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ON MEASURABLE FUNCTIONS WITH VANISHING DIFFERENCES

Abstract. It is shown (under suitable conditions on $H \subset \mathbf{R}$) that if $f: \mathbf{R} \to \mathbf{R}$ is a measurable function such that for an $n \in \mathbf{N}_0$ and every $h \in H$ we have $\Delta_h^{n+1} f(x) = 0$ almost everywhere on $\mathbf{R}$, then f is equal almost everywhere on $\mathbf{R}$ to a polynomial of degree at most n. In particular, every measurable polynomial function $f: \mathbf{R} \to \mathbf{R}$ is a polynomial. In fact, these (essentially known) results are here proved in a more general and more abstract form. The paper contains also a version of the Łomnicki-type theorem on measurable microperiodic functions.

Introduction. In the present paper we study (under conditions which will be specified later) measurable functions $f: X \to Y$ satisfying for every $h \in H$ the condition

$$(1) \qquad \Delta_h^{n+1} f(x) = 0 \quad \text{almost everywhere in } X$$

(n is here a fixed nonnegative integer). We will prove that such a function f is equal almost everywhere in X to a continuous polynomial function of order n. (See the Preliminaries section below for definitions).

Such a result clearly is related to the theorem of R. Ger [5] (cf. also [9; Theorem 17.7.2]) on almost polynomial functions. And indeed, in some instances our Theorem 3 is an immediate consequence of Ger's result, but in general the two results are independent of one another.

For $X = Y = \mathbf{R}$ and f Lebesgue measurable our theorem becomes a special case of much more general and difficult result in [7]. Also [1] contains related results.

For $n = 0$ we obtain a version of our earlier result [10] about measurable microperiodic functions. In the case $X = Y = \mathbf{R}$ and f Lebesgue measurable this version reduces to a result of R.P. Boas Jr. [2].

Concerning further references pertinent to the questions discussed in the present paper and, in particular, to the quoted results the reader is referred to [9] and [10].

Manuscript received December 10, 1990.

AMS (1991) subject classification: Primary 39B52. Secondary 26C99, 28A20.

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Preliminaries. Let X and Y be linear spaces over $\mathbf{Q}$ (or, what amounts to the same, commutative divisible groups), and let f denote an arbitrary function $f\colon X \to Y$. The difference operator Δ_h with the span $h \in X$ and its iterates Δ_h^n, $n = 1, 2, \dots$, are defined by the formulas

$$\text{(2)}\qquad \begin{cases} \Delta_h^1 f(x) = \Delta_h f(x) = f(x+h) - f(x), \\ \Delta_h^{n+1} f(x) = \Delta_h \Delta_h^n f(x), \quad n = 1, 2, \dots \end{cases}$$

The composition of operators $\Delta_{h_1}, \dots, \Delta_{h_n}$ is denoted simply by $\Delta_{h_1 \dots h_n}$.

It can easily be shown by induction that

$$\text{(3)}\qquad \Delta_h^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+kh).$$

The main properties of the difference operator may be found e.g. in [9; Chapter XV]. In particular, we have the following

LEMMA 1. *For every positive integer n and every $h_1, \dots, h_n \in X$ we have*

$$\Delta_{h_1 \dots h_n} f(x) = \sum_{\varepsilon_1, \dots, \varepsilon_n = 0}^{1} (-1)^{\varepsilon_1 + \dots + \varepsilon_n} \Delta_{h'}^n f(x+h'')$$

where

$$h' = -\sum_{j=1}^{n} \varepsilon_j h_j \frac{1}{j}, \qquad h'' = \sum_{j=1}^{n} \varepsilon_j h_j.$$

Let i be a positive integer. A function $\psi\colon X^i \to Y$ is called *i-additive* whenever it is additive in each variable, i.e. whenever the relation

$$\psi(x_1, \dots, x_{j-1}, x+y, x_{j+1}, \dots, x_i)$$
$$= \psi(x_1, \dots, x_{j-1}, x, x_{j+1}, \dots, x_i) + \psi(x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_i)$$

holds for every $x_1, \dots, x_i, x, y \in X$ and every $j = 1, \dots, i$. Function ψ is called *symmetric* iff

$$\psi(x_1, \dots, x_i) = \psi(x_{j_1}, \dots, x_{j_i})$$

for every $x_1, \dots, x_i \in X$ and every permutation $(j_1, \dots, j_i)$ of $(1, \dots, i)$. The function $\Psi\colon X \to Y$ arising from ψ by putting all the variable equal

$$\Psi(x) = \psi(x, \dots, x), \qquad x \in X,$$

is called the *diagonalization* of ψ. By a 0-additive function we understand any constant from Y. Every 0-additive is symmetric, and its diagonalization is again the same constant.

LEMMA 2. *Let $\Psi\colon X \to Y$ be the diagonalization of a symmetric n-additive function $\psi\colon X^n \to Y$ ($n \in \mathbf{N}$). For every integer $m \geqslant n$ and every $h_1, \dots, h_m \in X$ we have*

$$\Delta_{h_1 \dots h_m} \Psi(x) = \begin{cases} n!\,\psi(h_1, \dots, h_n) & \text{if } m = n, \\ 0 & \text{if } m > n. \end{cases}$$

In particular, for every $h \in X$,

$$\Delta_h^m \Psi(x) = \begin{cases} n!\,\Psi(h) & \text{if } m = n, \\ 0 & \text{if } m > n. \end{cases}$$

COROLLARY 1. *Let* $n \in \mathbf{N}_0$ *and let* Ψ_i *be the diagonalization of a symmetric i-additive function,* $i = 0, \ldots, n$. *Put*

$$f(x) = \sum_{i=0}^{n} \Psi_i(x). \tag{4}$$

Then

$$\Delta_h^{n+1} f(x) = 0 \quad \text{for all } x, h \in X. \tag{5}$$

A function $f\colon X \to Y$ fulfilling (5) is called a *polynomial function of order* n ($n \in \mathbf{N}_0$). Corollary 1 states that every function of form (4) is a polynomial function of order n. The converse is also true: if $f\colon X \to Y$ is a polynomial function of order n, then there exist symmetric i-additive functions $\psi_i\colon X^i \to Y$, $i = 0, \ldots, n$, such that (4) holds, where Ψ_i denotes the diagonalization of ψ_i, $i = 0, \ldots, n$. The classical reference is [11], but several other proofs of this fact have been found since: cf. [9].

When $X = Y = \mathbf{R}$, the function $\psi\colon \mathbf{R}^i \to \mathbf{R}$,

$$\psi(x_1, \ldots, x_i) = c x_1 \ldots x_i, \quad x_1, \ldots, x_i \in \mathbf{R},$$

($c \in \mathbf{R}$) is symmetric and i-additive, and its diagonalization is the monomial cx^i. By Corollary 1 every polynomial $f\colon \mathbf{R} \to \mathbf{R}$ is a polynomial function of every order $\geqslant$ the degree of f. The converse is true under mild regularity assumption; cf. [9] and also Proposition 3 and Corollary 4 in the last section of the present paper.

Before we proceed further with our main result we prove a variant (in the spirit of [2]; cf. also [7]) of a theorem [10] on measurable microperiodic functions.

1. In this section we assume that (cf. [10]):

(i) X is a separable semitopological group (i.e. the group operation is separately continuous with respect to either variable; cf. [6]).

