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ON SPACES WITH VECTOR STRUCTURE

Abstract. In [1], using the notion of linear space of translations of the set over the field, n-dimensional Klein spaces over arbitrary field were defined. In [2] the definition of vector structure over the field was given and used to introduce the concept of n-dimensional generalized elementary Klein space.

The aim of present paper is to define (Section 1) and state some of the properties of, so called, spaces with vector structure, without the use of Klein's ideas. In Section 2 it is shown that affine and Euclidean space are the examples of such spaces. Other examples are the elliptic and projective space. Using the notion of vector structure, in Section 3 the definition of tangent bundle is given and some properties of it are observed, with the aim to introduce (Section 4) the concept of m-dimensional hyperplane in spaces with vector structure.

1. Vector structure of a set over the field. The group of transformations $\mathcal{T}(M)$ of a nonempty set M will be called a *group of translations* of this set iff it acts straightly transivitely, i.e. for any $p, q \in M$ there exists one and only one $\tau \in \mathcal{T}(M)$ such that $\tau(p) = q$.

Let $\mathbf{K}$ be an arbitrary field, with zero and unity denoted by 0 and 1, respectively. Note that Abelian group of translations $\mathcal{T}(M)$ with outer operation

$$\cdot : \mathbf{K} \times \mathcal{T}(M) \to \mathcal{T}(M), \tag{1.1}$$

satisfying, for all $a, b \in \mathbf{K}$ and $\tau_1, \tau_2, \tau \in \mathcal{T}(M)$, conditions:

$$\begin{aligned} a\cdot(\tau_1 \circ \tau_2) &= (a\cdot\tau_1)\circ(a\cdot\tau_2), \\ (a+b)\cdot\tau &= (a\cdot\tau)\circ(b\cdot\tau), \\ (ab)\cdot\tau &= a\cdot(b\cdot\tau), \\ 1\cdot\tau &= \tau, \end{aligned} \tag{1.2}$$

is a linear space over $\mathbf{K}$.

According to the definition given in [1], the Abelian group of translations $\mathcal{T}(M)$ with outer operation (1.1) satisfying conditions (1.2) will be called a *linear space of translations* of the set M over the field $\mathbf{K}$ and denoted by $\mathcal{T}(M, \mathbf{K})$.

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In [2] the notion of the group of translations and the linear space of translations of the set M will be generalized as follows.

The group of transformations $\mathscr{T}_D(M)$ of the set M will be called a *group of quasi-translations* of this set with quasi-domain D ($\emptyset \neq D \subset M$), iff it acts straightly transitively on D and for all $\tau \in \mathscr{T}_D(M)$ the condition $\tau|_{M\setminus D} = \mathrm{id}_{M\setminus D}$ holds. Abelian group of quasi-translations $\mathscr{T}_D(M)$ with outer operation:

$$\cdot : \mathbf{K} \times \mathscr{T}_D(M) \to \mathscr{T}_D(M),$$

satisfying for all $a, b \in \mathbf{K}$, $\tau_1, \tau_2, \tau \in \mathscr{T}_D(M)$ conditions (1.2) we will call a *linear space of quasi-translation* of the set M over the field $\mathbf{K}$, and denote by $\mathscr{T}_D(M, \mathbf{K})$.

In particular, when $D = M$, the linear space of quasi-translations is a linear space of translations of M over $\mathbf{K}$.

DEFINITION 1.1. Let $\{\mathscr{T}_D(M, \mathbf{K})\}_{D\in\Lambda}$ be a system of linear spaces of quasi-translations of M over $\mathbf{K}$, where Λ is a family of quasi-domains of these spaces, and let $\{\mathscr{A}_p(M)\}_{p\in M}$ be a system of groups of transformations of M, satisfying, for all $p \in M$ and $\alpha \in \mathscr{A}_p(M)$ the equality $\alpha(p) = p$. The pair

$$(\{\mathscr{T}_D(M, \mathbf{K})\}_{D\in\Lambda}, \{\mathscr{A}_p(M)\}_{p\in M}) \tag{1.3}$$

is called a *vector structure* of the set M over the field $\mathbf{K}$ iff the following axioms remain true:

