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QUASIORTHOGONAL PROJECTIVE SPACES

Abstract. In the preceding paper (see [2]) we defined and investigated quasibilinear functionals on vector spaces, quasiorthogonal and weakly quasiorthogonal vector spaces. In the present paper we give certain applications of these concepts in projective geometry.

In Section 1 we define quasiorthogonal projective spaces (Definition 1) and give analytical representation of these structures (Theorem 1). Theorem 1 may be treated here as an analytical definition of quasiorthogonal projective spaces. In Section 2 the polarity and duality of quasiorthogonal projective spaces is investigated and the main results are given in Theorem 2.

Quasiorthogonal projective spaces are some generalization of real projective spaces with general projective metrics (see [4]). This is shown in Section 3 (Theorem 3).

1. Basic notions. Let us consider an $(n+1)$-dimensional vector space V over a commutative field F of characteristic different from 2 and a relation $\sim\ \subset V\times V$ defined by the condition

$$u \sim v :\Leftrightarrow \exists\ \lambda, \mu \in F\backslash\{0\} \quad (\lambda u = \mu v).$$

The factor space

$$P(V) := (V\backslash\{\Theta\})/\sim$$

is (see [1]) an n-dimensional projective space over F and for arbitrary k-dimensional vector subspace $U < V$ an image $\pi(U\backslash\{\Theta\})$ is a $(k-1)$-dimensional projective subspace of $P(V)$, where π denotes a canonical projection of $V\backslash\{\Theta\}$ onto $P(V)$. Projective subspaces are also projective spaces and 0-dimensional subspaces are called *points* while $(n-1)$-dimensional subspaces are called *hyperplanes* of $P(V)$. The set of all hyperplanes of $P(V)$ is denoted here by $\mathscr{H}(P(V))$. Since we may identify structures $\mathscr{H}(P(V))$ and $P(V^*)$, where V^* denotes a conjugate vector space, then we put

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$$P^*(V) := P(V^*) = \mathscr{H}(P(V))$$

and we say that $P^*(V)$ is *dual* to the $P(V)$ projective space. Analogically we adopt

$$P^{**}(V) := (P^*(V))^* = P(V).$$

In [2] we defined weakly quasiorthogonal vector spaces as structures $(W, \perp)$, where W is some vector space of finite dimension and $\perp$ is a relation of orthogonality of vectors determined by some quasibilinear functional on W. Now we adopt the following:

DEFINITION 1. A structure $(P(\overset{*}{V}), \perp)$ is called a *quasiorthogonal projective space* (or shortly a qps) if and only if $(V^*, \perp_0)$ is a weakly quasiorthogonal vector space and

$$H \perp G :\Leftrightarrow (\pi^{-1}(H) \cup \{\Theta\}) \perp_0 (\pi^{-1}(G) \cup \{\Theta\})$$

for every $H, G \in \mathscr{H}(P(V))$.

A hyperplane $H \in \mathscr{H}(P(V))$ is said to be *singular* iff

$$\forall G \in \mathscr{H}(P(V)) \quad (H \perp G).$$

A quasiorthogonal projective space $(P(V), \perp)$ is said to be *nondegerated* (*degenerated, totally degenerated*) iff there is no singular hyperplane of $P(V)$ (there is a singular hyperplane of $P(V)$, all hyperplanes of $P(V)$ are singular).

A hyperplane $H \in \mathscr{H}(P(V))$ is said to be *isotropic* iff $H \perp H$.

Since each nonzero vector $u \in V$ determines some 1-dimensional vector subspace $\mathrm{Lin}(u)$ of V then we can define a mapping $\varphi: V\backslash\{\Theta\} \to P(V)$ as follows:

$$\varphi(u) := \pi(\mathrm{Lin}(u)\backslash\{\Theta\}) \quad \text{for } u \in V\backslash\{\Theta\}.$$

A tuple $\langle a^0, \ldots, a^{n+1}\rangle$ of points of n-dimensional projective space $P(V)$ is said to be (see e.g. [1]) a *co-ordinate $(n+2)$-frame* of $P(V)$ iff there exists a basis $\langle e^1, \ldots, e^{n+1}\rangle$ of V such that $a^0 = \varphi(e^1 + \ldots + e^{n+1})$ and $a^i = \varphi(e^i)$ for $i = 1, \ldots, n+1$. Each fixed co-ordinate $(n+2)$-frame $\langle a^0, \ldots, a^{n+1}\rangle$ of $P(V)$ uniquely determines homogeneous coordinates $(p_1, \ldots, p_{n+1})_\sim$ of any point $p \in P(V)$ and $(H^1, \ldots, H^{n+1})_\sim$ of any hyperplane $H \in \mathscr{H}(P(V))$. Now by virtue of results of [2] we can formulate the following:

THEOREM 1. *Let $P(V)$ be an n-dimensional projective space over a commutative field F of characteristic different from 2. A structure $(P(V)), \perp)$ is a quasiorthogonal projective space if and only if there exist integer numbers $r = r(\perp)$, $n_0, \ldots, n_r$, scalars $\lambda_1, \ldots, \lambda_{n+1} \in F$ and a co-ordinate $(n+2)$-frame $\langle a^0, \ldots, a^{n+1}\rangle$ of $P(V)$, such that*