Although we do not assume that the group is commutative, we use the additive notation because of the connection of the present section with the rest of the paper. Observe that in a semitopological group translations are homeomorphisms.

(ii) Y is a separable metric space.

(iii) $H \subset X$ is countable and dense subsemigroup of X.

(iv) $\mathcal{M}$ is a σ-algebra of subsets of X. A function $f\colon X \to Y$ is said to be *$\mathcal{M}$-measurable* iff $f^{-1}(V) \in \mathcal{M}$ for every open set $V \subset Y$.

(v) $\mathcal{N} \subset \mathcal{M}$ is a *proper σ-ideal*, i.e., a non-empty family of subsets $A \in \mathcal{M}$ of X fulfilling the conditions (cf. [9], [10])

1. $X \notin \mathcal{N}$.

2. If $A_1 \subset A_2$ and $A_2 \in \mathcal{N}$, then also $A_1 \in \mathcal{N}$.

3. If $A_i \in \mathcal{N}$ for $i = 1, 2, \ldots$, then also $\bigcup_{i=1}^{\infty} A_i \in \mathcal{N}$.

Let $\Phi(x)$ be a condition depending on a parameter $x \in X$. We say that $\Phi(x)$ holds $\mathcal{N}-$(a.e.) in X iff $\Phi(x)$ holds in $X \setminus A$, where $A \in \mathcal{N}$.

(vi) The following analogue of Smital's lemma (cf. [9], [10]) holds true: (s) If $B \in \mathcal{M} \setminus \mathcal{N}$, $D \subset X$ is dense in X and $B+D \in \mathcal{M}$, then $X \setminus (B+D) \in \mathcal{N}$.

REMARK 1. The most important examples of $\mathcal{M}$, $\mathcal{N}$ fulfilling (iv)—(vi) are as follows (cf. [10]).

I. $X = \mathbf{R}^N$, $\mathcal{M}$ is the family of all Lebesgue measurable subsets of $\mathbf{R}^N$, and $\mathcal{N}$ is the family of all subsets of $\mathbf{R}^N$ of N-dimensional Lebesgue measure zero.

In the sequel these particular $\mathcal{M}$ and $\mathcal{N}$ (in $X = \mathbf{R}^N$) will be denoted by $\mathcal{M}_N$ and $\mathcal{N}_N$, respectively.

II. X is a locally compact topological group with a complete right Haar measure μ defined on a σ-algebra $\mathcal{M}$ of subsets of X, and $\mathcal{N}$ is the family of all subsets of X of measure μ zero.

III. X is a second category semitopological group. $\mathcal{M}$ is the family of all Baire subsets of X, and $\mathcal{N}$ is the family of all first category subsets of X.

Now we prove the following

THEOREM 1. *Let hypotheses* (i)—(vi) *be fulfilled. If an $\mathcal{M}$-measurable function $f\colon X \to Y$ fulfils for every $h \in H$ the condition*

$$f(x+h) = f(x) \quad \mathcal{N}-(a.e.), \tag{6}$$

then there exists a $c \in Y$ such that

$$f(x) = c \quad \mathcal{N}-(a.e.). \tag{7}$$

Proof. We may assume without loss of generality (possibly replacing H by $H \cup \{0\}$) that

$$0 \in H. \tag{8}$$

For every $h \in H$ write

$$X_h := \{x \in X\colon f(x+h) = f(x)\} \tag{9}$$

and put

$$X^* := \{x \in X\colon f(x+h) = f(x) \quad \text{for every } h \in H\} = \bigcap_{h \in H} X_h. \tag{10}$$

Condition (6) says that for every $h \in H$

$$X \setminus X_h \in \mathcal{N},$$

whence

$$X \setminus X^* = \bigcup_{h \in H} (X \setminus X_h) \in \mathcal{N}, \tag{11}$$

since H is countable. Relation (11) implies in particular that $X^* \in \mathcal{M}$ and $X^* \notin \mathcal{N}$ so that

(12) $$X^* \in \mathscr{M} \setminus \mathscr{N}.$$

We are going to show that

(13) $$X^* + H = X^*.$$

Take arbitrary $x \in X^*$ and $h \in H$. For every $h' \in H$ we have $h + h' \in H$, whence by (10)

$$f((x+h)+h') = f(x+(h+h')) = f(x) = f(x+h),$$

and again by (10) $x+h \in X^*$. Thus $X^* + H \subset X^*$. The converse inclusion results from (8).

Further the proof runs very much like in [10]. Let $R \subset Y$ be a countable and dense set, and for every $m \in \mathrm{N}$ and $r \in R$ let K_r^m denote the open ball in Y centred at r and with the radius 2^{-m}. For every fixed $m \in \mathrm{N}$ we have

$$\bigcup_{r \in R} [X^* \cap f^{-1}(K_r^m)] = X^*,$$

thus in view of (12) there exists an $r_m \in R$ such that

$$B_m := X^* \cap f^{-1}(K_{r_m}^m) \notin \mathscr{N}.$$

On the other hand, $B_m \in \mathscr{M}$ since f is $\mathscr{M}$-measurable and by (12) $X^* \in \mathscr{M}$. Consequently

(14) $$B_m \in \mathscr{M} \setminus \mathscr{N}.$$

Take arbitrary $x \in B$ and $h \in H$. According to (10) and to the definition of B_m we have $f(x+h) = f(x) \in K_{r_m}^m$. Moreover, $x + h \in X^*$ by virtue of (13). Thus $x + h \in B_m$, that is, $B_m + H \subset B_m$ and since the converse inclusion results from (8), we actually have

(15) $$B_m + H = B_m.$$

Now, (14), (15) and (s) imply

(16) $$X \setminus B_m = X \setminus (B_m + H) \in \mathscr{N}.$$

Now we put

$$B := \bigcap_{m \in \mathrm{N}} B_m \subset X, \qquad C := \bigcap_{m \in \mathrm{N}} K_{r_m}^m \subset Y.$$

By (16)

(17) $$X \setminus B = \bigcup_{m \in \mathrm{N}} (X \setminus B_m) \in \mathscr{N}.$$

In particular, $B \neq \varnothing$ and since evidently

(18) $$f(B) \subset C,$$

also $C \neq 0$. On the other hand, it is clear that C cannot contain two distinct points. Consequently C is a singleton: $C = \{c\}$ with a $c \in Y$. Relation (7) results now from (18) and (17).

The assumption about the algebraic structure of H is not so arbitrary as it might seem. This can be seen from Lemma 3 below. The set family $\mathcal{N}$ is said to be *invariant under (right) translations* iff $A+d\in\mathcal{N}$ for every $A\in\mathcal{N}$ and $d\in X$.

LEMMA 3. *Let X be an arbitrary group, $Y\neq 0$ an arbitrary set, $f\colon X\to Y$ an arbitrary function, and let $\mathcal{N}$ be a proper σ-ideal of subset of X (i.e. a family of subsets of X fulfilling conditions* 1—3 *of* (v)) *invariant under right translations. Then the set*

$$H^*:=\{h\in X\colon\ f(x+h)=f(x)\quad \mathcal{N}-(\text{a.e.})\}\tag{19}$$

is a subgroup of X.

Proof. Clearly $0\in H^*$, so $H^*\neq 0$. For $h\in X$ we have $h\in H^*$ if and only if $X\setminus X_h\in\mathcal{N}$, where X_h is defined by (9).

