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QUASI-JENSEN FUNCTIONS

Abstract. There is defined quasi-Jensen function as a solution of a certain functional inequality which generalizes the classical Jensen equation: $f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}$. The introduced inequality is analogous to the inequality which defines J. Tabor's quasi-additive functions. The main result of this paper is to show strong relationship between quasi-Jensen and quasi-additive functions.

0. The subject matter of the present considerations was proposed to me by Professor J. Tabor and it is connected with his papers [2] and [3]. In particular, the paper [3] deals with functions defined on a vector space X taking their values in a normed space Y and satisfying, with some $\varepsilon \geqslant 0$, the condition

(1) $\quad \|f(x+y)-f(x)-f(y)\| \leqslant \varepsilon \cdot \min\{\|f(x+y)\|, \|f(x)+f(y)\|\} \quad \text{for } x, y \in X,$

or in other words the conjunction of inequalities

(2) $\quad \|f(x+y)-f(x)-f(y)\| \leqslant \varepsilon \|f(x+y)\| \quad \text{for } x, y \in X,$

and

(3) $\quad \|f(x+y)-f(x)-f(y)\| \leqslant \varepsilon \|f(x)+f(y)\| \quad \text{for } x, y \in X.$

Inequality (1) is, of course, a generalization of the Cauchy equation. Functions satisfying (1) are called *quasi-additive* functions. The purpose of these considerations is to put forward and to characterize a condition which would be a similar type generalization of the Jensen equation

$$f\left(\frac{x+y}{2}\right)-\frac{f(x)+f(y)}{2}=0 \quad \text{for } x, y \in X.$$

We are going to show the relationships between quasi-additive functions and functions satisfying the generalized Jensen condition mentioned above.

Manuscript received October 26, 1988, and in final form June 12, 1992.

AMS (1992) subject classification: Primary 39B72.

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From now on X will denote a vector space and Y will denote a vector space equipped with norm $\|\cdot\|$.

1. In the first part of the paper we will be considering functions defined on X which take their values in Y ($f: X \to Y$) satisfying, with some fixed $\varepsilon \geqslant 0$, inequality

$$(4)\quad \left\| f\left(\frac{x+y}{2}\right) - \frac{f(x)+f(y)}{2} \right\| \leqslant \varepsilon \cdot \min\left\{ \left\| f\left(\frac{x+y}{2}\right)\right\|, \left\| \frac{f(x)+f(y)}{2} \right\| \right\}, \quad x, y \in X,$$

or equivalently, the conjunction of two inequalities

$$(5)\quad \left\| f\left(\frac{x+y}{2}\right) - \frac{f(x)+f(y)}{2} \right\| \leqslant \varepsilon \left\| f\left(\frac{x+y}{2}\right)\right\| \quad \text{for } x, y \in X,$$

and

$$(6)\quad \left\| f\left(\frac{x+y}{2}\right) - \frac{f(x)+f(y)}{2} \right\| \leqslant \varepsilon \left\| \frac{f(x)+f(y)}{2} \right\| \quad \text{for } x, y \in X.$$

LEMMA 1. *If* $0 \leqslant \varepsilon < 1$ *and* $f: X \to Y$ *satisfies inequality* (5), *then* f *satisfies also inequality*

$$\left\| f\left(\frac{x+y}{2}\right) - \frac{f(x)+f(y)}{2} \right\| \leqslant \frac{\varepsilon}{1-\varepsilon} \left\| \frac{f(x)+f(y)}{2} \right\| \quad \textit{for } x, y \in X.$$

Proof. Using properties of norm and inequality (5) we get

$$\left\| f\left(\frac{x+y}{2}\right)\right\| - \left\| \frac{f(x)+f(y)}{2} \right\| \leqslant \left\| f\left(\frac{x+y}{2}\right) - \frac{f(x)+f(y)}{2} \right\| \leqslant \varepsilon \left\| f\left(\frac{x+y}{2}\right)\right\|,$$

and therefore

$$(1-\varepsilon)\left\| f\left(\frac{x+y}{2}\right)\right\| \leqslant \left\| \frac{f(x)+f(y)}{2} \right\|,$$

so in other words

$$\left\| f\left(\frac{x+y}{2}\right)\right\| \leqslant \frac{1}{1-\varepsilon} \left\| \frac{f(x)+f(y)}{2} \right\|.$$

