rho\big(3a(\underline{f}(F)-\underline{f}(G))\big)+g(w).$$

Taking $W=W_1\cap W_{F,G}$ we obtain $\rho\big(a(\underline{f}(T_w(F))-\underline{f}(F))\big)<\varepsilon$ for all $w\in W$. We prove analogously that there exists a $W\in\mathscr{W}$ such that $\rho\big(a(\overline{f}(T_w(F))-\overline{f}(F))\big)<\varepsilon$ for every $w\in W$. Hence $T_w(F)\xrightarrow[\varphi,\mathscr{W}]{}F$ because $W_0=W\cap W\in\mathscr{W}$.

3. The application. Let now $W = \mathbf{N}$ and let the filter $\mathscr{W}$ consist of all sets $W \subset W$ which are complements of finite sets.

Let now φ_i be convex for $i \in W$. Let for every $w \in W$ $K_{w,j}$: $\mathbf{R} \to \mathbf{R}$ for $j \in W$, and let $K_{w,j}(0) = 0$ for all $w, j \in W$. We define for all $F \in X_\varphi^1$ and $i \in W$

$$(T_w(F))(i) = \Big\{\sum_{j=0}^{i} K_{w,i-j}(f(j)):\ f(j) \in F(j),\ \ j = 0, 1, \ldots, i\Big\},$$

(1)
$$T_w(F) = ((T_w(F))(i))_{i=0}^{\infty}.$$

We shall call K a *semisingular kernel*, if the following conditions are satisfied, where

$$L_{w,i} = \sup_{u \neq v} \frac{|K_{w,i}(u) - K_{w,i}(v)|}{|u-v|}:$$

(i) $L(w) = (\sum_{i=0}^{\infty} L_{w,i}) \leqslant \sigma < \infty$,

(ii) $L_{w,j}/L(w) \xrightarrow{\mathscr{W}} 0$ for $j = 1, 2, \ldots$

If moreover $(1/c)K_{w,0}(c) \xrightarrow{\mathscr{W}} 1$ for every $c \neq 0$, then K will be called a *singular kernel.*

DEFINITION 5. The sequence $(\varphi_i)_{i=0}^{\infty}$ is called τ_+*-bounded* if there exist constants $k_1, k_2 \geqslant 1$ and a double-sequence $(\varepsilon_{i,j})$ such that $\varphi_{i+j}(u) \leqslant k_1 \varphi_i(k_2 u) + \varepsilon_{i,j}$ for $u \geqslant 0$, $i, j \in W$, where $\varepsilon_{i,j} \geqslant 0$, $\varepsilon_{i,0} = 0$, $\varepsilon_j = \sum_{i=0}^{\infty} \varepsilon_{i,j} \to 0$ as $j \to \infty$, $\varepsilon = \sup \varepsilon_j < \infty$.

THEOREM 1. *If K is a semisingular kernel such that $K_{w,i}(s) \geqslant K_{w,i}(t)$ for all $i, w \in W$ and $s > t$, $\varphi = (\varphi_i)_{i=0}^{\infty}$ is τ_+-bounded, then T_w: $X_\varphi^1 \to X_\varphi^1$ for every $w \in W$ and the family T of operators defined by* (1) *is $\mathscr{W}$-bounded.*

Proof. It is easy to see that

$$\underline{f}(T_w(F))(i) = \sum_{j=0}^{i} K_{w,i-j}(\underline{f}(F)(j))$$

and

$$\bar{f}(T_w(F))(i) = \sum_{j=0}^{i} K_{w,i-j}(\bar{f}(F)(j)).$$

$T_w(F)(i)$ is convex because $K_{w,i}$ is continuous for all $w, i \in W$. Let $c > 0$ be arbitrary. Then for $F, G \in X_\varphi^1$ we have (see the proof of Theorem 1 in [6])

$$\rho\big(c(\underline{f}(T_w(F)) - \underline{f}(T_w(G)))\big) \leqslant \sum_{i=0}^{\infty} \varphi_i\Big(c \sum_{j=0}^{i} L_{w,j} |\underline{f}(F)(i-j) - \underline{f}(G)(i-j)|\Big)$$
$$\leqslant k_1 \rho\big(ck_2\sigma(\underline{f}(F) - \underline{f}(G))\big) + g(w),$$