Take arbitrary h', $h''\in H^*$ so that

$$X\setminus X_{h'}\in\mathcal{N}\ \text{ and }\ X\setminus X_{h''}\in\mathcal{N}.\tag{20}$$

Hence also

$$X\setminus[X_{h''}+(h''-h')]=(X\setminus X_{h''})+(h''-h')\in\mathcal{N}.\tag{21}$$

For $x\in X_{h''}+(h''-h')$ we have $x+h'-h'\in X_{h''}$, whence by (9)

$$f(x+h')=f((x+h'-h'')+h'')=f(x+h'-h'').\tag{22}$$

Further, for $x\in X_{h'}$ we have, also by (9),

$$f(x+h')=f(x).\tag{23}$$

Relations (22) and (23) imply that for $x\in X_{h'}\cap[X_{h''}+(h''-h')]$ we have

$$f(x+(h'-h''))=f(x),$$

which shows that $X_{h'}\cap[X_{h''}+(h''-h')]\subset X_{h'-h''}$. This yields according to (20) and (21)

$$X\setminus X_{h'-h''}\subset(X\setminus X_{h'})\cup(X\setminus[X_{h''}+(h''-h')])\in\mathcal{N}.$$

Consequently $X\setminus X_{h'-h''}\in\mathcal{N}$, that is, $h'-h''\in H^*$. This means that H^* is a subgroup of X.

REMARK 2. It follows from Theorem 1 that under hypotheses (i)—(vi) if, moreover, $\mathcal{N}$ is invariant under right translations and $f\colon X\to Y$ is an $\mathcal{M}$-measurable function fulfilling (6) for every $h\in H$, then for the set (19) we have

$$H^*=X.\tag{24}$$

Indeed, then there are a $c\in Y$ and a set $B\subset X$ such that $X\setminus B\in\mathcal{N}$ and $f(x)=c$ for $x\in B$. Then also, for arbitrary $h\in X$, we have $f(x+h)=c$ for $x\in B-h$ and $X\setminus(B-h)=[(X\setminus B)-h]\in\mathcal{N}$. Consequently $f(x+h)=c=f(x)$ for $x\in B\cap(B-h)\subset X_h$, whence $X\setminus X_h\in\mathcal{N}$ and $h\in H^*$. This implies (24).

We terminate this section with a version of Theorem 1 in the case where $X=\mathbf{R}$.

THEOREM 2. *Let* $X = \mathbf{R}$, *let* Y *fulfil* (ii), *and let hypotheses* (iv)—(vi) *be fulfielled with* $X = \mathbf{R}$. *Further assume that* $\mathscr{N}$ *is invariant under translations and* $H \subset \mathbf{R}$ *is a dense set. If an* $\mathscr{M}$*-measurable function* $f\colon \mathbf{R} \to Y$ *fulfils condition* (6) *for every* $h \in H$, *then there exists a* $c \in Y$ *such that* (7) *holds.*

Proof. In view of Lemma 3 we may assume that H is a group (a subgroup of the additive group of the reals). We will distinguish two cases.

Case 1. There exist in H two incommensurable (rationally independent) real numbers a, b. Then the set

$$\tilde{H} = \{x \in \mathbf{R}\colon\ x = ka + lb,\ k, l \in \mathbf{Z}\} \tag{25}$$

is contained in H:

$$\tilde{H} \subset H. \tag{26}$$

$\tilde{H}$ is a countable and dense subgroup of $\mathbf{R}$ and by (26) relation (6) holds for every $h \in \tilde{H}$. Theorem 1 (with $\tilde{H}$ in place of H) implies the existence of a $c \in Y$ with the property (7).

Case 2. Any two members of H are rationally dependent. Then there exists an $a \in \mathbf{R} \setminus \{0\}$ such that

$$H \subset a\mathbf{Q}.$$

Consequently H is countable and again (7) results from Theorem 1.

Theorem 2 with $Y = \mathbf{R}$ and $\mathscr{M} = \mathscr{M}_1$, $\mathscr{N} = \mathscr{N}_1$ (cf. Remark 1) was proved by R.P. Boas Jr. [2]. Our Theorems 1 and 2 may be regarded as genaralizations of the latter result.

2. Now we pass to the case of the general n in (1). To this aim we must strengthen considerably our hypotheses. In this section we assume that:

(vii) X is a linear space over $\mathbf{Q}$, endowed with a topology such that X becomes a separable topological space and the mapping

$$(\lambda, x, z) \mapsto \lambda x + z, \qquad \lambda \in \mathbf{Q}, \quad x, z \in X,$$

is separately continuous with respect to each variable (a semilinear topology; cf. [8]).

Whenever we refer to a subset of X as a group we have in mind the additive structure of X. (Thus X is a commutative divisible separable semitopological group). Observe that for every fixed $\lambda \in \mathbf{Q} \setminus \{0\}$ and $z \in X$ the mapping $X \ni x \mapsto \lambda x + z \in X$ is a homeomorphism.

(viii) Y is a linear space over $\mathbf{Q}$, endowed with a topology such that Y becomes a separable and metrizable topological space and the mapping

$$(\mu, y, w) \mapsto \mu y + w, \qquad \mu \in \mathbf{Q}, \quad y, w \in Y, \tag{27}$$

is jointly continuous with respect to the triple (μ, y, w) in $\mathbf{Q} \times Y \times Y$ (a linear topology).

In other words, Y is a linear space over $\mathbf{Q}$, endowed with a topology such that Y becomes a T_0 topological space satisfying the second axiom of countability and mapping (27) is continuous in $\mathbf{Q}\times Y\times Y$ (cf. [4; p. 537, Exercise 8.1.6(a)]).

(ix) $E\subset X$ is a countable and dense linear subspace of X over $\mathbf{Q}$ with the property:

(P) For every positive integer i, every dense subgroup G of E and every symmetric i-additive function $\psi\colon G^i\to Y$ there exists a (necessarily unique) continuous symmetric i-additive function $\hat{\psi}\colon X^i\to Y$ such that $\hat{\psi}=\psi$ on G^i.

(x) $H\subset X$ is a subgroup of X such that the set

$$H_0 = H\cap E \tag{28}$$

is dense in X.

Concerning the set classes $\mathscr{M}$ and $\mathscr{N}$ we assume that besides (iv)—(vi) they fulfil also the following conditions.

(xi) We have

$$\lambda A+z\in\mathscr{N},\qquad \lambda B+z\in\mathscr{M}$$

for every $\lambda\in\mathbf{Q}$, $z\in X$, $A\in\mathscr{N}$ and $B\in\mathscr{M}$.

(xii) $\mathscr{M}$ contains all Borel subsets of X. In other words, every continuous function $f\colon X\to Y$ is $\mathscr{M}$-measurable.

REMARK 3. All the examples of $\mathscr{M},\mathscr{N}$ given in Remark 1 fulfil also conditions (xi) and (xii).

We start with a lemma.