Owing to the last inequality we get from (5)

$$\left\| f\left(\frac{x+y}{2}\right) - \frac{f(x)+f(y)}{2} \right\| \leqslant \varepsilon \left\| f\left(\frac{x+y}{2}\right)\right\| \leqslant \frac{\varepsilon}{1-\varepsilon} \left\| \frac{f(x)+f(y)}{2} \right\|,$$

which completes the proof.

One can prove analogously

LEMMA 2. *If* $0 \leqslant \varepsilon < 1$ *and* $f: X \to Y$ *satisfies inequality* (6), *then* f *satisfies also the inequality*

$$\left\| f\left(\frac{x+y}{2}\right) - \frac{f(x)+f(y)}{2} \right\| \leqslant \frac{\varepsilon}{1-\varepsilon} \left\| f\left(\frac{x+y}{2}\right)\right\| \quad \textit{for } x, y \in X.$$

Let us notice that Lemma 1 and Lemma 2 establish a certain kind of equivalence between conditions (5) and (6).

PROPOSITION 1. *If* $0 \leqslant \varepsilon < 1$, $f: X \to Y$ *satisfies condition* (5), *and* $f(0) = 0$, *then* f *satisfies inequality*

$$\|f(x+y)-f(x)-f(y)\| \leqslant \frac{2\varepsilon}{1-\varepsilon}\|f(x+y)\| \quad \textit{for } x, y \in X. \tag{7}$$

Proof. Putting $y = 0$ into (5) we get

$$\left\|f\left(\frac{x}{2}\right)-\frac{f(x)}{2}\right\| \leqslant \varepsilon\left\|f\left(\frac{x}{2}\right)\right\|. \tag{8}$$

Using above inequality we can write

$$\left\|f\left(\frac{x}{2}\right)\right\|-\left\|\frac{f(x)}{2}\right\| \leqslant \left\|f\left(\frac{x}{2}\right)-\frac{f(x)}{2}\right\| \leqslant \varepsilon\left\|f\left(\frac{x}{2}\right)\right\|,$$

and hence

$$\left\|f\left(\frac{x}{2}\right)\right\| \leqslant \frac{1}{2(1-\varepsilon)}\|f(x)\|. \tag{9}$$

Now putting $x+y$ for x into (8) and (9) we get

$$\left\|f\left(\frac{x+y}{2}\right)-\frac{f(x+y)}{2}\right\| \leqslant \varepsilon\left\|f\left(\frac{x+y}{2}\right)\right\|, \tag{10}$$

and

$$\left\|f\left(\frac{x+y}{2}\right)\right\| \leqslant \frac{1}{2(1-\varepsilon)}\|f(x+y)\|. \tag{11}$$

Using in turn inequalities (10), (5), (11) we get for every $x, y \in X$

$$\begin{aligned}
&\|f(x+y)-f(x)-f(y)\| \\
&\leqslant 2\left\|f\left(\frac{x+y}{2}\right)-\frac{f(x+y)}{2}\right\|+2\left\|f\left(\frac{x+y}{2}\right)-\frac{f(x)+f(y)}{2}\right\| \\
&\leqslant 2\varepsilon\left\|f\left(\frac{x+y}{2}\right)\right\|+2\varepsilon\left\|f\left(\frac{x+y}{2}\right)\right\| = 4\varepsilon\left\|f\left(\frac{x+y}{2}\right)\right\| \leqslant \frac{2\varepsilon}{1-\varepsilon}\|f(x+y)\|,
\end{aligned}$$

which completes the proof.