(i) $$1 \leqslant r \leqslant n+1,$$

(ii) $$0 = n_0 < n_1 \ldots < n_{r-1} < n_r = n+1,$$

(iii) *if $r > 1$ then* $\forall j \in \{1, \ldots, r-1\} \quad \exists i \in \{n_{j-1}+1, \ldots, n_j\} \quad (\lambda_i \neq 0)$

and for every $H, G \in \mathscr{H}(P(V))$

(iv) $\quad H \perp G \Leftrightarrow \exists j \in \{1,\dots,r\} \quad (\forall i \in \{n_j+1,\dots,n+1\} \quad (H^i = G^i = 0))$
$\wedge \exists i \in \{n_{j-1}+1,\dots,n_j\} \quad (H^i \neq 0 \vee G^i \neq 0) \wedge \Sigma^j \lambda_i H^i G^i = 0),$

where $\Sigma^j \lambda_i H^i G^i$ *denotes the summation over all* i *with* $i \in \{n_{j-1}+1,\dots,n_j\}$.

Moreover, $(P(V), \perp)$ *is nondegenerated iff*

(v) $$\forall i \in \{1,\dots,n+1\} \quad (\lambda_i \neq 0)$$

and totally degenerated iff

(vi) $$\forall i \in \{1,\dots,n+1\} \quad (\lambda_i = 0\}.$$

REMARK. In this paper we adopt the following convention:

$$\forall n,m,k \in \mathbf{Z} \quad (k \in \{n,\dots,m\} :\Leftrightarrow n \leqslant k \leqslant m),$$

where $\mathbf{Z}$ denotes the set of integer numbers. Moreover, the symbol Σ^j always denotes the summation over all i with $i \in \{n_{j-1}+1,\dots,n_j\}$.

This theorem gives an analytical representation of any qps. A co-ordinate $(n+2)$-frame $\langle a^0,\dots,a^{n+1}\rangle$ of $P(V)$ such that the relation $\perp$ is described by the formula (iv) is called here *orthogonal* (with respect to $\perp$). Since by Theorem 1 we may represent any relation $\perp$ in some co-ordinate $(n+2)$-frame $\langle a^0,\dots,a^{n+1}\rangle$ by a suitable sequence $((\lambda_1,\dots,\lambda_{n_1}),\dots,(\lambda_{n_{r-1}+1},\dots,\lambda_{n+1}))$, then we say that this sequence is a *canonical form* of $\perp$ *with respect* to $\langle a^0,\dots,a^{n+1}\rangle$ and we write $\langle\perp\rangle_{\langle a^0,\dots,a^{n+1}\rangle} = ((\lambda_1,\dots,\lambda_{n_1}),\dots,(\lambda_{n_{r-1}+1},\dots,\lambda_{n+1}))$. Since (see [2]) two congruent quasibilinear functionals determine the same relation of orthogonality, then we have:

COROLLARY 1. *If* $(P(V), \perp)$ *is qps and* $\langle\perp\rangle_{\langle a^0,\dots,a^{n+1}\rangle} = ((\lambda_1,\dots,\lambda_{n_1}),\dots,(\lambda_{n_{r-1}+1},\dots,\lambda_{n+1}))$ *then* $\langle\perp\rangle_{\langle a^0,\dots,a^{n+1}\rangle} = ((\varepsilon_1\lambda_1,\dots,\varepsilon_1\lambda_{n_1}),\dots,(\varepsilon_r\lambda_{n_{r-1}+1},\dots,\varepsilon_r\lambda_{n+1}))$ *for every* $\varepsilon_1,\dots,\varepsilon_r \in F\backslash\{0\}$.

It is obvious that the concept of quasiorthogonal projective space is some generalization of the concept of an orthogonal projective space (see e.g. [3], [5]) and we have

COROLLARY 2. *A qps* $(P(V)), \perp)$ *is an orthogonal projective space if and only if* $r(\perp) = 1$.

Additionally, it is clear that we also obtain:

COROLLARY 3. *If* $(P(V), \perp)$ *is a qps then the relation* $\perp$ *is symmetric, i.e.* $H \perp G \Leftrightarrow G \perp H$ *for every* $H, G \in \mathscr{H}(P(V))$.

Now let us assume that $\langle\perp\rangle_{\langle a^0,\dots,a^{n+1}\rangle} = ((\lambda_1,\dots,\lambda_{n_1}),\dots,(\lambda_{n_{r-1}+1},\dots,\lambda_{n+1}))$ and let us put

$$T_0 := P(V),$$

$$T_j := \{p \in P(V)\colon \forall i \in \{1,\dots,n_j\}\ (p_i = 0)\} \quad \text{for } j = 1,\dots,r.$$

It is evident that $T_0,\dots,T_r$ are projective subspaces of $P(V)$, $P(V) = T_0 \supsetneq T_1 \supsetneq \dots \supsetneq T_{r-1} \supsetneq T_r = \varnothing$ and $H \perp G \Leftrightarrow \exists j \in \{1,\dots,r\}$ $(T_j \subset H \cap G \wedge T_{j-1} \not\subset H \cap G \wedge \Sigma^j \lambda_i H^i G^i = 0)$ for $H, G \in \mathscr{H}(P(V))$. Let us define a relation $\perp_1 \subset \mathscr{H}(P(V)) \times \mathscr{H}(P(V))$ as follows:

$$H\perp_1 G :\Leftrightarrow \Sigma^r \lambda_i H^i G^i = 0 \quad \text{for } H, G \in \mathscr{H}(P(V)).$$