LEMMA 4. *Let hypotheses* (vii), (viii), (iv) *and the part of* (xi) *concerning* $\mathscr{M}$ *be fulfilled. If the functions* $f,f_1,f_2\colon X\to Y$ *are* $\mathscr{M}$*-measurable, then also (for every fixed* $\lambda,\mu\in\mathbf{Q}$, $z\in X$ *and* $w\in Y$*) the functions* $g_1,g_2\colon X\to Y$ *given by*

$$g_1(x)=\mu f(\lambda x+z)+w,\qquad g_2(x)=f_1(x)+f_2(x),\qquad x\in X,$$

are $\mathscr{M}$*-measurable.*

Proof. Since every constant function from X into Y is $\mathscr{M}$-measurable we may assume that $\lambda\neq 0$ and $\mu\neq 0$. For every open set $V\subset Y$ we have

$$g_1^{-1}(V)=\frac{1}{\lambda}\left[f^{-1}\left(\frac{1}{\mu}(V-w)\right)-z\right],$$

whence the $\mathscr{M}$-measurability of g_1 results.

The space Y satisfies the second axiom of countability, consequently it has a countable neighbourhood base $\mathscr{U}$. The $\mathscr{M}$-measurability of g_2 is now a consequence of the formula

$$g_2^{-1}(V)=\bigcup_{\substack{U_1,U_2\in\mathscr{U}\\ U_1+U_2\subset V}} f_1^{-1}(U_1)\cap f_2^{-1}(U_2)$$

valid for every open set $V\subset Y$.

COROLLARY 2. *Under conditions of Lemma 4, if $f\colon X \to Y$ is an $\mathcal{M}$-measurable function, then for every positive integer n and every fixed $h_1, \dots, h_n \in X$ the function $\Delta_{h_1 \dots h_n} f$ is $\mathcal{M}$-measurable.*

Proof. For $n = 1$ this is true by virtue of (2) and Lemma 4. Now use induction on n.

Now we are going to prove our main result.

THEOREM 3. *Let hypotheses* (iv)—(xii) *be fulfilled and let n be a non-negative integer. If an $\mathcal{M}$-measurable function $f\colon X \to Y$ satisfies for every $h \in H$ the condition*

$$\Delta_h^{n+1} f(x) = 0 \quad \mathcal{N}-(\text{a.e.}), \tag{29}$$

then there exists a continuous polynomial function $\varphi\colon X \to Y$ of order n such that

$$f(x) = \varphi(x) \quad \mathcal{N}-(\text{a.e.}). \tag{30}$$

Proof. First observe that (replacing, if necessary, H by the set H_0 given by (28)) we may assume that H is a countable and dense subgroup of X fulfilling the condition $H \subset E$.

For $n = 0$ Theorem 3 is a consequence of Theorem 1. Now assume that Theorem 3 it true with n replaced by $n-1$ ($n \in \mathrm{N}$) and let $f\colon X \to Y$ be an $\mathcal{M}$-measurable function fulfilling for every $h \in H$ condition (29). Put

$$G = (n+1)!\, H. \tag{31}$$

G is a countable and dense subgroup of X fulfilling the condition

$$G \subset E. \tag{32}$$

Moreover, for every $h \in G$ and every positive integer $j \leqslant n+1$ we have $\frac{h}{j} \in H$. Thus it follows from (29) by virtue of Lemma 1 that for every $h_1, \dots, h_{n+1} \in G$ the function f satisfies the condition

$$\Delta_{h_1 \dots h_{n+1}} f(x) = 0 \quad \mathcal{N}-(\text{a.e.}). \tag{33}$$

For every $h_1, \dots, h_n \in G$ we define a function $g_{h_1 \dots h_n}\colon X \to Y$ by the formula

$$g_{h_1 \dots h_n}(x) := \Delta_{h_1 \dots h_n} f(x), \qquad x \in X. \tag{34}$$

Corollary 2 guarantes that $g_{h_1 \dots h_n}$ is $\mathcal{M}$-measurable, and by (33)

$$\Delta_h g_{h_1 \dots h_n}(x) = 0 \quad \mathcal{N}-(\text{a.e.})$$

for every $h \in G$. By virtue of Theorem 1 $g_{h_1 \dots h_n}$ is constant $\mathcal{N}-$(a.e.) in X (the constant, however, depends on $h_1, \dots, h_n$):

$$g_{h_1 \dots h_n}(x) = c(h_1, \dots, h_n) \quad \mathcal{N}-(\text{a.e.}).$$

In other words, for every $h_1, \dots, h_n \in G$, we have in view of (34)

$$\Delta_{h_1 \dots h_n} f(x) = c(h_1, \dots, h_n) \quad \text{for } x \in X \setminus A[h_1, \dots, h_n], \tag{35}$$

where

$$(36)\qquad A[h_1, \dots, h_n] \in \mathcal{N}.$$

Fix arbitrary $h_1, \dots, h_n \in G$ and an arbitrary permutation $(i_1, \dots, i_n)$ of $(1, \dots, n)$. According to (36) there exists an x in X such that

$$x \in X \setminus (A[h_1, \dots, h_n] \cup A[h_{i_1}, \dots, h_{i_n}]),$$

whence by (35)

$$c(h_1, \dots, h_n) = \Delta_{h_1 \dots h_n} f(x) = \Delta_{h_{i_1} \dots h_{i_n}} f(x) = c(h_{i_1}, \dots, h_{i_n}),$$

This shows that c is a symmetric function of its variables. Moreover, for every $u, v, h_2, \dots, h_n \in G$ we can find an x in X such that

$$x \in X \setminus (A[u+v, h_2, \dots, h_n] \cup A[u, h_2, \dots, h_n] \cup (A[v, h_2, \dots, h_n] - u)).$$

Thus by (2) and (35)

$$\begin{aligned}
&c(u+v, h_2, \dots, h_n) - c(u, h_2, \dots h_n) - c(v, h_2, \dots, h_n) \\
&\quad = \Delta_{u+v, h_2, \dots, h_n} f(x) - \Delta_{u, h_2, \dots, h_n} f(x) - \Delta_{v, h_2, \dots, h_n} f(x+u) \\
&\quad = \Delta_{h_2, \dots, h_n} [\Delta_{u+v} f(x) - \Delta_u f(x) - \Delta_v f(x+u)] \\
&\quad = \Delta_{h_2 \dots h_n} [f(x+u+v) - f(x) - f(x+u) + f(x) - f(x+u+v) + f(x+u)] = 0.
\end{aligned}$$

Consequently c is additive in the first variable, and due to the symmetry c is actually additive in each variable. Consequently $c: G^n \to Y$ is a symmetric n-additive function. According to (P) (cf., in particular, (31) and (32)) there exists a continuous symmetric n-additive function $\psi: X^n \to Y$ such that

$$(37)\qquad \psi(h_1, \dots, h_n) = c(h_1, \dots, h_n) \quad \text{for } h_1, \dots, h_n \in G.$$

Let $\hat{\Psi}: X \to Y$ be the diagonalization of ψ and write $\Psi(x) = \dfrac{1}{n!} \hat{\Psi}(x), x \in X$. For $h \in G$ we have by Lemma 2, (37), (35) and (36)

$$(38)\qquad \Delta_h^n \Psi(x) = \Delta_h^n f(x) \quad \mathcal{N}-\text{(a.e.)}.$$

$\Psi: X \to Y$ is a continuous, and hence $\mathcal{M}$-measurable function. By Lemma 4 also the function $f - \Psi$ is $\mathcal{M}$-measurable and for every $h \in G$ we have in view of (38)

$$\Delta_h^n [f(x) - \Psi(x)] = 0 \quad \mathcal{N}-\text{(a.e.)}.$$

By the induction hypothesis there exists a continuous polynomial function $\varphi_0: X \to Y$ of order $n-1$ such that

$$f(x) - \Psi(x) = \varphi_0(x) \quad \mathcal{N}-\text{(a.e.)}.$$

Hence we obtain (30), where $\varphi(x) := \Psi(x) + \varphi_0(x)$ clearly is a continuous polynomial function of order n (cf., in particular, Lemma 2). Induction completes the proof.