Let us notice that if $\varepsilon < 1/3$, then $\frac{2\varepsilon}{1-\varepsilon} < 1$. Now, using Proposition 1 of [2] we conclude that inequality (7) implies

$$\|f(x+y)-f(x)-f(y)\| \leqslant \frac{2\varepsilon}{1-3\varepsilon}\|f(x)+f(y)\| \quad \text{for } x, y \in X.$$

That is why we can state

PROPOSITION 2. *If* $0 \leqslant \varepsilon < 1/3, f: X \to Y$ *satisfies* (5), *and* $f(0) = 0$, *then* f *is a quasi-additive function with* $\varepsilon' = \dfrac{2\varepsilon}{1-3\varepsilon}$ *i.e.,*

$$\|f(x+y)-f(x)-f(y)\| \leqslant \varepsilon' \cdot \min\{\|f(x+y)\|, \|f(x)+f(y)\|\} \quad \textit{for } x, y \in X.$$

Futhermore, if $\varepsilon < 1/5$, *then* $\varepsilon' < 1$.

PROPOSITION 3. *If* $0 \leqslant \varepsilon < 1$ *and* $f: X \to Y$ *satisfies inequality* (2), *then for* $\varepsilon' = \dfrac{2\varepsilon}{1-\varepsilon}$ f *satisfies inequality* (5) *i.e.,*

$$\left\|f\left(\frac{x+y}{2}\right)-\frac{f(x)+f(y)}{2}\right\| \leqslant \frac{2\varepsilon}{1-\varepsilon}\left\|f\left(\frac{x+y}{2}\right)\right\| \quad \textit{for } x, y \in X. \tag{12}$$

Proof. Putting $\dfrac{x+y}{2}$ for x and y into inequality (2) we get

$$\left\|f(x+y)-2f\left(\frac{x+y}{2}\right)\right\| \leqslant \varepsilon \|f(x+y)\|, \tag{13}$$

and hence

$$\left\|\frac{f(x+y)}{2}-f\left(\frac{x+y}{2}\right)\right\| \leqslant \varepsilon \left\|\frac{f(x+y)}{2}\right\|. \tag{14}$$

From inequality (2) we can also get

$$\left\|\frac{f(x+y)}{2}-\frac{f(x)+f(y)}{2}\right\| \leqslant \varepsilon \left\|\frac{f(x+y)}{2}\right\|. \tag{15}$$

Finally using inequality (13) we get

$$\|f(x+y)\|-2\left\|f\left(\frac{x+y}{2}\right)\right\| \leqslant \left\|f(x+y)-2f\left(\frac{x+y}{2}\right)\right\| \leqslant \varepsilon \|f(x+y)\|,$$

and therefore

$$\|f(x+y)\| \leqslant \frac{2}{1-\varepsilon}\left\|f\left(\frac{x+y}{2}\right)\right\|. \tag{16}$$

Now, from inequalities (14), (15), (16), we have

$$\begin{aligned}\left\|f\left(\frac{x+y}{2}\right)-\frac{f(x)+f(y)}{2}\right\| &\leqslant \left\|f\left(\frac{x+y}{2}\right)-\frac{f(x+y)}{2}\right\|+\left\|\frac{f(x+y)}{2}-\frac{f(x)+f(y)}{2}\right\| \\ &\leqslant \varepsilon\left\|\frac{f(x+y)}{2}\right\|+\varepsilon\left\|\frac{f(x+y)}{2}\right\| = \varepsilon\|f(x+y)\| \\ &\leqslant \frac{2\varepsilon}{1-\varepsilon}\left\|f\left(\frac{x+y}{2}\right)\right\|,\end{aligned}$$

which completes the proof.

Of course if $\varepsilon < 1/3$, then $\frac{2\varepsilon}{1-\varepsilon} < 1$, and so by Lemma 1 inequality (12) implies

$$\left\| f\left(\frac{x+y}{2}\right) - \frac{f(x)+f(y)}{2} \right\| \leqslant \frac{2\varepsilon}{1-3\varepsilon} \left\| \frac{f(x)+f(y)}{2} \right\| \quad \text{for } x, y \in X.$$

So we have

PROPOSITION 4. *If* $0 \leqslant \varepsilon < 1/3$ *and* $f\colon X \to Y$ *satisfies* (2), *then for* $\varepsilon' = \frac{2\varepsilon}{1-3\varepsilon}$ *f satisfies inequality* (4) *i.e.,*

$$\left\| f\left(\frac{x+y}{2}\right) - \frac{f(x)+f(y)}{2} \right\| \leqslant \varepsilon' \cdot \min\left\{ \left\| f\left(\frac{x+y}{2}\right) \right\|, \left\| \frac{f(x)+f(y)}{2} \right\| \right\} \quad \textit{for } x, y \in X.$$

Moreover, if $\varepsilon < 1/5$, *then* $\varepsilon' < 1$.