$$\rho\big(c(\bar{f}(T_w(F))-\bar{f}(T_w(G)))\big) \leqslant k_1\rho\big(ck_2\sigma(\bar{f}(F)-\bar{f}(G))\big)+g(w),$$

where $g(w)=\dfrac{1}{L(w)}\sum_{j=1}^{\infty} L_{w,j}\varepsilon_j \xrightarrow{\mathscr{W}} 0$. So $T_w: X^1_\varphi \to X^1_\varphi$ and T is $\mathscr{W}$-bounded.

Now, given a kernel K and a number $c \neq 0$, let us denote

$$x^j_w(c) = (\underbrace{0, 0, \ldots, 0}_{j+1 \text{ times}}, K_{w,1}(c), K_{w,2}(c), \ldots).$$

Moreover, let us write

$$e_k = (\delta_{i,k})_{i=0}^{\infty} \quad \text{with } \delta_{i,k}=1 \text{ for } i=k, \quad \delta_{i,k}=0 \text{ for } i\neq k,$$

$$E_k = (\Delta_{i,k})_{i=0}^{\infty} \quad \text{with } \Delta_{i,k}=[0,1] \text{ for } i=k, \quad \Delta_{i,k}=0 \text{ for } i\neq k.$$

LEMMA 2. *If* $F = c_0e_0+\underline{c}_0E_0+\ldots+c_ne_n+\underline{c}_nE_n$, *then for every* $b>0$

$$\rho\big(b2^{-1}(n+1)^{-1}(\underline{f}(T_w(F))-\underline{f}(F))\big) \leqslant \frac{3}{2}\sum_{j=0}^{n}\rho\big(bx^j_w(d_j)\big)+\frac{1}{2}\sum_{j=0}^{n}\varphi_j\big(b\,|K_{w,0}(d_j)-d_j|\big)$$

and

$$\rho\big(b2^{-1}(n+1)^{-1}(\bar{f}(T_w(F))-\bar{f}(F))\big) \leqslant \frac{3}{2}\sum_{j=0}^{n}\rho\big(bx^j_w(\underline{d}_j)\big)+\frac{1}{2}\sum_{j=0}^{n}\varphi_j\big(b\,|K_{w,j}(\underline{d}_j)-\underline{d}_j|\big),$$

where

$$d_j=\begin{cases} c_j+\underline{c}_j & \textit{for } \underline{c}_j\leqslant 0,\\ c_j & \textit{for } \underline{c}_j>0,\end{cases} \qquad \underline{d}_j=\begin{cases} c_j & \textit{for } \underline{c}_j\leqslant 0,\\ c_j+\underline{c}_j & \textit{for } \underline{c}_j>0.\end{cases}$$

Proof. It is easily seen that

$$\underline{f}(T_w(F))(i)-\underline{f}(F)(i)=\begin{cases}\sum_{j=0}^{i}K_{w,i-j}(d_j)-d_i & \text{for } i\leqslant n,\\ \sum_{j=0}^{n}K_{w,i-j}(d_j) & \text{for } i>n,\end{cases}$$

and

$$\bar{f}(T_w(F))(i)-\bar{f}(F)(i)=\begin{cases}\sum_{j=0}^{i}K_{w,i-j}(\underline{d}_j)-\underline{d}_i & \text{for } i\leqslant n,\\ \sum_{j=0}^{n}K_{w,i-j}(\underline{d}_j) & \text{for } i>n.\end{cases}$$

So the proof is quite analogous to that of Lemma in [6] and we omit it.

We easy obtain (see [5, 8.13 and 8.14]) the following

LEMMA 3. *Let* $\varphi = (\varphi_i)_{i=0}^{\infty}$ *satisfy the condition* (δ_2). *Let* $F \in X_{\varphi}^{1}$ *and* $F = (F(i))_{i=0}^{\infty}$. *Let* F_w *be such that* $F_w(i) = F(i)$ *for* $i = 0, 1, \dots, w$, $F_w(i) = 0$ *for* $i > w$ *for every* $w \in W$, *then* $F_w \xrightarrow[\varphi,\mathscr{W}]{} F$.