REMARK 4. One could believe that without the measurability assumption (29) still implies (30) with a (not necessarily continuous) polynomial function $\varphi: X \to Y$ of order n. However, it is not so, as may be seen from Example 1 below.

Similarly, one could reasonable conjecture that if the equality in (29) holds for all $x \in X$, then also the equality in (30) holds for all $x \in X$. And again, in general it is not true, the conjecture being disproved by Example 2 below.

Before proceeding with the announced examples we prove a lemma.

LEMMA 5. *Let hypotheses* (vii), (viii), (v) *and the part of* (xi) *concerning* $\mathcal{N}$ *be fulfilled, and suppose that we are given a set* $H \subset X$, *functions* $f, g: X \to Y$ *and an* $m \in \mathbf{N}$ *such that*

$$H \notin \mathcal{N} \tag{39}$$

and

$$\Delta_h^m f(x) = \Delta_h^m g(x) \quad \textit{for all } x \in X,\ h \in H. \tag{40}$$

If

$$f(x) = g(x) \quad \mathcal{N}-(\text{a.e.}), \tag{41}$$

then actually $f = g$ *on* X.

Proof. The proof is standard. Write (41) as

$$f(x) = g(x) \quad \text{for } x \in T, \tag{42}$$

where $X \setminus T \in \mathcal{N}$. Take an arbitrary $x \in X$ and write

$$S := \bigcap_{j=1}^{m} \frac{1}{j}(T - x). \tag{43}$$

We have

$$X \setminus S = \bigcup_{j=1}^{m} \left[X \setminus \frac{1}{j}(T-x) \right] = \bigcup_{j=1}^{m} \left[\frac{1}{j}(X \setminus T) - \frac{1}{j} x \right] \in \mathcal{N}.$$

In view of (39) we get hence $H \cap S \neq \emptyset$. Take an $h \in H \cap S$. It follows from (43) that

$$x + jh \in T \quad \text{for } j = 1, \ldots, m,$$

whence by (42)

$$f(x+jh) = g(x+jh) \quad \text{for } j = 1, \ldots, m,$$

and by (40), in view of formula (3), we obtain $f(x) = g(x)$.

In the examples that follow $X = Y = \mathbf{R}$, $\mathcal{M} = \mathcal{M}_1$, $\mathcal{N} = \mathcal{N}_1$ (cf. Remark 1). We write *measurable* instead of $\mathcal{M}_1$-measurable and *almost everywhere* instead of $\mathcal{N}_1-(\text{a.e.})$.

EXAMPLE 1. Assuming the continuum hypothesis, W. Sierpiński [12; p. 135] constructed a nonmeasurable function $\sigma: \mathbf{R} \to \mathbf{R}$ such that for every $h \in \mathbf{R}$

$$\Delta_h \sigma(x) = 0 \quad \text{almost everywhere.} \tag{44}$$

(By the way, this shows that the measurability assumption in Theorems 1 and 2 is essential). By induction

(45) $$\Delta_{h_1 \dots h_i} \sigma(x) = 0 \quad \text{almost everywhere}$$

for arbitrary $h_1, \dots, h_i \in \mathbf{R}$ and $i \in \mathrm{N}$.

Fix an $n \in \mathrm{N}$ and a polynomial $P\colon \mathbf{R} \to \mathbf{R}$ of degree n, and write $f = P + \sigma$. For every $x, h_1, \dots, h_{n+1} \in \mathbf{R}$ we have (cf. the Preliminaries section)

(46) $$\Delta_{h_1 \dots h_{n+1}} P(x) = 0,$$

whence it follows in view of (45) that for every $h \in \mathbf{R}$

(47) $$\Delta_h^{n+1} f(x) = 0 \quad \text{almost everywhere.}$$

Suppose that there exists a (discontinuous) polynomial function $\varphi\colon \mathbf{R} \to \mathbf{R}$ of order n such that

(48) $$f(x) = \varphi(x) \quad \text{almost everywhere.}$$

By virtue of Lemma 1 the function φ fulfils for every $x, h_1, \dots, h_{n+1} \in \mathbf{R}$ the condition

(49) $$\Delta_{h_1 \dots h_{n+1}} \varphi(x) = 0.$$

Now fix arbitrarily an $h \in \mathbf{R}$. We have by (44) and (48)

$$\Delta_h P(x) = \Delta_h \varphi(x) \quad \text{almost everywhere,}$$

whereas by (46) and (49)

$$\Delta_{h'}^{n}(\Delta_h P(x)) = 0 = \Delta_{h'}^{n}(\Delta_h \varphi(x)) \quad \text{for all } x, h' \in \mathbf{R}.$$

According to Lemma 5 $\Delta_h P = \Delta_h \varphi$ in $\mathbf{R}$, whence $\Delta_h \sigma = 0$ in $\mathbf{R}$. This being true for every $h \in \mathbf{R}$, it follows that $\sigma = \text{const}$, a contradiction. Consequently (48) cannot be true.

EXAMPLE 2. Fix an $n \in \mathrm{N}$ and a polynomial $\varphi\colon \mathbf{R} \to \mathbf{R}$ of degree n, and define the function $f\colon \mathbf{R} \to \mathbf{R}$ by

(50) $$f(x) := \begin{cases} 0 & \text{for } x \in \mathbf{Q}, \\ \varphi(x) & \text{for } x \in \mathbf{R} \setminus \mathbf{Q}. \end{cases}$$

For every $x, h \in \mathbf{Q}$ we have $x + jh \in \mathbf{Q}$ for $j = 0, \dots, n+1$, whence by (3) and (50)

$$\Delta_h^{n+1} f(x) = \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} f(x+jh) = 0,$$

whereas for $x \in \mathbf{R} \setminus \mathbf{Q}$, $h \in \mathbf{Q}$ we have $x + jh \in \mathbf{R} \setminus \mathbf{Q}$ for $j = 0, \dots, n+1$, whence by (3) and (50)

$$\begin{aligned} \Delta_h^{n+1} f(x) &= \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} f(x+jh) = \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} \varphi(x+jh) \\ &= \Delta_h^{n+1} \varphi(x) = 0. \end{aligned}$$

Thus with $H = \mathbf{Q}$ the function f fulfils the condition

$$\Delta_h^{n+1} f(x) = 0 \quad \text{for all } x \in \mathbf{R},\ h \in H,$$

but the equality $f(x) = \varphi(x)$ holds only almost everywhere, and not everywhere in $\mathbf{R}$.

Such an example would not be possible if the set H were large enough.

THEOREM 4. *Let hypotheses* (iv)—(xii) *and condition* (39) *be fulfilled and let n be a nonnegative integer. If an $\mathscr{M}$-measurable function $f: X \to Y$ satisfies the condition*

$$(51) \qquad \Delta_h^{n+1} f(x) = 0 \quad \textit{for all } x \in X,\ h \in H,$$

then f is continuous polynomial function of order n.