Now we are going to present some properties of functions satisfying (4).

LEMMA 3. *Let us assume that* $f\colon X \to Y$ *satisfies condition* (4). *If there exists an* $x_0 \in X$ *such that* $f(-x_0) = -f(x_0)$, *then the function is odd.*

Proof. Putting $x = x_0$ and $y = -x_0$ into (6) we obtain $\|f(0)\| = 0$. Now, for each $x \in X$ we have

$$\left\| f\left(\frac{x+(-x)}{2}\right) - \frac{f(x)+f(-x)}{2} \right\| \leqslant \varepsilon \left\| f\left(\frac{x+(-x)}{2}\right) \right\|,$$

hence

$$\left\| \frac{f(x)+f(-x)}{2} \right\| \leqslant 0,$$

and therefore

$$f(-x) = -f(x),$$

which completes the proof.

PROPOSITION 5. *Let us assume that* $f\colon X \to Y$ *satisfies inequality* (4) *with* $0 \leqslant \varepsilon < 1$. *Let us define function* $g\colon X \to Y$ *as follows:*

$$g(x) := \frac{f(x)+f(-x)}{2} \quad \textit{for } x \in X.$$

Then the function g is bounded. Moreover, unless g vanishes, values of g are separated from zero.

Proof. Putting $-x$ for y into (5) and (6) we get respectively

$$\left\| f(0) - \frac{f(x)+f(-x)}{2} \right\| \leqslant \varepsilon \|f(0)\| \quad \text{for } x \in X,$$

and

$$\left\| f(0) - \frac{f(x)+f(-x)}{2} \right\| \leqslant \varepsilon \left\| \frac{f(x)+f(-x)}{2} \right\| \quad \text{for } x \in X.$$

So, in accordance with the assumed notation

(17) $$\|f(0)-g(x)\| \leqslant \varepsilon \|f(0)\| \quad \text{for } x \in X,$$

and

(18) $$\|f(0)-g(x)\| \leqslant \varepsilon \|g(x)\| \quad \text{for } x \in X.$$

Using inequality (17) we get

$$\|f(0)\| - \|g(x)\| \leqslant \|f(0)-g(x)\| \leqslant \varepsilon \|f(0)\|,$$

and

$$\|g(x)\| - \|f(0)\| \leqslant \|f(0)-g(x)\| \leqslant \varepsilon \|f(0)\|.$$

Therefore

$$\|g(x)\| \geqslant (1-\varepsilon) \|f(0)\|,$$

and

$$\|g(x)\| \leqslant (1+\varepsilon)\|f(0)\|.$$

In similar way, using (18) we get

$$\|f(0)\| - \|g(x)\| \leqslant \|f(0)-g(x)\| \leqslant \varepsilon \|g(x)\|,$$

and

$$\|g(x)\| - \|f(0)\| \leqslant \|f(0)-g(x)\| \leqslant \varepsilon \|g(x)\|.$$

Hence

$$\|g(x)\| \geqslant \frac{1}{1+\varepsilon} \|f(0)\|,$$

and

$$\|g(x)\| \leqslant \frac{1}{1-\varepsilon} \|f(0)\|.$$

Therefore we can write

$$\frac{1}{1+\varepsilon} \|f(0)\| \leqslant \|g(x)\| \leqslant (1+\varepsilon) \|f(0)\|.$$

In the case where $f(0) = 0$ (by Lemma 3 it is equivalent to the fact that f is odd) from the last inequality we obtain $g \equiv 0$, which completes the proof.