It is evident that $H\perp_1 G \Leftrightarrow (T_{r-1} \subset H \vee T_{r-1} \subset G \vee T_{r-1} \not\subset H \cup G \wedge H \perp G)$ for $H, G \in \mathscr{H}(P(V))$ and $(P(V), \perp_1)$ is an orthogonal projective space. Moreover, if $T_{r-1} \neq P(V\ )$ (i.e. $r(\perp) \neq 1$) then T_{r-1} is a singular subspace of $(P(V), \perp_1)$. The set $P(T_{r-1}) := \{H \in \mathscr{H}(P(V)):\ T_{r-1} \subset H\} = \{H \in \mathscr{H}(P(V)):\ \forall i \in \{n_{r-1}+1, \ldots, n+1\}\ (H^i = 0)\}$ is an $(n_{r-1}-1)$-dimensional projective space whose points are hyperplanes of $P(V)$ and putting

$$H \perp_2 G :\Leftrightarrow \Sigma^{r-1} \lambda_i H^i G^i = 0 \quad \text{for } H, G \in P(T_{r-1})$$

and defining a relation $\perp\!\!\!\perp_2 \subset \mathscr{H}(P(V)) \times \mathscr{H}(P(V))$ as a union

$$\perp\!\!\!\perp_2 := (\perp_1 \setminus P(T_{r-1}) \times P(T_{r-1})) \cup \perp_2$$

we easily find that $H \perp\!\!\!\perp_2 G \Leftrightarrow ((H \notin P(T_{r-1}) \vee G \notin P(T_{r-1})) \wedge H \perp_1 G \vee H \in P(T_{r-1}) \wedge G \in P(T_{r-1}) \wedge H \perp_2 G)$ for $H, G \in \mathscr{H}(P(V))$, $(P^*(T_{r-1}), \perp_2)$ is an orthogonal projective space, $(P(V), \perp\!\!\!\perp_2)$ is a qps, $r(\perp\!\!\!\perp_2) = 2$ and if $T_{r-2} \neq P(V)$ then T_{r-2} is a singular subspace of $(P(V), \perp\!\!\!\perp_2)$. Continuing this procedure for $j = 1, \ldots, r$ we obtain orthogonal projective spaces $(P(V), \perp_1)$, $(P^*(T_{r-1}), \perp_2), \ldots, (P^*(T_1), \perp_r)$ and quasiorthogonal projective spaces $(P(V), \perp\!\!\!\perp_1)$, $(P(V), \perp\!\!\!\perp_2), \ldots, (P(V), \perp\!\!\!\perp_r)$ such that $\perp\!\!\!\perp_1 := \perp_1, \perp\!\!\!\perp_r := \perp$, $r(\perp\!\!\!\perp_1) = 1$, $r(\perp\!\!\!\perp_2) = 2, \ldots, r(\perp\!\!\!\perp_r) = r$, $P(T_r) = P(V)$, $P(T_{r-1}) = \{H \in \mathscr{H}(P(V)):\ T_{r-1} \subset H\}, \ldots, P(T_1) = \{H \in \mathscr{H}(P(V)):\ T_1 \subset H\}$, and for every $j \in \{2, \ldots, r\}$ we have:

$$H \perp_j G \Leftrightarrow \Sigma^j \lambda_i H^i G^i = 0 \quad \text{for } H, G \in P(T_{r+1-j}),$$

$$\perp\!\!\!\perp_j = (\perp\!\!\!\perp_{j-1} \setminus P(T_{r+1-j}) \times P(T_{r+1-j})) \cup \perp_j,$$

$$\forall H \in P(T_{r-j}) \quad \forall G \in P(T_{r+1-j}) \quad (H \perp\!\!\!\perp_{j-1} G).$$

This shows the principle concept of quasiorthogonal projective spaces and the method of construction of such structures. To construct a qps $(P(V), \perp)$ it is sufficient to choose a sequence of subspaces $T_0, \ldots, T_r$ such that $P(V) = T_0 \supsetneq T_1 \supsetneq \ldots \supsetneq T_{r-1} \supsetneq T_r = \varnothing$, and next to define relations $\perp_1 \subset \mathscr{H}(P(V)) \times \mathscr{H}(P(V))$, $\perp_2 \subset P(T_{r-1}) \times P(T_{r-1}), \ldots, \perp_r \subset P(T_1) \times P(T_1)$ such that $(P(V), \perp_1)$, $(P^*(T_{r-1}), \perp_2), \ldots, (P^*(T_1), \perp_r)$ are orthogonal projective spaces. Then $(P(V), \perp\!\!\!\perp_r)$ is a qps and $r(\perp\!\!\!\perp_r) = r$.