Proof. Condition (51) implies (29) (the exceptional sets being empty), thus according to Theorem 3 there exists a continuous polynomial function $\varphi: X \to Y$ of order n such that (30) holds. On the other hand, since φ is a polynomial function of order n, we have in particular

$$(52) \qquad \Delta_h^{n+1} \varphi(x) = 0 \quad \text{for all } x \in X,\ h \in H.$$

Relations (30), (39), (51) and (52) show by virtue of Lemma 5 that $f = \varphi$, that is, f is a continuous polynomial function of order n.

COROLLARY 3. *Let hypotheses* (iv)—(ix) *and* (xi)—(xii) *be fulfilled. Then every $\mathscr{M}$-measurable polynomial function $f: X \to Y$ is continuous.*

This results from Theorem 4 on taking $H = X$.

REMARK 5. It could seem that the condition $H \notin \mathscr{N}$ is considerably weaker than $H = X$, but in many cases it is not true. Suppose that the following form of the theorem of Steinhaus is valid in X:

(H) If $A, B \in \mathscr{M} \setminus \mathscr{N}$, then $\operatorname{int}(A+B) \neq \varnothing$.

(This is certainly the case for all examples of $\mathscr{M}$, $\mathscr{N}$ listed in Remark 1; cf. [10]). Let $H \subset X$ be a dense subgroup of X. If $H \in \mathscr{M} \setminus \mathscr{N}$, then by (H)

$$\operatorname{int} H = \operatorname{int}(H+H) \neq \varnothing,$$

whence $H = H+H = X$ since H is dense in X.

REMARK 6. Under conditions (vii) and (v), if, moreover, $\mathscr{N}$ is invariant under translations, we have $\operatorname{int} A = \varnothing$ for every set $A \in \mathscr{N}$. Indeed, suppose that $\operatorname{int} A \neq \varnothing$ for an $A \in \mathscr{N}$, and let D be a countable and dense subset of X. We have

$$A+D = \bigcup_{d \in D} (A+d) \in \mathscr{N}$$

since D is countable and $A+d \in \mathscr{N}$ for every d. On the other hand, $A+D = X$ since $\operatorname{int} A \neq \varnothing$ and D is dense in X. Consequently $X \in \mathscr{N}$, a contradiction.

Thus if a condition $\Phi(x)$ is fulfilled $\mathscr{N}-$(a.e.), then it is fulfilled on a dense subset of X. In particular, if two continuous functions are equal $\mathscr{N}-$(a.e.), then actually they coincide in the whole of X.

It follows that the continuous polynomial function φ occuring in Theorem 3 is determined uniquely. The uniqueness of a polynomial function $\varphi: X \to Y$ fulfilling (30) (without appealing to continuity) may be obtained from Lemma 5.

3. In the present section we discuss some particular cases of Theorem 3. Of course, the most interesting and important instances of X, Y fulfilling (vii) or (viii) are $\mathbf{R}$ (or, more generally, $\mathbf{R}^N$) and $\mathbf{C}$.

(a) $X = \mathbf{R}$, $Y = \mathbf{K}$ ($\mathbf{K}$ stands for $\mathbf{R}$ or $\mathbf{C}$). We start with a lemma.

LEMMA 6. *For every real number $a \neq 0$ the set $E = a\mathbf{Q}$ is a countable and dense linear subspace of* $\mathbf{R}$ over $\mathbf{Q}$ *with property* (P) ($X = \mathbf{R}$, $Y = \mathbf{K}$).

Proof. Only (P) requires a proof. Let G be a dense subgroup of E. We may assume that $a \in G$. For otherwise take an $a' \in G\setminus\{0\} \subset E = a\mathbf{Q}$. Thus there exists an $r \in \mathbf{Q}\setminus\{0\}$ such that $a' = ar$, whence $a = a'r^{-1}$ and $E = a\mathbf{Q} = a'(r^{-1}\mathbf{Q}) = a'\mathbf{Q}$ and $a' \in G$.

One can prove by induction that if $\psi_i: G^i \to \mathbf{K}$ is a symmetric i-additive function, then there exists a $c_i \in \mathbf{K}$ such that

$$\psi_i(t_1, \ldots, t_i) = c_i t_1 \ldots t_i \quad \text{for all } t_1, \ldots, t_i \in G. \tag{53}$$

Clearly the function $\hat{\psi}_i: \mathbf{R}^i \to \mathbf{K}$ given by

$$\hat{\psi}_i(x_1, \ldots, x_i) = c_i x_1 \ldots x_i \quad \text{for all } x_1, \ldots, x_i \in \mathbf{R} \tag{54}$$

is a continuous extension of ψ_i onto $\mathbf{R}^i$, and $\hat{\psi}_i$ is symmetric and i-additive.

COROLLARY 4. *If $f: \mathbf{R} \to \mathbf{K}$ is a continuous polynomial function of order n ($n \in \mathbf{N}$), then f is a polynomial (in a real variable x with coefficients from* $\mathbf{K}$) *of degree at most n.*

Proof. It follows from the theorem of Mazur-Orlicz [11] (cf. the Preliminaries section) that f can be written in form (4), where for $j = 0, \ldots, n$ the function Ψ_j is the diagonalization of a symmetric j-additive function $\hat{\psi}_j: \mathbf{R}^j \to \mathbf{K}$. It is enough to show that for $j = 1, \ldots, n$ the function Ψ_j is a monomial

$$\Psi_j(x) = c_j x^j, \quad x \in \mathbf{R}, \tag{55}$$

with a $c_j \in \mathbf{K}$. (For $j = 0$ (55) is trivial). Suppose this has already been proved for $j = i+1, \ldots, n$ $(1 \leqslant i \leqslant n)$ and write

$$F_i(x) := f(x) - \sum_{j=i+1}^{n} c_j x^j$$

($F_n = f$) so that

$$F_i(x) = \sum_{j=0}^{i} \Psi_j(x). \tag{56}$$

The function F_i is continuous and in view of (2) $\Delta_{x_1 \ldots x_i} F_i(x)$ is a continuous function of $x_1, \ldots, x_i$ in $\mathbf{R}$. But according to (56) and Lemma 2

$$\Delta_{x_1 \ldots x_i} F_i(x) = i!\,\hat{\psi}_i(x_1, \ldots, x_i)$$

and thus $\hat{\psi}_i$ is continuous.

Now put $G := \mathbf{Q}$ and $\psi_i = \hat{\psi}_i|_{G^i}$. We have (53), whence (54) results in view of the continuity of $\hat{\psi}_i$ and ultimately we get (55) for $j = i$. Thus (55) is valid for $j = 0, \ldots, n$.

Now we assume that

(xiii) $H \subset \mathbf{R}$ is a subgroup of $\mathbf{R}$ and there exists an $a \in \mathbf{R}$ such that the set $H \cap (a\mathbf{Q})$ is dense in $\mathbf{R}$.

