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ADDITIVE FUNCTIONS WITH BIG GRAPHS

Abstract. In this note we show that there exists a collection containing $\mathfrak{c}$ additive functions with big graphs such that $f(x) \neq g(x)$ for every f, g in the collection ($f \neq g$) and every $x \in \mathbf{R}^n \setminus \{\vec{0}\}$.

A function f: $\mathbf{R}^n \to \mathbf{R}$ is called *additive* if it satisfies Cauchy's functional equation, i.e. if $f(x+y) = f(x)+f(y)$ for all $x, y \in \mathbf{R}^n$.

Professor Marek Kuczma (Katowice, Poland), in his book, published in 1985, entitled "An Introduction to the Theory of Functional Equations and Inequalities", [2], presents an up to date and very comprehensive study of additive functions.

If f: $\mathbf{R}^n \to \mathbf{R}$ is discontinuous additive function then the graph of f (symbolically $\mathrm{Gr}(f)$) is dense in $\mathbf{R}^{n+1}$ (see [2]). An additive function f: $\mathbf{R}^n \to \mathbf{R}$ is called an *additive function with small graph* if $f(\mathbf{R}^n)$ is countable. On pages 286 and 287 in [2] it is shown that every additive function having a small graph is discontinuous, the set $\mathrm{Gr}(f)$ is of measure zero and of the first category in $\mathbf{R}^{n+1}$ and is not connected.

Let P: $\mathbf{R}^{n+1} \to \mathbf{R}^n$ be the projection: if $p \in \mathbf{R}^{n+1}$ and $p = (x, y)$, $x \in \mathbf{R}^n$, $y \in \mathbf{R}$, then $P(p) = x$. An additive function f: $\mathbf{R}^n \to \mathbf{R}$ is said to have a *big graph* if for every Borel set $F \subset \mathbf{R}^{n+1}$ such that $P(F)$ has cardinal $\mathfrak{c}$ (the cardinal of continuum) we have $F \cap \mathrm{Gr}(f) \neq \emptyset$. In [2], starting on page 287 several results about additive functions with big graph are presented. In particular it is shown that there exist additive functions with big graphs. This is a result of Jones and can be found in [1].

In this note we will extend the last mentioned result. We will show that there exists a collection containing $\mathfrak{c}$ additive functions with big graphs such that $f(x) \neq g(x)$ for every f, g in the collection ($f \neq g$) and every $x \in \mathbf{R}^n$ different from the zero vector.

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In the following ω_c will denote the smallest ordinal having the cardinal c. $\mathbf{R}^n \setminus \{\vec{0}\}$ can be written in the form $\mathbf{R}^n \setminus \{\vec{0}\} = \{t_a\}_{a<\omega_c}$. Let $\mathscr{F}$ be the family of all Borel sets $F \subset \mathbf{R}^{n+1}$ such that the cardinal of $P(F)$ is c. The cardinal of $\mathscr{F}$ is c and hence $\mathscr{F}$ can be written in the form $\mathscr{F} = \{F_a\}_{a<\omega_c}$(see pg. 287 in [2]). The proof of the theorem mentioned in the last paragraph uses transfinite induction and is more complicated than the proof of Jones result. We now proceed to the proof of our theorem.

THEOREM. *There exists a collection $\{f_a\}_{a<\omega_c}$ of additive functions on $\mathbf{R}^n$ into $\mathbf{R}$ such that each f_a has big graph and such that for each $a < b < \omega_c$ and each $x \in \mathbf{R}^n \setminus \{\vec{0}\}$, $f_a(x) \neq f_b(x)$.*

Proof. Following Kuczma, if $A \subset \mathbf{R}^n$, $E(A)$ will denote the vector subspace (over $\mathbf{Q}$, the rationals) generated by A.

We will now define by transfinite induction, for each a, $a < \omega_c$, two sequences $\{(x_{ba}, y_{ba})\}_{b<\omega_c}$ and $\{(u_{ba}, v_{ba})\}_{b<\omega_c}$ of points in $\mathbf{R}^{n+1}$. The first is to assure that the graph will be "big" and the second is to get a Hamel basis in an appropriate place. This will be done considering larger and larger squares of the double indices a and b.

Step 1. Let (x_{11}, y_{11}) be an arbitrary point in F_1, with $x_{11} \neq \vec{0}$. Let u_{11} be the first t_a (from the sequence $\{t_a\}_{a<\omega_c}$) not in $E(\{x_{11}\})$. Let v_{11} be an arbitrary real number (fixed).