2. Polarity and duality. Consider an arbitrary n-dimensional qps $(P(V), \perp)$. Each point $p \in P(V)$ such that any hyperplane passing through p is orthogonal to a given hyperplane $H \in \mathscr{H}(P(V))$ is called a *polar of* H. The set of all polars of a given hyperplane H is denoted here by the symbol $p(H)$, i.e.

$$p(H) := \{p \in P(V):\ \ \forall G \in \mathscr{H}(P(V))\ (p \in G \Rightarrow H \perp G)\} \quad \text{for } H \in \mathscr{H}(P(V)).$$

Denoting by p^* a pencil of hyperplanes passing through p i.e.

$$p^* := \{H \in \mathscr{H}(P(V)):\ p \in H\} \quad \text{for } p \in P(V),$$

we may write

(1) $$p(H) = \{p\in P(V)\colon\ \forall G\in p^* \quad (H\perp G)\} \quad \text{for } H\in\mathscr{H}(P(V)).$$

Analogically we define a *polar hyperplane* of a given point $p\in P(V)$ and a set $P(p)$ of all polar hyperplanes of p by the formula

(2) $$P(p) := \{H\in\mathscr{H}(P(V))\colon\ \forall G\in p^* \quad (H\perp G)\} \quad \text{for } p\in P(V).$$

In other words we have the equivalence $H\in P(p)\Leftrightarrow p\in p(H)$. Now we have the following:

LEMMA 1. *Let* $\langle\perp\rangle_{\langle a^0,\dots,a^{n+1}\rangle} = ((\lambda_1,\dots,\lambda_{n_1}),\dots,(\lambda_{n_{r-1}+1},\dots,\lambda_{n+1}))$, $H\in\mathscr{H}(P(V)), 1\leqslant j\leqslant r, \forall i\in\{n_j+1,\dots,n+1\}$ $(H^i = 0)$ *and* $\exists\, i\in\{n_{j-1}+1,\dots,n_j\}$ $(H^i\neq 0)$. *Then*

(i) *if* $\forall i\in\{n_{j-1}+1,\dots,n_j\}$ $(\lambda_i\neq 0)$ *then* $p\in p(H)$ *iff*

$$\forall i\in\{1,\dots,n_{j-1}\} \quad (p_i = 0)$$

and

$$\exists\,\rho\in F\backslash\{0\} \quad \forall i\in\{n_{j-1}+1,\dots,n_j\} \quad (p_i = \rho\lambda_i H^i),$$

(ii) *if* $n_{j-1}+1\leqslant k_j < n_j$, $\forall i\in\{n_{j-1}+1,\dots,k_j\}$ $(\lambda_i\neq 0)$ *and*

$$\forall i\in\{k_j+1,\dots,n_j\} \quad (\lambda_i = 0)$$

then

(a) *if* $\exists\, i\in\{n_{j-1}+1,\dots,k_j\}$ $(H^i\neq 0)$ *then* $p\in p(H)$ *iff*

$$\forall i\in\{1,\dots,n_{j-1}\} \quad (p_i = 0), \quad \forall i\in\{k_j+1,\dots,n_j\} \quad (p_i = 0)$$

and

$$\exists\,\rho\in F\backslash\{0\} \quad \forall i\in\{n_{j-1}+1,\dots,k_j\} \quad (p_i = \rho\lambda_i H^i),$$

(b) *if* $\forall i\in\{n_{j-1}+1,\dots,k_j\}$ $(H^i = 0)$ *then* $p(H) = P(V)$,

(iii) *if* $j = r$ *and* $\forall i\in\{n_{j-1}+1,\dots,n+1\}$ $(\lambda_i = 0)$ *then* $p(H) = P(V)$.

Proof. (i) "$\Rightarrow$". According to our assumptions there exists an $s\in\{n_{j-1}+1,\dots,n_j\}$ such that $H^s\neq 0$. Let $p\in p(H)$ and $p = (p_1,\dots,p_{n+1})_\sim$. Then from Theorem 1 and (1) it follows that $p_s\neq 0$, since if this were not true we could put $G^s = 1$ and $G^i = 0$ for $i\in\{1,\dots,n+1\}\backslash\{s\}$ and obtain $G\in p^*$ and $H\not\perp G$, contradicting the assumptions. Now let us suppose that $j\geqslant 2$ and $p_t\neq 0$ for some $t\in\{1,\dots,n_{j-1}\}$. Putting $G^s = p_t$, $G^t = -p_s$ and $G^i = 0$ for $i\in\{1,\dots,n+1\}\backslash\{t,s\}$ we again find that $G\in p^*$ and $H\not\perp G$. This contradiction proves that:

1. $$\forall i\in\{1,\dots,n_{j-1}\} \quad (p_i = 0\}.$$

Now, if $n_j - n_{j-1} = 1$, then $s = n_{j-1}+1, p_{n_{j-1}+1}\neq 0$ and putting $\rho = (\lambda_{n_{j-1}+1}H^{n_{j-1}+1})^{-1}p_{n_{j-1}+1}$ we have $p_{n_{j-1}+1} = \rho\lambda_{n_{j-1}+1}H^{n_{j-1}+1}$. Let us assume additionally that $n_j - n_{j-1} > 1$. Then for any $t\in n_{j-1}+1,\dots,n_j\}$, taking into account a hyperplane G such that $G^s = p_t$, $G^t = -p_s$, $G^i = 0$ for $i\in\{1,\dots,n+1\}\backslash\{t,s\}$ we see that $G\in p^*$ and $H\perp G$ iff $p_s = \rho\lambda_s H^s$ and $p_t = \rho\lambda_t H^t$ for $\rho\in F\backslash\{0\}$. In this way we obtain