THEOREM 2. *Let* $\varphi = (\varphi_i)_{i=0}^{\infty}$ *satisfy the condition* (δ_2). *Let* K *be a singular kernel such that* $\rho(bx_w^j) \xrightarrow{\mathscr{W}} 0$ *for every* $j \in \mathrm{N}$ *and all* $b > 0$. *Let the assumptions of Theorem 1 hold. Then* $T_w(F) \xrightarrow[\varphi,\mathscr{W}]{} F$ *for every* $F \in X_{\varphi}^{1}$.

Proof. Let $S = \{c_0 e_0 + \underline{c}_0 E_0 + \dots + c_n e_n + \underline{c}_n E_n \colon n \in \mathrm{N}\}$. From the assumptions and from Lemma 2 we easily obtain that $T_w(F) \xrightarrow[\varphi,\mathscr{W}]{} F$ for every $F \in S$. From the assumptions and from Lemma 3, $S_{\varphi,\mathscr{W}} = X_{\varphi}^{1}$, so, from Lemma 1 and Theorem 1, $T_w(F) \xrightarrow[\varphi,\mathscr{W}]{} F$ for every $F \in X_{\varphi}^{1}$.

4. A generalization of General lemma. Let

$$X_{\varphi} = \{F \in X \colon \underline{f}(F), \bar{f}(F) \in l^{\varphi}\}.$$

REMARK 1. *If* $F, G \in X_{\varphi}$ *and* $a \in \mathbf{R}$, *then* $F + G \in X_{\varphi}$.

Proof. Let $F, G \in X_{\varphi}$ and $a \in \mathbf{R}$. If $F(i)$ and $G(i)$ are compact, then $F(i) + G(i)$ and $aF(i)$ are compact. $\underline{f}(F+G)(i) = \underline{f}(F)(i) + \underline{f}(G)(i)$ and $\bar{f}(F+G)(i) = \bar{f}(F)(i) + \bar{f}(G)(i)$ for every $i \in \mathrm{N}$, so $F + G \in X_{\varphi}$. If $a = 0$, then $aF \in X_{\varphi}$. If $a > 0$, then $\underline{f}(aF)(i) = a\underline{f}(F)(i)$, $\bar{f}(aF)(i) = a\bar{f}(F)(i)$. If $a < 0$, then $\underline{f}(aF)(i) = a\bar{f}(F)(i)$, $\bar{f}(aF)(i) = a\underline{f}(F)(i)$. So $aF \in X_{\varphi}$.

Let

$$d(A, B) = \max\{\max_{x \in A} \min_{y \in B} |x - y|,\ \max_{y \in B} \min_{x \in A} |x - y|\}$$

for all compact nonempty $A, B \subset \mathbf{R}$.

For all $F, G \in X_{\varphi}$ we define the function $D(F, G)$ by the formula

$$D(F, G)(i) = d(F(i), G(i)) \quad \text{for every } i \in \mathrm{N}.$$

Now, we introduce the function O by the formula

$$O(i) = 0 \quad \text{for every } i \in \mathrm{N}.$$

REMARK 2. *If* $F, G \in X_{\varphi}$, *then* $D(F, G) \in l^{\varphi}$.

Proof. Let $F, G \in X_{\varphi}$. We have for every $a > 0$

$$\rho(aD(F, G)) \leqslant \rho(a(D(F, O) + D(G, O))) \leqslant \rho(2aD(F, O)) + \rho(2aD(G, O))$$
$$\leqslant \rho(4a\underline{f}(F)) + \rho(4a\bar{f}(F)) + \rho(4a\underline{f}(G)) + \rho(4a\bar{f}(G)).$$

So $D(F, G) \in l^{\varphi}$.