Step 2. $E(\{x_{11}, u_{11}\})$ has cardinality less than c (in fact this set is countable). Therefore there exists a point (x_{21}, y_{21}) in F_2 such that $x_{21} \notin E(x_{11}, u_{11}\})$. Let u_{21} be the first t_a not in $E(\{x_{11}, u_{11}, x_{21}\})$. Let v_{21} be an arbitrary real number. We now proceed to define 4 pairs for the function f_2. $E(\{x_{11}, u_{11}, x_{21}, u_{21}\})$ has cardinality less than c. Therefore there exists a point (x_{12}, y_{12}) in F_1 such that $x_{12} \notin E(\{x_{11}, u_{11}, x_{21}, u_{21}\})$. Let u_{12} be the first t_a not in $E(\{x_{12}\})$. Now we want to select v_{12} so that f_2 can be defined (so far) on $E(\{x_{12}, u_{12}\})$ differs, for each non-zero vector, from f_1 defined on $E(\{x_{11}, u_{11}, x_{21}, u_{21}\})$. For every $s, t \in \mathbf{Q}$ there exists at most one 4-tuple (u, v, w, x) of elements from $\mathbf{Q}$ such that $sx_{12}+tu_{12} = ux_{11}+vu_{11}+wx_{21}+xu_{21}$. There is exactly one choice of v_{12} such that $sy_{12}+tv_{12}$ equals $uy_{11}+vv_{11}+wy_{21}+xv_{21}$. Since $E(\{x_{12}, u_{12}\})$ has cardinality less than c, less than c choices of v_{12} have to be avoided. Therefore, there exists a real number v_{12} such that if $sx_{12}+tu_{12} = ux_{11}+vu_{11}+wx_{21}+xu_{21}$ is not the zero vector, where $s, t, u, v, w, x \in \mathbf{Q}$, then $sy_{12}+tv_{12}$ is not equal to $uy_{11}+vv_{11}+wy_{21}+xv_{21}$. Therefore the pair (u_{12}, v_{12}) has been defined. Similarly, there exists a point (x_{22}, y_{22}) in F_2 such that $x_{22} \notin E(\{x_{11}, u_{11}, x_{21}, u_{21}, x_{12}, u_{12}\})$. Let u_{22} be the first t_a not in $E(\{x_{12}, u_{12}, x_{22}\})$. Arguing as we did about the existence of v_{12} it can be shown that there exists a real number v_{22} such that if $sx_{12}+tu_{12}+ux_{22}+vu_{22} = wx_{11}+xu_{11}+yx_{21}+zu_{21}$ is not the zero vector, where $s, t, u, v, w, x, y, z \in \mathbf{Q}$, then $sy_{12}+tv_{12}+uy_{22}+vv_{22}$ is not equal to $wy_{11}+xv_{11}+yy_{21}+zv_{21}$. This completes step 2.

The process started in steps one and two can be continued by transfinite induction. Namely, suppose that $a < \omega_c$ and for each b, d less than a the pairs (x_{bd}, y_{bd}) have been defined in such a way that:

(1) $(x_{bd}, y_{bd}) \in F_b$ for each b, d less than a.

(2) $x_{bd} \notin E(\{x_{ed}: e < b\} \cup \{u_{ed}: e < b\})$ for each b, d less than a.

(3) u_{bd} is the first t_a not in $E(\{x_{ed}: e \leqslant b\} \cup \{u_{ed}: e < b\})$ for each b, d less than a.

(4) If g, h are less than $a, g \neq h$, and $i_1, \ldots, i_m, j_1, \ldots, j_n, e_1, \ldots, e_s, k_1, \ldots, k_t$ are less than a and

$$u_1 x_{i_1 g} + u_2 x_{i_2 g} + \ldots + u_m x_{i_m g} + v_1 u_{j_1 g} + v_2 u_{j_2 g} + \ldots + v_n u_{j_n g}$$
$$= w_1 x_{e_1 h} + w_2 x_{e_2 h} + \ldots + w_s x_{e_s h} + z_1 u_{k_1 h} + z_2 u_{k_2 h} + \ldots + z_t u_{k_t h}$$

is a non-zero vector, where

$u_1, \ldots, u_m, v_1, \ldots, v_n\ w_1, \ldots, w_s, z_1, \ldots, z_t$ are all in $\mathbf{Q}$, then

$$u_1 y_{i_1 g} + u_2 y_{i_2 g} + \ldots + u_m y_{i_m g} + v_1 v_{j_1 g} + v_2 v_{j_2 g} + \ldots + v_n v_{j_n g}$$

is not equal to as above

$$w_1 y_{e_1 h} + w_2 y_{e_2 h} + \ldots + w_s y_{e_s h} + z_1 v_{k_1 h} + z_2 v_{k_2 h} + \ldots + z_t v_{k_t h}.$$