2. $\quad \exists\, \rho \in F\backslash\{0\} \quad \forall i \in \{n_{j-1}+1, \ldots, n_j\} \quad (p_i = \rho \lambda_i H^i).$

"$\Leftarrow$". We assume that $p = (0, \ldots, 0, \lambda_{n_{j-1}+1} H^{n_{j-1}+1}, \ldots, \lambda_{n_j} H^{n_j}, p_{n_j+1}, \ldots, p_{n+1})_\sim$ and $G \in p^*$. Hence $\Sigma^j \lambda_i H^i G^i + \sum_{i=n_j+1}^{n+1} p_i G^i = 0$. Now, if $G^i \neq 0$ for some $i \in \{n_j+1, \ldots, n+1\}$ then from Theorem 1 we have $H \perp G$ because $\forall\, i \in \{n_j+1, \ldots, n+1\}$ $(H^i = 0)$. In the opposite case, if $\forall i \in \{n_j+1, \ldots, n+1\}$ $(G^i = 0)$ then $\Sigma^j \lambda_i H^i G^i = 0$ but this equation gives $H \perp G$ as well. Thus $H \perp G$ for every $G \in p^*$, i.e. $p \in p(H)$.

The proof of the case (ii) (a) is similar to the proof of (i) and in the cases (ii) (b) and (iii) it is sufficient to observe that H is a singular hyperplane, i.e. $\forall\, G \in \mathscr{H}(P(V))$ $(H \perp G)$.

REMARK. For the simplicity of notation we shall denote the canonical form of a relation $\perp$ by $(*)$, i.e. the symbol $(*)$ always denotes here the equation $\langle \perp \rangle_{\langle a^0, \ldots, a^{n+1} \rangle} = ((\lambda_1, \ldots, \lambda_{n_1}), \ldots, (\lambda_{n_{r-1}+1}, \ldots, \lambda_{n+1}))$.

Now, from Lemma 1 we can easily deduce the following:

LEMMA 2. *If* $(*)$, $\forall i \in \{1, \ldots, n+1\}$ $(\lambda_i \neq 0)$, $1 \leqslant j \leqslant r$, $p = (0, \ldots, p_{n_{j-1}+1}, \ldots, p_{n+1})_\sim$ *and* $\exists\, i \in \{n_{j-1}+1, \ldots, n_j\}$ $(p_i \neq 0)$ *then* $H \in P(p)$ *if and only if* $\exists\, \rho \in F\backslash\{0\}$ $\forall i \in \{n_{j-1}, \ldots, n_j\}$ $(H^i = (\lambda_i)^{-1} p_i)$ *and* $\forall i \in \{n_j+1, \ldots, n+1\}$ $(H^i = 0)$.

We may say that points $p, q \in P(V)$ are conjugate with respect to a relation $\perp$ in a qps $(P(V), \perp)$ and we may write $p \perp^* q$ if and only if p lies on some polar hyperplane of q, i.e.

$$(3) \qquad p \perp^* q \;:\Leftrightarrow\; \exists\, H \in P(p) \;\; (q \in H) \quad \text{for } p, q \in P(V).$$

From (1), (3) and Lemma 2 we directly obtain the following

LEMMA 3. *If* $(*)$ *and* $\forall i \in \{1, \ldots, n+1\}$ $(\lambda_i \neq 0)$ *then* $\forall p, q \in P(V)$ $(p \perp^* q \Leftrightarrow (\exists\, i \in \{1, \ldots, n_1\}$ $(p_i \neq 0 \vee q_i \neq 0) \wedge \Sigma^1 (\lambda_i)^{-1} p_i q_i = 0 \vee \exists\, j \in \{2, \ldots, r\}$ $(\forall i \in \{1, \ldots, n_{j-1}\}$ $(p_i = q_i = 0) \wedge \exists\, i \in \{n_{j-1}+1, \ldots, n_j\}$ $(p_i \neq 0 \vee q_i \neq 0) \wedge \Sigma^j (\lambda_i)^{-1} p_i q_i = 0)))$.

These lemmas describe the polarity and the relation $\perp^*$ in any non-degenerated qps. We may now consider the case of a degenerated qps. First we have the following:

LEMMA 4. *If* $(*)$, $j \in \{1, \ldots, r\}$, $n_{j-1} < s \leqslant n_j$, $\lambda_s = 0$ *and* $\forall i \in \{1, \ldots, n+1\} \backslash \{s\}$ $(\lambda_i \neq 0)$ *then*

(i) *if* $j = 1$ *then* $p \perp^* q \Leftrightarrow p_s q_s = 0$ *for* $p, q \in P(V)$,

(ii) *if* $j > 1$ *then* $p \perp^* q \Leftrightarrow (\exists\, i \in \{1, \ldots, n_{j-1}\}$ $(p_i \neq 0 \vee q_i \neq 0) \vee \forall i \in \{1, \ldots, n_{j-1}\}$ $(p_i = q_i = 0) \wedge p_s q_s = 0))$.