Now we will give some sufficient condition in order to inequality (5), with some fixed $\varepsilon > 0$, holds.

PROPOSITION 6. *Let ε be fixed positive number. Let us assume that $f: X \to Y$ is bounded. Then there exists a $y_0 \in Y$ such that the function $g: X \to Y$, defined as follows:*

$$g(x) := f(x) + y_0 \quad \textit{for } x \in X,$$

satisfies inequality (5).

Proof. The fact that f is bounded means that there exists an $M \geqslant 0$ such that

$$\|f(x)\| \leqslant M \text{ for } x \in X.$$

Let us fix a $y_0 \in Y$ such that $\|y_0\| = c := \frac{2+\varepsilon}{\varepsilon} M$. From the definition of the function g arises that

$$\|g(x)\| = \|f(x)+y_0\| \leqslant \|f(x)\|+\|y_0\| \leqslant M+c.$$

We also have

$$\|y_0\|-\|g(x)\| \leqslant \|g(x)-y_0\| = \|f(x)\| \leqslant M,$$

and hence

$$\|g(x)\| \geqslant \|y_0\|-M,$$

that is

$$\|g(x)\| \geqslant c-M.$$

So we have

$$c-M \leqslant \|g(x)\| \leqslant c+M \quad \text{for } x \in X.$$

Simultaneously, we have

$$\left\|g\left(\frac{x+y}{2}\right)-\frac{g(x)+g(y)}{2}\right\| = \left\|f\left(\frac{x+y}{2}\right)-\frac{f(x)+f(y)}{2}\right\|$$

$$\leqslant \left\|f\left(\frac{x+y}{2}\right)\right\|+\frac{\|f(x)\|+\|f(y)\|}{2} \leqslant 2M \quad \text{for } x, y \in X.$$

Therefore

$$\left\|g\left(\frac{x+y}{2}\right)-\frac{g(x)+g(y)}{2}\right\| \leqslant 2M = \varepsilon(c-M) \leqslant \varepsilon\left\|g\left(\frac{x+y}{2}\right)\right\| \quad \text{for } x, y \in X,$$

which completes the proof.

If we assume that $\varepsilon < 1$, then by Lemma 1 we obtain that inequality (6) is also satisfied.

Last proposition shows that the class of functions satyisfying inequality (4) is quite large. Simultaneously, as we will show in the Example 1, a translation of quasi-additive function can be out of this class, unlike to the case of additive and Jensen functions (see e.g., [1]). As we look for generalization of the Jensen equation the condition (4) does not seem to be satisfying.

EXAMPLE 1. Let us consider the function $f\colon \mathbf{R} \to \mathbf{R}$ defined as follows:

$$f(x) = \begin{cases} x & \text{for } x \in [0, 21] \cup [26, +\infty), \\ \frac{2}{3}x + 7 & \text{for } x \in (21, 24], \\ \frac{3}{2}x - 13 & \text{for } x \in (24, 26), \end{cases}$$

and

$$f(x) = -f(-x) \quad \text{for } x < 0.$$

The function defined in this way satisfies inequality (1) with $\varepsilon = 1/2$ (see Example 2 in [2]). Let us put $c = 21$, and let us define

$$g(x) := f(x) + c \quad \text{for } x \in \mathbf{R}.$$

Elementary calculations show that for $x = -22$ and $y = -20$ the left side in condition (4) equals 1/6, and the right one equals zero. It means that g does not satisfy inequality (4).

2. With respect to the last remarks we change a little the subject of our investigations. From now on we will be considering functions $f\colon X \to Y$ satisfying the following condition for some $\varepsilon \geqslant 0$:

there exists an $x_0 \in X$ such that

$$\text{(19)} \quad \left\| f\left(\frac{x+y}{2}\right) - \frac{f(x)+f(y)}{2} \right\| \leqslant \varepsilon \cdot \min\left\{ \left\| f\left(\frac{x+y}{2}\right) - f(x_0) \right\|, \left\| \frac{f(x)+f(y)}{2} - f(x_0) \right\| \right\} \quad \text{for } x, y \in X.$$

Of course it is still a generalization of the Jensen equation. Unlike as in the case of condition (4), the error in the realisation of the Jensen equation is measured ow with respect to $\left\| f\left(\frac{x+y}{2}\right) - f(x_0) \right\|$ and $\left\| \frac{f(x)+f(y)}{2} - f(x_0) \right\|$ i.e., with respect to the distance between $f\left(\frac{x+y}{2}\right)$ or $\frac{f(x)+f(y)}{2}$, and some initial value $f(x_0)$.