Lemma 6, Theorem 3 and Corollary 4 imply the following

PROPOSITION 1. *Let hypotheses* (iv)—(vi), (xi), (xii) (*with* $X = \mathbf{R}$) *and* (xiii) *be fulfilled and let* n *be a nonnegative integer. If an* $\mathcal{M}$*-measurable function* $f: \mathbf{R} \to \mathbf{K}$ *satisfies for every* $h \in H$ *condition* (29), *then there exists a polynomial* φ *(in a real variable, with coefficients from* $\mathbf{K}$*) of degree not exceeding* n *such that* (30) *holds.*

As it has been pointed out in Remarks 1 and 3, hypotheses (iv)—(vi) and (xi)—(xii) ($X = \mathbf{R}$) are fulfilled, e.g., by the family of all Lebesgue measurable subsets of $\mathbf{R}$ or that of all Baire subsets of $\mathbf{R}$ as $\mathcal{M}$, and by the family of all subsets of $\mathbf{R}$ of Lebesgue measure zero or that of all first category subsets of $\mathbf{R}$, respectively, as $\mathcal{N}$. As to (xiii), it is certainly fulfilled whenever H is a linear subspace of $\mathbf{R}$ over $\mathbf{Q}$. Another example of an H fulfilling (xiii) is furnished by the set of all dyadic numbers. On the other hand, (xiii) is not fulfilled by $H = \tilde{H}$ given by (25), where a, b are incommensurable real numbers. Actually, it can be inferred from the argument in the proof of Theorem 2 that if H is a dense subgroup of $\mathbf{R}$, then H fulfils either (xiii) or

(xiii)′ $H \subset \mathbf{R}$ is a subgroup of $\mathbf{R}$ and there exist incommensurable $a, b \in \mathbf{R}$ such that (26) with (25) holds.

The two conditions do not exclude each other. For instance, $H = \mathbf{R}$ fulfils both (xiii) and (xiii)′.

Unfortunately, we have not been able to prove Proposition 1 with (xiii) replaced by (xiii)′. The Proposition 2 below (cf. the sentence immediately before Example 1), however, is a consequence of a much more general result of J. H. B. Kemperman [7].

PROPOSITION 2. *Let* $H \subset \mathbf{R}$ *fulfil* (xiii)′ *and let* n *be a nonnegative integer. If a measurable function* $f: \mathbf{R} \to \mathbf{R}$ *fulfils for every* $h \in H$ *condition* (47), *then there exists a real polynomial* φ *of degree not excending* n *and such that* (48) *holds.*

Extending this result to the case of functions $f: \mathbf{R} \to \mathbf{K}$ presents no difficulties.

J.A. Baker [1] proved that if a function $f: \mathbf{R} \to \mathbf{C}$ satisfies for certain $m, n \in \mathbf{N}$ and incommensurable $a, b \in \mathbf{R}$ the condition

$$\Delta_a^m f(x) = \Delta_b^n f(x) = 0 \quad \text{for all } x \in \mathbf{R},$$

and is Lebesgue integrable on an interval of length ma, then there exists a polynomial φ (in a real variable, with complex coefficients) of degree at most $m-1$ such that

$$\Delta_a^m \varphi(x) = \Delta_b^n \varphi(x) = 0 \quad \text{for all } x \in \mathbf{R}$$

and (48) holds.

Taking $X = Y = \mathbf{R}$ and choosing suitable $\mathcal{M}, \mathcal{N}$, we obtain from Corollaries 3 and 4 the following result essentially due to Z. Ciesielski [3] (cf. also [9]).

PROPOSITION 3. *Every Lebesgue measurable or Baire measurable polynomial function $f\colon \mathbf{R} \to \mathbf{R}$ (of order n) is continuous, and hence it is a real polynomial (of degree at most n).*

(b) $X = \mathbf{R}^N$, $Y = \mathbf{R}^M$ $(M, N \in \mathbf{N})$. Let $e_1, \ldots, e_N \in \mathbf{R}^N$ be linearly independent over $\mathbf{R}$ (a base of $\mathbf{R}^N$ over $\mathbf{R}$) and put

$$E := \{x \in \mathbf{R}^N\colon\ x = \lambda_1 e_1 + \ldots + \lambda_N e_N,\ \lambda_1, \ldots, \lambda_N \in \mathbf{Q}\}. \tag{57}$$

LEMMA 7. *Let E be given by (57) with $e_1, \ldots, e_N \in \mathbf{R}^N$ linearly independent over $\mathbf{R}$ and let $G \subset E$ be a dense subgroup of E. If $\psi\colon G \to \mathbf{R}$ is an additive function, then there exist real constants $c_1, \ldots, c_N$ such that for $t = \tau_1 e_1 + \ldots + \tau_N e_N \in G$*

$$\psi(t) = \psi(\tau_1 e_1 + \ldots + \tau_N e_N) = c_1 \tau_1 + \ldots + c_N \tau_N. \tag{58}$$

Proof. There exist $e'_1, \ldots, e'_N \in G$ linearly independent over $\mathbf{R}$ (otherwise G could not be dense in E). We have, since $G \subset E$,

$$e'_j = \lambda_{j_1} e_1 + \ldots + \lambda_{j_N} e_N, \quad j = 1, \ldots, N, \tag{59}$$

with rational λ_{jk}, $j, k = 1, \ldots, N$, whence also

$$e_j = \mu_{j_1} e'_1 + \ldots \mu_{j_N} e'_N, \quad j = 1, \ldots, N, \tag{60}$$

with rational μ_{jk}, $j, k = 1, \ldots, N$. Relations (57), (59) and (60) imply that

$$E = \{x \in \mathbf{R}^N\colon\ x = \mu_1 e'_1 + \ldots + \mu_N e'_N,\ \mu_1, \ldots, \mu_N \in \mathbf{Q}\}.$$

Thus in the sequel we assume that $e_1, \ldots, e_N \in G$.

For every $h \in G$ and $k \in \mathbf{Z}$ we have $kh \in G$ and $\psi(kh) = k\psi(h)$. An arbitrary $t \in G$ can be written as $t = \tau_1 e_1 + \ldots + \tau_N e_N$ with $\tau_1, \ldots, \tau_N \in \mathbf{Q}$. Choose a $q \in \mathbf{N}$ such that $p_j = q\tau_j \in \mathbf{Z}$ for $j = 1, \ldots, N$. We have by the additivity of ψ, since $p_1 e_1 + \ldots + p_N e_N \in G$,

$$\begin{aligned} q\psi(t) = \psi(qt) &= \psi(p_1 e_1 + \ldots + p_N e_N) = p_1 \psi(e_1) + \ldots + p_N \psi(e_N) \\ &= q\tau_1 \psi(e_1) + \ldots + q\tau_N \psi(e_N), \end{aligned}$$

whence (58) results with $c_j := \psi(e_j)$, $j = 1, \ldots, N$.

LEMMA 8. *For every $e_1, \ldots, e_N \in \mathbf{R}^N$ linearly independent over $\mathbf{R}$ the set E given by (57) is a countable and dense linear subspace of $\mathbf{R}^N$ over $\mathbf{Q}$ with property* (P) $(X = \mathbf{R}^N,\ Y = \mathbf{R}^M)$.

Proof. Again only (P) requires a proof. Note that we may restrict ourselves to $M = 1$, since each of the M components of ψ may be considered separately.