Proof. (i). We consider three cases:

I. If $q_s = 0$, then putting $H^s = 1$ and $H^i = 0$ for $i \in \{1, \ldots, n+1\} \backslash \{s\}$ from Theorem 1 we obtain $H \perp G$ for every $G \in \mathscr{H}(P(V))$, hence for any $p \in P(V)$ we have $H \in P(p)$ and consequently $p \perp^* q$.

II. If $q_s \neq 0$ and $p_s = 0$, then putting $H^i = (\lambda_i)^{-1} p_i$ for $i \in \{1, \dots, n_1\} \setminus \{s\}$, $H^s = -(q_s)^{-1}((\lambda_1)^{-1} p_1 q_1 + \dots + (\lambda_{s-1})^{-1} p_{s-1} q_{s-1} + (\lambda_{s+1})^{-1} p_{s+1} q_{s+1} + \dots + (\lambda_{n_1})^{-1} p_{n_1} q_{n_1})$ and $H^i = 0$ for $i \in \{n_1+1, \dots, n+1\}$ we obtain $q \in H$ and $H \perp G$ for every $G \in p^*$ and consequently again $p \perp^* q$.

III. Let $p_s \neq 0$ and $q_s \neq 0$. There is $l \in \{n_{r-1}+1, \dots, n+1\}$ such that $l \neq s$ and there is an $H \in q^*$ such that $H^l \neq 0$. Putting $G^l = 1$, $G^s = -p_l(p_s)^{-1}$ and $G^i = 0$ for $i \in \{1, \dots, n+1\} \setminus \{l, s\}$ we obtain $p \in G$ and $H \not\perp G$ because $\lambda_l H^l \neq 0$. Hence $p \not\perp^* q$ and also $q \not\perp^* p$ because we can exchange p for q.

The proof of (ii) is analogous.

The next step is the following:

LEMMA 5. *If* $(*)$ *and* $\exists\, k, l \in \{1, \dots, n+1\}$ $(k \neq l \wedge \lambda_k = \lambda_l = 0)$, *then* $\forall p, q \in P(V)$ $(p \perp^* q)$.

Proof. Let us assume $\lambda_k = \lambda_l = 0$ and $l \neq k$ for some $l, k \in \{1, \dots, n+1\}$. It is evident that for every point $q \in P(V)$ there exists a hyperplane $H \in q^*$ such that $H^i = 0$ for $i \in \{1, \dots, n+1\} \setminus \{k, l\}$. From Theorem 1 we obtain $\forall G \in \mathscr{H}(P(V))$ $(H \perp G)$ and consequently $p \perp^* q$ for every $p \in P(V)$.

The Lemmas 3, 4 and 5 imply the following:

LEMMA 6. *If* $(P(V), \perp)$ *is a qps then* $\forall p, q \in P(V)$ $(p \perp^* q \Leftrightarrow q \perp^* p)$.

It was stated in Section 1 that $P(V)$ and $\mathscr{H}(P(V))$ are treated here as mutually dual projective spaces. Now for any co-ordinate $(n+2)$-frame $\langle a^0, \dots, a^{n+1} \rangle$ of $P(V)$ we define a dual co-ordinate $(n+2)$-frame

$$\langle a^0, \dots, a^{n+1} \rangle^* := \langle A_0, \dots, A_{n+1} \rangle$$

of $\mathscr{H}(P(V)) = P^*(V)$ putting $A_0 = (1, \dots, 1)_\sim$ and

$$A_j^i = \begin{cases} 1 & \text{when } i = n+2-j, \\ 0 & \text{when } i \neq n+2-j, \end{cases} \qquad \text{for } i, j \in \{1, \dots, n+1\},$$

where $(A_j^1, \dots, A_j^{n+1})_\sim$ are homogeneous coordinates of the hyperplane A_j with respect to $\langle a^0, \dots, a^{n+1} \rangle$ for $j = 0, 1, \dots, n+1$. It is well known that if a point $p \in P(V)$ has homogeneous coordinates $(p_1, \dots, p_{n+1})_\sim$ with respect to $\langle a^0, \dots, a^{n+1} \rangle$ then with respect to $\langle a^0, \dots, a^{n+1} \rangle^*$ it has coordinates $(p^1, \dots, p^{n+1})_\sim$ where $p^i = p_{n+2-i}$ for $i = 1, \dots, n+1$. This property, Theorem 1 and Lemmas 3, 4 and 5 allow the following lemma to be formulated:

LEMMA 7. *If* $(P(V), \perp)$ *is a qps then* $(P^*(V), \perp^*)$ *is a qps. Furthermore, if* $(*)$ *then*

(i) *if* $\lambda_i \neq 0$ *for* $i \in \{1, \dots, n+1\}$ *then* $\langle \perp^* \rangle_{\langle a^0, \dots, a^{n+1} \rangle^*} = (((\lambda_{n+1})^{-1}, \dots, (\lambda_{n_{r-1}+1})^{-1}), \dots, ((\lambda_{n_1})^{-1}), \dots, (\lambda_1)^{-1})$,