It is obvious that inequality (19) can be written as a conjunction of inequalities

$$\text{(20)} \quad \left\| f\left(\frac{x+y}{2}\right) - \frac{f(x)+f(y)}{2} \right\| \leqslant \varepsilon \left\| f\left(\frac{x+y}{2}\right) - f(x_0) \right\| \quad \text{for } x, y \in X,$$

and

$$\text{(21)} \quad \left\| f\left(\frac{x+y}{2}\right) - \frac{f(x)+f(y)}{2} \right\| \leqslant \varepsilon \left\| \frac{f(x)+f(y)}{2} - f(x_0) \right\| \quad \text{for } x, y \in X.$$

DEFINITION 1. Function $f: X \to Y$ is called *quasi-Jensen function* iff there exists an $x_0 \in X$ and $\varepsilon \geqslant 0$ such that inequality (19) is satisfied.

LEMMA 4. *Having given a function $f: X \to Y$ satisfying, for $\varepsilon \geqslant 0$ and for some $x_0 \in X$, inequality* (20) *or* (21) *we define the function $g: X \to Y$ as follows:*

$$g(x) := f(x+x_0) - f(x_0) \quad \textit{for } x \in X. \tag{22}$$

Then

a) *if f satisfies* (20), *then g satisfies* (5),
b) *if f satisfies* (21), *then g satisfies* (6),
c) *if f satisfies* (19), *then g satisfies* (4),

Proof. It arises from the definition of the function g that

$$\left\| g\left(\frac{x+y}{2}\right) - \frac{g(x)+g(y)}{2} \right\| = \left\| f\left(\frac{x+x_0+y+x_0}{2}\right) - \frac{f(x+x_0)+f(y+x_0)}{2} \right\|.$$

Assuming that f satisfies inequality (20) we get

$$\left\| f\left(\frac{x+x_0+y+x_0}{2}\right) - \frac{f(x+x_0)+f(y+x_0)}{2} \right\|$$

$$\leqslant \varepsilon \left\| f\left(\frac{x+x_0+y+x_0}{2}\right) - f(x_0) \right\| = \varepsilon \left\| f\left(\frac{x+y}{2} + x_0\right) - f(x_0) \right\| = \varepsilon \left\| g\left(\frac{x+y}{2}\right) \right\|.$$

That is to say, we showed that

$$\left\| g\left(\frac{x+y}{2}\right) - \frac{g(x)+g(y)}{2} \right\| \leqslant \varepsilon \left\| g\left(\frac{x+y}{2}\right) \right\| \quad \text{for } x, y \in X,$$

which completes the proof in case a). In case b) the proof runs similarly as in case a), and case c) is a corollary from a) and b).

LEMMA 5. *Assume that $f: X \to Y$ satisfies inequality* (20) *for some $0 \leqslant \varepsilon < 1$. Then f satisfies also inequality*

$$\left\| f\left(\frac{x+y}{2}\right) - \frac{f(x)+f(y)}{2} \right\| \leqslant \frac{\varepsilon}{1-\varepsilon} \left\| \frac{f(x)+f(y)}{2} - f(x_0) \right\| \quad \textit{for } x, y \in X.$$