DEFINITION 2′. A family $T = (T_w)_{w \in \mathscr{W}}$ of operators $T_w \colon X_{\varphi} \to X_{\varphi}$ will be called $(d, \mathscr{W})$ – *bounded* if there exist positive constants k_1, k_2 and a function $g \colon W \to \mathbf{R}_+$ such that $g(w) \xrightarrow{\mathscr{W}} 0$ and for $F, G \in X_{\varphi}$ there is a set $W_{F,G} \in \mathscr{W}$ for which

$$\rho\big(aD(T_w(F), T_w(G))\big) \leqslant k_1\rho\big(ak_2D(F, G)\big)+g(w) \quad \text{for all } w\in W_{F,G},\ a>0.$$

DEFINITION 3′. Let $F_w \in X_\varphi$ for every $w\in W$ and let $F\in X_\varphi$. We write $F_w \xrightarrow[d,\varphi,\mathscr{W}]{} F$ if for every $\varepsilon>0$ and every $a>0$ there is a $W\in\mathscr{W}$ such that $\rho\big(aD(F_w, F)\big)<\varepsilon$ for every $w\in W$.

DEFINITION 4′. Let $S\subset X_\varphi$. We denote

$$S_{d,\varphi,\mathscr{W}} = \{F\in X_\varphi\colon F_w \xrightarrow[d,\varphi,\mathscr{W}]{} F \text{ for some } F_w\in S,\ w\in W\}.$$

LEMMA 1′. *Let* $S\subset X_\varphi$ *and let* $T=(T_w)_{w\in W}$ *be* $(d,\mathscr{W})$*-bounded. If* $T_w(F)\xrightarrow[d,\varphi,\mathscr{W}]{} F$ *for every* $F\in S$, *then* $T_w(F)\xrightarrow[d,\varphi,\mathscr{W}]{} F$ *for every* $F\in S_{d,\varphi,\mathscr{W}}$.

Proof. Let $a,\varepsilon>0$ be arbitrary and let $F\in S_{d,\varphi,\mathscr{W}}$ be given. Then there exist $G\in S$ and $W_1\in\mathscr{W}$ such that $\rho\big(3ak_2D(T_w(F), F)\big)<\varepsilon/6k_1$, $\rho\big(3aD(T_w(G), G)\big)<\varepsilon/6$, $\rho\big(3aD(F, G)\big)<\varepsilon/6$, $g(w)<\varepsilon/6$ for every $w\in W_1$, where we may assume $k_1\geqslant 1$. Let $W_{F,G}$ be chosen for T and F, G according to the definition of $(d,\mathscr{W})$-boundedness. We have

$$\begin{aligned}\rho\big(aD(T_w(F), F)\big) &\leqslant \rho\big(3aD(T_w(F), T_w(G))\big)+\rho\big(3aD(T_w(G), G)\big)+\rho\big(3aD(F, G)\big)\\ &\leqslant k_1\rho\big(3ak_2D(F, G)\big)+g(w)+\rho\big(3aD(F, G)\big)+\rho\big(3aD(T_w(G), G)\big).\end{aligned}$$

Taking $W=W_1\cap W_{F,G}$ we obtain $\rho\big(aD(T_w(F), F)\big)<\varepsilon$ for all $w\in W$.

5. The application. Let $\varphi, W, \mathscr{W}$ be such that as in section 3. Let for every $w\in W K_{w,j}\colon \mathbf{R}\to\mathbf{R}$ for $j\in\mathrm{N}$ and let $K_{w,j}(0)=0$ for all $w, j\in W$. We define for all $F\in X_\varphi$ and all compact nonempty $A\subset\mathbf{R}$ the operators $\mathbf{K}_{w,i}$, T_w by the formulas

$$K_{w,i}(A)=\{K_{w,i}(x)\colon x\in A\} \quad \text{for all } w, i\in W, \tag{2}$$

$$(T_w(F))(i)=\sum_{j=0}^{i}\mathbf{K}_{w,i-j}(F(j)), \quad T_w(F)=\big((T_w(F))(i)\big)_{i=0}^{\infty} \tag{3}$$

for all $i, w\in W$.

We shall call K *d-semisingular kernel*, if the following conditions are satisfied:

(i) $L(w)=\big(\sum_{i=0}^{\infty}L_{w,i}\big)\leqslant\sigma<\infty$,

(ii) $L_{w,j}/L(w)\xrightarrow{\mathscr{W}}0$ for $j=1,2,\ldots$

where

$$L_{w,i}=\sup_{A\neq B}\frac{d\big(\mathbf{K}_{w,i}(A), \mathbf{K}_{w,i}(B)\big)}{d(A, B)}$$

for all $w\in W$ and $i\in\mathrm{N}$, with compact nonempty $A, B\subset\mathbf{R}$.