(ii) *if there is an* $s \in \{1, \dots, n_1\}$ *such that* $\lambda_s = 0$ *and* $\lambda_i \neq 0$ *for* $i \in \{1, \dots, n+1\} \setminus \{s\}$ *then* $\langle \perp^* \rangle_{\langle a^0, \dots, a^{n+1} \rangle^*} = ((\mu_1, \dots, \mu_{n+1}))$ *where* $\mu_{n+2-s} = 1$ *and* $\mu_i = 0$ *for* $i \in \{1, \dots, n+1\} \setminus \{n+2-s\}$,

(iii) *if there are* $j \in \{1, \dots, r-1\}$ *and* $s \in \{n_j+1, \dots, n_{j+1}\}$ *such that* $\lambda_s = 0$ *and* $\lambda_i \neq 0$ *for* $i \in \{1, \dots, n+1\} \setminus \{s\}$ *then* $\langle \perp^* \rangle_{\langle a^0, \dots, a^{n+1} \rangle^*} = ((\mu_1, \dots, \mu_{n+2-n_j}, (0, \dots, 0))$ *where* $\mu_{n+2-s} = 1$ *and* $\mu_i = 0$ *for* $i \in \{1, \dots, n+2-n_j\} \setminus \{n+2-s\}$,

(iv) *if there are* $s, l \in \{1, \dots, n+1\}$ *such that* $\lambda_l = \lambda_s = 0$ *and* $l \neq s$ *then* $\langle \perp^* \rangle_{\langle a^0, \dots, a^{n+1} \rangle^*} = ((0, \dots, 0))$.

According to our conventions we have $(P(V))^{**} = (P^*(V))^* = P(V)$ and $(\langle a^0, \dots, a^{n+1} \rangle^*)^* = \langle a^0, \dots, a^{n+1} \rangle$ and there is a relation

$$\perp^{**} := (\perp^*)^* \subset \mathscr{H}(P(V)) \times \mathscr{H}(P(V)).$$

A direct consequence of Theorem 1 and Lemma 7 is the following:

THEOREM 2. *If* $(P(V), \perp)$ *is a quasiorthogonal projective space then* $(P^*(V), \perp^*)$ *and* $(P(V), \perp^{**})$ *are also quvsiorthogonal projective spaces. The equation* $\perp = \perp^{**}$ *is satisfied if and only if* $(P(V), \perp)$ *is either nondegenerated or totally degenerated or is a 1-dimensional quasiorthogonal projective space. If* $(P(V), \perp)$ *is a degenerated quasiorthogonal projective space and* $\dim P(V) > 1$ *then* $(P(V), \perp^{**})$ *is a totally degenerated quasiorthogonal projective space.*

3. The connection between real quasiorthogonal projective spaces and projective spaces with general projective metrics. Let us consider an n-dimensional projective space $P(\mathbf{R}^{n+1})$ with an arbitrary co-ordinate $(n+2)$-frame $\langle a^0, \dots, a^{n+1} \rangle$ and let us fix arbitrary integer numbers $r, n_0, n_1, \dots, n_r, l_1, \dots, l_r$ such that $1 \leqslant r \leqslant n+1$ and $0 = n_0 < l_1 \leqslant n_1 < l_2 \leqslant \dots \leqslant n_{r-1} < l_r \leqslant n_r = n+1$. Let us put

$$\varepsilon_i := \begin{cases} 1 & \text{when } i \in \bigcup_{j=1}^{r} \{n_{j-1}+1, \dots, l_j\}, \\ -1 & \text{when } i \in \bigcup_{j=1}^{r} \{l_j+1, \dots, n_j\}, \end{cases}$$

$T_0 := P(\mathbf{R}^{n+1})$ and $T_j := \{p \in P(\mathbf{R}^{n+1}):\ \forall i \in \{1, \dots, n_j\}\ (p_i = 0\}$ for $j = 1, \dots, r$. Now defining the quadric surface $Q := \{p \in P(\mathbf{R}^{n+1}):\ \sum_{i=1}^{n_1} \varepsilon_i (p_i)^2 = 0\}$ and the set $P' := P(\mathbf{R}^{n+1}) \backslash Q$ we may define the following subsets of the Cartesian product $P' \times P'$:

$$D_1 := \{(p, q) \in P' \times P':\ \forall \lambda, \mu \in \mathbf{R}\ (\forall i \in \{1, \dots, n_1\} (\lambda p_i + \mu q_i = 0) \Rightarrow \lambda = \mu = 0)\},$$

$$D_r := \{(p, q) \in P' \times P':\ \exists\, \lambda \in \mathbf{R} \backslash \{0\}\ \ \forall i \in \{1, \dots, n_{r-1}\}\ \ (p_i = \lambda q_i)\},$$

$$D_j := \{(p, q) \in P' \times P':\ \exists\, \lambda \in \mathbf{R} \backslash \{0\}\ \ (\forall i \in \{1, \dots, n_{j-1}\}\ \ (p_i = \lambda q_i)$$
$$\wedge\ \exists\, i \in \{n_{j-1}+1, \dots, n_j\}\ (p_i \neq \lambda q_i))\}\quad \text{for } j \in \{2, \dots, r-1\}.$$