Proof. Define the function $g: X \to Y$ as in (22). Using Lemma 4 we state that g satisfies inequality (5). That is why we can use Lemma 1 and state

$$\left\| g\left(\frac{x+y}{2}\right) - \frac{g(x)+g(y)}{2} \right\| \leqslant \frac{\varepsilon}{1-\varepsilon} \left\| \frac{g(x)+g(y)}{2} \right\| \quad \text{for } x, y \in X,$$

which means

$$\left\| f\left(\frac{x+x_0+y+x_0}{2}\right) - \frac{f(x+x_0)+f(y+x_0)}{2} \right\|$$

$$\leqslant \frac{\varepsilon}{1-\varepsilon} \left\| \frac{f(x+x_0)+f(y+x_0)}{2} - f(x_0) \right\| \quad \text{for } x, y \in X.$$

Putting into above inequality $x-x_0$ and $y-x_0$ in place of x and y respectively we obtain

$$\left\|f\left(\frac{x+y}{2}\right)-\frac{f(x)+f(y)}{2}\right\| \leqslant \frac{\varepsilon}{1-\varepsilon}\left\|\frac{f(x)+f(y)}{2}-f(x_0)\right\| \quad \text{for } x, y \in X,$$

which completes the proof.

Proceeding similarly as above, using Lemma 4 and Lemma 2 one can show the following

LEMMA 6. *If $f: X \to Y$ satisfies inequality* (21) *for* $0 \leqslant \varepsilon < 1$, *then f satisfies also inequality*

$$\left\|f\left(\frac{x+y}{2}\right)-\frac{f(x)+f(y)}{2}\right\| \leqslant \frac{\varepsilon}{1-\varepsilon}\left\|f\left(\frac{x+y}{2}\right)-f(x_0)\right\| \quad \textit{for } x, y \in X.$$

Lemma 5 and Lemma 6 permit us to state that conditions (20) and (21) are equivalent to a certain degree.

THEOREM 1. *If $f: X \to Y$ satisfies* (20) *for* $0 \leqslant \varepsilon < 1$ *and some* $x_0 \in X$, *then the function* $g: X \to Y$, *defined by formula* (22), *satisfies inequality* (2) *with* $\varepsilon' = \dfrac{2\varepsilon}{1-\varepsilon}$ *i.e.,*

$$\|g(x+y)-g(x)-g(y)\| \leqslant \frac{2\varepsilon}{1-\varepsilon}\|g(x+y)\| \quad \textit{for } x, y \in X.$$

Proof. By Lemma 4 function g satisfies inequality (5). Furthermore, it results from the definition of g that $g(0) = 0$. In this way the assumptions of Proposition 1 are satisfied and we can easily obtain our result.

Similarly one can prove, using Lemma 4 and Proposition 2 the following

THEOREM 2. *If $f: X \to Y$ satisfies* (20) *for some* $x_0 \in X$ *and* $0 \leqslant \varepsilon < 1/3$, *then the function g, defined by* (22), *satisfies inequality* (1) *with* $\varepsilon' = \dfrac{2\varepsilon}{1-3\varepsilon}$ *i.e.,*

$$\|g(x+y)-g(x)-g(y)\| \leqslant \varepsilon' \cdot \min\{\|g(x+y)\|, \|g(x)+g(y)\|\} \quad \textit{for } x, y \in X.$$

In particular Theorem 2 implies that each quasi-Jensen function (with sufficiently small ε) can be obtained by a translation of a quasi-additive function. It is similar to the case of the Jensen functions and additive functions (see e.g. [1]).

In the end we will show that translation by some vector in $X \times Y$ of a quasi-additive function is a quasi-Jensen function.

THEOREM 3. *Let $g: X \to Y$ satisfies inequality* (2) *for* $0 \leqslant \varepsilon < 1$. *Fix arbitrary* $x_0 \in X$ *and* $y_0 \in Y$. *Then the function $f: X \to Y$, defined as follows:*

$$f(x) := g(x-x_0)+y_0 \quad \textit{for } x \in X,$$

satisfies inequality (20) *with* $\varepsilon' = \dfrac{2\varepsilon}{1-\varepsilon}$ *i.e.,*

$$\left\|f\left(\frac{x+y}{2}\right)-\frac{f(x)+f(y)}{2}\right\| \leqslant \varepsilon' \left\|f\left(\frac{x+y}{2}\right)-f(x_0)\right\| \quad \textit{for } x, y \in X.$$