If moreover $d\big(\mathbf{K}_{w,0}(A), A\big)\xrightarrow{\mathscr{W}}0$ for every compact nonempty $A\subset\mathbf{R}$, then K will be called the *d-singular kernel*.

THEOREM 1′. *Let K be a d-semisingular kernel. Let $\varphi = (\varphi_i)_{i=0}^{\infty}$ be τ_+-bounded. If $K_{w,i}(s) \geq K_{w,i}(t)$ for all $w, i \in W$ and $s > t$, then $T_w: X_\varphi \to X_\varphi$ for every $w \in W$ and the family T given by* (3) *is $(\boldsymbol{d}, \mathscr{W})$-bounded.*

Proof. It is easy to see that $T_w: l^\varphi \to l^\varphi$ for every $w \in W$ (see [6, Theorem 1]). $K_{w,i}$ is continuous for all $w, i \in W$ so we easily obtain that $T_w: X_\varphi \to X_\varphi$ for every $w \in W$. Now, we prove that T is $(\boldsymbol{d}, \mathscr{W})$-bounded. Let $a > 0$ be arbitrary. Then for $F, G \in X_\varphi$ we have (see also [6, the proof of Theorem 1])

$$\begin{aligned}
\rho\big(aD(T_w(F), T_w(G))\big) &= \sum_{i=0}^{\infty} \varphi_i\Big(a\boldsymbol{d}\Big(\sum_{j=0}^{i} \boldsymbol{K}_{w,i-j}(F(j)), \sum_{j=0}^{i} \boldsymbol{K}_{w,i-j}(G(j))\Big)\Big) \\
&\leq \sum_{i=0}^{\infty} \varphi_i\Big(a\Big(\sum_{j=0}^{i} \boldsymbol{d}\big(\boldsymbol{K}_{w,i-j}(F(j)), \boldsymbol{K}_{w,i-j}(G(j))\big)\Big)\Big) \\
&= \sum_{i=0}^{\infty} \varphi_i\Big(a\Big(\sum_{j=0}^{i} \boldsymbol{d}\big(\boldsymbol{K}_{w,j}(F(i-j)), \boldsymbol{K}_{w,j}(G(i-j))\big)\Big)\Big) \\
&\leq \sum_{i=0}^{\infty} \varphi_i\Big(a\Big(\sum_{j=0}^{i} L_{w,j}\boldsymbol{d}\big(F(i-j), G(i-j)\big)\Big)\Big) \\
&\leq \frac{1}{L(w)} \sum_{j=0}^{\infty} \sum_{i=j}^{\infty} L_{w,j}\varphi_j\big(aL(w)\,\boldsymbol{d}(F(i-j), G(i-j))\big) \\
&= \frac{1}{L(w)} \sum_{j=0}^{\infty} L_{w,j} \sum_{i=0}^{\infty} \varphi_{i+j}\big(aL(w)\,\boldsymbol{d}(F(i), G(i))\big) \\
&\leq k_1 \rho\big(ak_2\sigma D(F, G)\big) + g(w),
\end{aligned}$$

where

$$g(w) = \frac{1}{L(w)} \sum_{j=1}^{\infty} L_{w,j}\varepsilon_j \xrightarrow{\mathscr{W}} 0.$$

Now, let us write $E_k = (A_{i,k})_{i=0}^{\infty}$ with $A_{i,k} = A_k$ for $i = k$, $k \in \mathbf{N}$, where $A_k \subset \mathbf{R}$ and A_k is compact nonempty for every $k \in \mathbf{N}$ and $A_{i,k} = 0$ for $i \neq k$. Moreover, let us write for every compact nonempty $A \subset \mathbf{R}$

$$x_w^j(A) = (\underbrace{0, 0, \ldots, 0}_{j+1 \text{ times}}, \boldsymbol{K}_{w,1}(A), \boldsymbol{K}_{w,2}(A), \ldots). \tag{4}$$