It is easy to verify that $D_1 \cup \dots \cup D_r = P' \times P'$ and $D_i \cap D_j = \varnothing$ for $i \neq j$, $i, j \in \{1, \dots, r\}$. Moreover, if $\overline{pq}$ denotes the projective line passing through distinct points p, q, then $D_1 = \{(p, q) \in P' \times P':\ p \neq q \wedge \overline{pq} \cap T_1 = \varnothing\}$, $D_r = \{(p, q) \in P' \times P':\ p = q \vee p \neq q \wedge \overline{pq} \cap T_{r-1} = \varnothing\}$ and $D_j = \{(p, q) \in P' \times P':\ p \neq q \wedge \overline{pq} \cap T_{j-1} \neq \varnothing \wedge \overline{pq} \cap T_j = \varnothing\}$ for $j \in \{2, \dots, r-1\}$.

Now we may define mappings $\delta_1 : D_1 \to C$, $\delta_2 : D_2 \to C, \ldots, \delta_r : D_r \to C$ as follows:

$$\cos^2 \frac{\delta_1(p,q)}{\rho} = \frac{(\Sigma^1 \varepsilon_i p_i q_i)^2}{(\Sigma^1 \varepsilon_i (p_i)^2)(\Sigma^1 \varepsilon_i (q_i)^2)} \quad \text{for } (p,q) \in D_1,$$

$$(\delta_j(p,q))^2 = \Sigma^j \varepsilon_i (p_i - q_i)^2 \quad \text{for } (p,q) \in D_j \text{ and } j \in \{2, \ldots, r\},$$

where ρ is some fixed nonzero real or pure imaginary number, the homogeneous coordinates of points p, q are normalized by the condition:

$$(p,q) \in D_j \Rightarrow (\Sigma^1 \varepsilon_i (p_i)^2 = \pm \rho^2 \wedge \forall i \in \{1, \ldots, n_{j-1}\} \ (p_i = q_i))$$

for $j \in \{2, \ldots, r\}$, and for every $j \in \{1, \ldots, r\}$ and $(p,q) \in D_j$ we adopt $\operatorname{Im} \delta_j(p,q) \geqslant 0$. The sequence $(\delta_1, \ldots, \delta_r)$ is said to be (see [4]). a *general projective metric on* $P(\mathbf{R}^{n+1})$ and the structure $(P(\mathbf{R}^{n+1}), (\delta_1, \ldots, \delta_r))$ is called either a *semi-elliptic projective space* $S^{n_1, \ldots, n_{r-1}}$ (when $l_i = n_i$ for every $i \in \{1, \ldots, r\}$) or a *semi-hyperbolic projective space* ${}^{l_1, \ldots, l_i}S^{n_1, \ldots, n_{r-1}}$ (when $l_i \neq n_i$ for some $i \in \{1, \ldots, r\}$).

It is evident that each authomorphism of the structure $(P(\mathbf{R}^{n+1}), (\delta_1, \ldots, \delta_r))$ (an isometry) is a projective transformation $\varphi : P(\mathbf{R}^{n+1}) \to P(\mathbf{R}^{n+1})$ such that $\varphi(T_1) = T_1, \ldots, \varphi(T_{r-1}) = T_{r-1}$, $\varphi(Q) = Q$ and $\delta_i(p,q) = \delta_i(\varphi(p), \varphi(q))$ for every $i \in \{1, \ldots, r\}$ and $(p,q) \in D_i$. Moreover, we can define similarities of $(P(\mathbf{R}^{n+1}), (\delta_1, \ldots, \delta_r))$ as projective transforations $\psi : P(\mathbf{R}^{n+1}) \to P(\mathbf{R}^{n+1})$ such that $\psi(T_1) = T_1, \ldots, \psi(T_{r-1}) = T_{r-1}, \psi(Q) = Q$ and $\forall p, q, s, t \in P' \forall j \in \{1, \ldots, r\}$ $((p,q) \in D_j \wedge (s,t) \in D_j \wedge \delta_j(p,q) = \delta_j(s,t) \Rightarrow \delta_j(\psi(p), \psi(q)) = \delta_j(\psi(s), \psi(t))$.

Now we can easily prove the following:

THEOREM 3. *If* $(P(\mathbf{R}^{n+1}), (\delta_1, \ldots, \delta_r))$ *is a real projective space with general projective metric* $(\delta_1, \ldots, \delta_r)$ *and* $\varphi : P(\mathbf{R}^{n+1}) \to P(\mathbf{R}^{n+1})$ *is an arbitrary projective transformation, then* φ *is a similarity of* $(P(\mathbf{R}^{n+1}), (\delta_1, \ldots, \delta_r))$ *if and only if* φ *is an automorphism of a quasiorthogonal space* $(P(\mathbf{R}^{n+1}), \perp)$, *where* $\langle \perp \rangle_{\langle a^0, \ldots, a^{n+1} \rangle} = ((\varepsilon_1, \ldots, \varepsilon_{n_1}), \ldots, (\varepsilon_{n_{r-1}+1}, \ldots, \varepsilon_{n+1}))$.

This last theorem shows that quasiorthogonal projective space is a common generalization of orthogonal projective spaces (see [3], [5]) and real projective spaces with general projective metrics (see [4]).

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