Proof. Evidently $g(0)=0$, so $f(x_0)=y_0$. From the definition of f we have for each $x, y \in X$

$$\left\|f\left(\frac{x+y}{2}\right)-\frac{f(x)+f(y)}{2}\right\| = \left\|g\left(\frac{x-x_0+y-x_0}{2}\right)-\frac{g(x-x_0)+g(y-x_0)}{2}\right\|.$$

The function g satisfies inequality (2), so using Proposition 3 we obtain

$$\left\|g\left(\frac{x-x_0+y-x_0}{2}\right)-\frac{g(x-x_0)+g(y-x_0)}{2}\right\| \leqslant \frac{2\varepsilon}{1-\varepsilon}\left\|g\left(\frac{x-x_0+y-x_0}{2}\right)\right\| \quad \textit{for } x, y \in X.$$

Now one can easily notice that

$$\left\|g\left(\frac{x-x_0+y-x_0}{2}\right)\right\| = \left\|g\left(\frac{(x+y)}{2}-x_0\right)\right\| = \left\|f\left(\frac{x+y}{2}\right)-y_0\right\| = \left\|f\left(\frac{x+y}{2}\right)-f(x_0)\right\|.$$

Therefore we obtain

$$\left\|f\left(\frac{x+y}{2}\right)-\frac{f(x)+f(y)}{2}\right\| \leqslant \frac{2\varepsilon}{1-\varepsilon}\left\|f\left(\frac{x+y}{2}\right)-f(x_0)\right\| \quad \text{for } x, y \in X,$$

which completes the proof.

In analogous way, using Proposition 4 one can obtain

THEOREM 4. *If* $g\colon X \to Y$ *satisfies* (2) *for* $0 \leqslant \varepsilon < 1/3$, *then for arbitrary* $x_0 \in X$ *and* $y_0 \in Y$ *the function* $f\colon X \to Y$, *defined as in Theorem 3, satisfies inequality* (19) *with* $\varepsilon' = \dfrac{2\varepsilon}{1-3\varepsilon}$ *i.e.,*

$$\left\|f\left(\frac{x+y}{2}\right)-\frac{f(x)+f(y)}{2}\right\| \leqslant \varepsilon' \cdot \min\left\{\left\|f\left(\frac{x+y}{2}\right)-f(x_0)\right\|, \left\|\frac{f(x)+f(y)}{2}-f(x_0)\right\|\right\} \quad \textit{for } x, y \in X.$$

Finally, as a corollary from Theorems 1-4 we get

COROLLARY 1. *Let* $f\colon X \to Y$ *satisfies* (20) *for some* $0 \leqslant \varepsilon < 1/3$ *and some* $x_0 \in X$. *Put*

$$f_1(x) := f(x-x_1)+y_1 \quad \textit{for } x \in X,$$

where x_1 *and* y_1 *are arbitrary elements of* X *and* Y *respectively. Then*

$$\left\| f_1\left(\frac{x+y}{2}\right) - \frac{f_1(x)+f_1(y)}{2} \right\| \leqslant \frac{4\varepsilon}{1-3\varepsilon} \left\| f_1\left(\frac{x+y}{2}\right) - f_1(x_1) \right\| \quad \text{for } x, y \in X.$$

Furthermore if $0 \leqslant \varepsilon < 1/7$, *then*

$$\left\| f_1\left(\frac{x+y}{2}\right) - \frac{f_1(x)+f_1(y)}{2} \right\| \leqslant \frac{4\varepsilon}{1-7\varepsilon} \left\| \frac{f_1(x)+f_1(y)}{2} - f_1(x_1) \right\| \quad \text{for } x, y \in X.$$

In particular we can say that a translation of quasi-Jensen function (for sufficiently small ε) remains quasi-Jensen function.

Owing to strong connections between quasi-additive and quasi-Jensen functions, showed above, many properties of quasi-additive functions and sufficient conditions (proved in [2] and [3]) remain true in case of quasi-Jensen functions.

Acknowledgement. I am grateful to Professor J. Tabor for suggesting this problem and supervising my work.

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