LEMMA 2′. *If $F = E_0 + E_1 + \ldots + E_n$, then for every $b > 0$ the inequality*

$$\rho\big(b^{-1}(n+1)^{-1}D(T_w(F), F)\big) \leq \frac{3}{2}\sum_{j=0}^{n} \rho\big(bD(x_w^j(A_{j,j}), O)\big) + \frac{1}{2}\sum_{j=0}^{n} \varphi_j\big(b\boldsymbol{d}(\boldsymbol{K}_{w,0}(A_{j,j}), A_{j,j})\big)$$

holds.

Proof. For any $a > 0$ we have (see also [6, the proof of Lemma])

$$\rho\big(aD(T_w(F),F)\big)$$
$$= \sum_{j=0}^{n} \varphi_i\{ad(\sum_{j=0}^{i} K_{w,i-j}(A_{j,j}), A_{j,j})\} + \sum_{i=n+1}^{\infty} \varphi_i\{ad(\sum_{j=0}^{n} K_{w,i-j}(A_{j,j}), 0)\}$$
$$\leqslant \sum_{i=1}^{n} \varphi_i\{a(\sum_{j=0}^{i-1} d(K_{w,i-j}(A_{j,j}), 0) + d(K_{w,0}(A_{i,i}), A_{i,i}))\}$$
$$+\varphi_0(ad\,K_{w,0}(A_{0,0}), A_{0,0})) + \sum_{i=n+1}^{\infty} \varphi_i\{a(\sum_{j=0}^{n} d(K_{w,i-j}(A_{j,j}), 0))\}$$
$$\leqslant \frac{3}{2}\sum_{j=0}^{n}\sum_{i=1}^{\infty} \varphi_{i+j}\{2a(n+1)\,d(K_{w,i}(A_{j,j}), 0)\}$$
$$+\frac{1}{2}\sum_{i=0}^{n} \varphi_i\{2a(n+1)\,d(K_{w,0}(A_{i,i}), A_{i,i})\}.$$

We obtain the assertion after writing $b = 2a(n+1)$.

LEMMA 3′. *Let $\varphi = (\varphi_i)_{i=0}^{\infty}$ satisfy the condition (δ_2). Let $F \in X_\varphi$ and $F = (F(i))_{i=0}^{\infty}$. Let F_w be such that $F_w(i) = F(i)$ for $i = 0, 1, \ldots, w$ and $F_w(i) = 0$ for $i > w$ for every $w \in W$, then $F_w \xrightarrow[d,\varphi,\mathscr{W}]{} F$.*

Proof. We have for every $a > 0$

$$\rho(aD(F_w, F)) = \sum_{i=w+1}^{\infty} \varphi_i(ad(F(i), 0))$$
$$= \sum_{i=w+1}^{\infty} \varphi_i(a\max(|\underline{f}(F)(i)|, |\bar{f}(F)(i)|)) \overset{\mathscr{W}}{\to} 0.$$

THEOREM 2′. *Let the assumptions of Lemmas 2′, 3′ and Theorem 1′ hold. Let K be the d-singular kernel. If for every compact and nonempty $A \subset \mathbf{R}$ $\rho(bD(x_w^j(A), O)) \overset{\mathscr{W}}{\to} 0$ for all $b > 0$ and every $j \in \mathrm{N}$, then $T_w(F) \xrightarrow[d,\varphi,\mathscr{W}]{} F$ for every $F \in X_\varphi$.*

Proof. Let $S = \{E_0 + E_1 + \ldots + E_n\colon n \in \mathrm{N}\}$. From the assumptions $T_w(F) \xrightarrow[d,\varphi,\mathscr{W}]{} F$ for every $F \in S$ and $S_{d,\varphi,\mathscr{W}} = X_\varphi$ so we obtain the assertion from Theorem 1′.

REMARK 3. *If $T_w\colon X_\varphi^1 \to X_\varphi^1$ and the assumptions of Theorem 2′ hold, then $T_w(F) \xrightarrow[d,\varphi,\mathscr{W}]{} F$ for every $F \in X_\varphi^1$.*

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