We now define (in order) the pairs (x_{a1}, y_{a1}), (u_{a1}, v_{a1}); (x_{a2}, y_{a2}), (u_{a2}, v_{a2});...;(x_{ab}, y_{ab}), (u_{ab}, v_{ab});... for all $b < a$. Then we define (in order) (x_{1a}, y_{1a}), (u_{1a}, v_{1a}); (x_{2a}, y_{2a}), (u_{2a}, v_{2a});...; (x_{aa}, y_{aa}), (u_{aa}, v_{aa}). This can be done (since $a < \omega_c$ and an infinite set and its set of all finite subsets have the same cardinality) in such a way as to preserve properties (1) thru (4). Therefore, by transfinite induction, we obtain for each a, $a < \omega_c$, two sequences $(x_{ba}, y_{ba})_{b<\omega_c}$ and $(u_{ba}, v_{ba})_{b<\omega_c}$ of points in $\mathbf{R}^{n+1}$ with the following properties.

(a) $(x_{bd}, y_{bd}) \in F_b$ for each b, d less than ω_c.

(b) For each $a < \omega_c$ the set $\{x_{ba}: b < \omega_c\} \cup \{u_{ba}: b < \omega_c\}$ is a Hamel basis for $\mathbf{R}^n$.

(c) If g, h are less than ω_c, $g \neq h$, and $i_1, \ldots, i_m, j_1, \ldots, j_n, e_1, \ldots, e_s, k_1, \ldots, k_t$ are less than ω_c and

$$u_1 x_{i_1 g} + u_2 x_{i_2 g} + \ldots + u_m x_{i_m g} + v_1 u_{j_1 g} + v_2 u_{j_2 g} + \ldots + v_n u_{j_n g}$$
$$= w_1 x_{e_1 h} + w_2 x_{e_2 h} + \ldots + w_s x_{e_s h} + z_1 u_{k_1 h} + z_2 u_{k_2 h} + \ldots + z_t u_{k_t h}$$

is a non-zero vector, where

$u_1, \ldots, u_m, v_1, \ldots, v_n, w_1, \ldots, w_s, z_1, \ldots, z_t$ are all in $\mathbf{Q}$, then

$$u_1 y_{i_1 g} + u_2 y_{i_2 g} + \ldots + u_m y_{i_m g} + v_1 v_{j_1 g} + v_2 v_{j_2 g} + \ldots + v_n v_{j_n g}$$

is not equal to

$$w_1 y_{e_1 h} + w_2 y_{e_2 h} + \ldots + w_s y_{e_s h} + z_1 v_{k_1 h} + z_2 v_{k_2 h} + \ldots + z_t v_{k_t h}.$$

For each $a < \omega_c, f_a$ is defined to be the unique additive function satisfying: $f_a(x_{ba}) = y_{ba}$ and $f_a(u_{ba}) = v_{ba}$ *for each* $b < \omega_c$. Because of (a), (b) and (c) the collection of functions $\{f_a\}_{a<\omega_c}$ satisfies the conditions of our theorem.

REMARK. It is a trivial exercise to show that there exists a collection $\{f_a\}_{a<\omega_c}$ of additive functions on $\mathbf{R}^n$ into $\mathbf{R}$ such that each f_a has a small graph and such that for each $a < b < \omega_c$ and each $x \in \mathbf{R}^n \setminus \{\vec{0}\}$, $f_a(x) \neq f_b(x)$.

COROLLARY. Since every additive function f: $\mathbf{R}^n \to \mathbf{R}$ with either small graph or large graph has the property that all of this level sets are saturated non-measurable (a set is saturated non-measurable if both it and its complement have inner measure zero) and topologically saturated measurable (a set is topologically saturated non-measurable if neither it nor its complement contains a second category set having the Baire property), see [2] (pg. 297), it follows either from our remark or our theorem that *there exists a collection of subspaces of* $\mathbf{R}^n$, $\{E_a\}_{a<\omega_c}$, *such that each* E_a *is saturated non-measurable and topologically saturated non-measurable and* $E_a \cap E_b = \{\vec{0}\}$ *for each* a, b, $a \neq b$.

REFERENCES

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