Let $G \subset E$ be a dense subgroup of E. (As previously, we assume that $e_1, \ldots, e_N \in G$). Our Lemma will be proved when we show (induction on $i \in \mathbf{N}$) the assertion:

(*) *For every symmetric i-additive function $\psi_i\colon G^i \to \mathbf{R}$ there exists a continuous symmetric i-additive function $\hat{\psi}_i\colon (\mathbf{R}^N)^i \to \mathbf{R}$ such that $\hat{\psi}_i|_{G^i} = \psi_i$.*

For $i = 1$ (*) is true by virtue of Lemma 7: the function

$$\hat{\psi}(x) = \hat{\psi}(\xi_1 e_1 + \dots + \xi_N e_N) = c_1 \xi_1 + \dots + c_N \xi_N$$

is the desired extension of (58) onto $\mathbf{R}^N$. Now assume that (*) is true for an $i \in \mathrm{N}$ and let $\psi_{i+1}\colon G^{i+1} \to \mathbf{R}$ be a symmetric $(i+1)$-additive function. Fix arbitrarily $t_1, \dots, t_i \in G$ and write

$$\text{(61)} \qquad \psi(t) := \psi_{i+1}(t_1, \dots, t_i, t), \qquad t \in G.$$

Thus $\psi\colon G \to \mathbf{R}$ is an additive function. By virtue of Lemma 7 we have (58) where $c_j \in \mathbf{R}, j = 1, \dots, N$, depend, in fact, on $t_1, \dots, t_i$ previously fixed. Since by (58) and (61)

$$c_j(t_1, \dots, t_i) = \psi_{i+1}(t_1, \dots, t_i, e_j), \quad j = 1, \dots, N,$$

every $c_j\colon G^i \to \mathbf{R}$ is a symmetric i-additive function. By the induction hypothesis every c_j can be extended onto $(\mathbf{R}^N)^i$ to a continuous symmetric i-additive function $\hat{c}_j\colon (\mathbf{R}^N)^i \to \mathbf{R}$. Let $\hat{c} = (\hat{c}_1, \dots, \hat{c}_N) \in \mathbf{R}^N$ be the N-tuple of functions $\hat{c}_j$ so that $\hat{c}$ is a function $\hat{c}\colon (\mathbf{R}^N)^i \to \mathbf{R}^N$. The functon $\hat{\psi}_{i+1}\colon (\mathbf{R}^N)^{i+1} \to \mathbf{R}$

$$\hat{\psi}_{i+1}(x_1, \dots, x_i, x_{i+1}) = \hat{c}(x_1, \dots, x_i) \cdot x_{i+1}, \qquad x_1, \dots, x_{i+1} \in \mathbf{R}^N,$$

where dot denotes the scalar product, is a continuous $(i+1)$-additive extension of ψ_{i+1} onto $(\mathbf{R}^N)^{i+1}$. The symmetry of $\hat{\psi}_{i+1}$ results from that of ψ_{i+1} and from the continuity of $\hat{\psi}_{i+1}$.

Thus (*) is valid for $i+1$. This completes the induction and ends the proof of Lemma 8.

Our next hypothesis reads:

(xiv) $H \subset \mathbf{R}^N$ is a subgroup of $\mathbf{R}^N$ and there exist $e_1, \dots, e_N \in \mathbf{R}^N$ linearly independent over $\mathbf{R}$ such that the set $H \cap E$, where E is given by (59), is dense in $\mathbf{R}^N$.

Since every continuous polynomial function of order n from $\mathbf{R}^N$ into $\mathbf{R}$ is a real polynomial in N variables of degree at most n [9, Theorem 15.9.4], we obtain from Theorem 3 and Lemma 8

PROPOSITION 4. *Let hypotheses* (iv)—(vi), (xi), (xii) ($X = \mathbf{R}^N$) *and* (xiv) *be fulfilled and let n be a nonnegative integer. If an $\mathcal{M}$-measurable function $f\colon \mathbf{R}^N \to \mathbf{R}^M$ satisfies for every $h \in H$ condition* (29), *then each of the M components of f is equal $\mathcal{N}$ —(a.e.) to a real polynomial in N variables of degree at most n.*

(c) $X = Y = \mathbf{C}$. Since in the present paper we do not go beyond continuity (the analytic structure of $\mathbf{C}$ plays no role whatsoever), we may identify $\mathbf{C}$ with $\mathbf{R}^2$. Thus the present situation becomes the special case $M = N = 2$ of (b), in particular (cf. Lemma 8), for every $z_1, z_2 \in \mathbf{C}\setminus\{0\}$ such that $z_1/z_2 \notin \mathbf{R}$ the set

$$\text{(62)} \qquad E := \{z \in \mathbf{C}\colon z = \lambda_1 z_1 + \lambda_2 z_2,\ \lambda_1, \lambda_2 \in \mathbf{Q}\}$$

is a countable and dense linear subspace of $\mathbf{C}$ over $\mathbf{Q}$ with property (P) ($X = Y = \mathbf{C}$).

Observe that a continuous polynomial function from $\mathbf{C}$ into $\mathbf{C}$ is not necessarily a complex polynomial. In fact, let $f\colon \mathbf{C}\to\mathbf{C}$ be a continuous polynomial function of order n. According to [9; Theorem 15.9.4] either of the (real) functions $\operatorname{Re} f(z)$, $\operatorname{Im} f(z)$ is a polynomial in two real variables $u=\operatorname{Re} z$ and $v=\operatorname{Im} z$ of degree at most n. In other words, there exist real constants a_{jk} and b_{jk}, $j,k=0,\dots,n$, $j+k\leqslant n$, such that for $z=u+iv$

$$(63)\qquad \operatorname{Re} f(z)=\sum_{\substack{j,k=0\\ j+k\leqslant n}}^{n} a_{jk}u^jv^k,\qquad \operatorname{Im} f(z)=\sum_{\substack{j,k=0\\ j+k\leqslant n}}^{n} b_{jk}u^jv^k,\quad z=u+iv\in\mathbf{C}.$$

With $d_{jk}=a_{jk}+ib_{jk}\in\mathbf{C}$ relation (63) yields

$$(64)\qquad f(z)=f(u+iv)=\sum_{\substack{j,k=0\\ j+k\leqslant n}}^{n} d_{jk}u^jv^j,\qquad z\in\mathbf{C}.$$

Setting in (64) $u=\frac{1}{2}(z+\bar z)$, $v=\frac{1}{2}(z-\bar z)\mathrm{i}$ ($\bar z$ denotes the complex conjugate of z) we arrive at a similar expression, but with other coefficients $c_{jk}\in\mathbf{C}$:

$$(65)\qquad f(z)=\sum_{\substack{j,k=0\\ j+k\leqslant n}}^{n} c_{jk}z^j\bar z^k,\qquad z\in\mathbf{C}.$$

In this way we have proved

LEMMA 9. *If $f\colon \mathbf{C}\to\mathbf{C}$ is a continuous polynomial function of order n, then f has form* (65) or, *equivalently*, (64), *where c_{jk} and d_{jk} are complex constants, $j,k=0,\dots,n$, $j+k\leqslant n$.*

Now we assume that

(xv) $H\subset\mathbf{C}$ is a subgroup of $\mathbf{C}$ and there exist $z_1,z_2\in\mathbf{C}\setminus\{0\}$ such that $z_1/z_2\notin\mathbf{R}$ and the set $H\cap E$, where E is given by (62), is dense in $\mathbf{C}$.

PROPOSITION 5. *Let hypotheses* (iv)—(vi), (xi), (xii) ($X=\mathbf{C}$) *and* (xv) *be fulfilled and let n be a nonnegative integer. If an $\mathscr{M}$-measurable function $f\colon \mathbf{C}\to\mathbf{C}$ satisfies for every $h\in H$ condition* (29), *then there exists a continuous polynomial function $\varphi\colon \mathbf{C}\to\mathbf{C}$ of order n (cf. Lemma 9) such that* (30) *holds.*

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