V1. For all $D, D' \in \Lambda$, $D \neq D'$ *and each* $\tau \in \mathscr{T}_D(M, \mathbf{K})$, $\tau' \in \mathscr{T}_{D'}(M, \mathbf{K})$, $a \in \mathbf{K}$ the following conditions hold: $\tau(D') \in \Lambda$ and

$$\tau \circ \mathscr{T}_{D'}(M, \mathbf{K}) \circ \tau^{-1} = \mathscr{T}_{\tau(D')}(M, \mathbf{K}),$$

$$\tau \circ (a \cdot \tau') \circ \tau^{-1} = a \cdot (\tau \circ \tau' \circ \tau^{-1}).$$

V2. For each $p, q \in M$ there exists a quasi-domain $D \in \Lambda$ such that $p, q \in D$.

V3. For all $p \in M$, $D \in \Lambda$ and $\tau \in \mathscr{T}_D(M, \mathbf{K})$

$$\tau \circ \mathscr{A}_p(M) \circ \tau^{-1} = \mathscr{A}_{\tau(p)}(M).$$

V4. There exists a point $p \in M$ such that for every two quasi-domains D', D'' of the family

$$\Lambda_p := \{D \in \Lambda : \ p \in D\}$$

there exists one and only one transformation $\alpha \in \mathscr{A}_p(M)$ satisfying conditions:

(a) $\alpha(D') = D''$ and $\alpha \circ \mathscr{T}_{D'}(M, \mathbf{K}) \circ \alpha^{-1} = \mathscr{T}_{D''}(M, \mathbf{K})$,

(b) for each $a \in \mathbf{K}$ and $\tau' \in \mathscr{T}_{D'}(M, \mathbf{K})$ the relation

$$\alpha \circ (a \cdot \tau') \circ \alpha^{-1} = a \cdot (\alpha \circ \tau' \circ \alpha^{-1})$$

holds true,

(c) for each $\tau' \in \mathscr{T}_{D'}(M, \mathbf{K})$, $\tau'' \in \mathscr{T}_{D''}(M, \mathbf{K})$ such that $\tau'(p) = \tau''(p)$, quasi-translations $\alpha \circ \tau' \circ \alpha^{-1}$ and τ'' are linearly dependent.

The question, whether these axioms are independent was not considered.

Now, let

$$(1.4)\qquad \mathscr{T}_A(M,\mathbf{K}):=\bigcup_{D\in A}\mathscr{T}_D(M,\mathbf{K})$$

and consider, for $\tau\in\mathscr{T}_A(\mathrm{M},\mathbf{K})$, the function

$$(1.5)\qquad L^*_\tau:\mathscr{T}_A(M,\mathbf{K})\to\mathscr{T}_A(M,\mathbf{K}),\qquad L^*_\tau(\tilde{\tau})=\tau\circ\tilde{\tau}\circ\tau^{-1}.$$

As an immediate consequence of axioms V1—V4 we get the following two corollaries:

COROLLARY 1.1. *For each $\tau\in\mathscr{T}_A(M,\mathbf{K})$ mapping* (1.5) *is a bijection. Moreover, for any $D\in A$, its restriction $L^*_\tau|_{\mathscr{T}_D(M,\mathbf{K})}$ is a linear isomorphism of linear space $\mathscr{T}_D(M,\mathbf{K})$ onto $\mathscr{T}_{\tau(D)}(M,\mathbf{K})$.*

COROLLARY 1.2. *Conditions stated in axiom* V4 *are satisfied in any point $p\in M$.*

Using these corollaries and axioms V1—V4, one can prove (cf. [2]) that the following corollary is true.

COROLLARY 1.3. *All linear spaces of the system $\{\mathscr{T}_D(M,\mathbf{K})\}_{D\in A}$ are isomorphic.*

In virtue of Corollary 1.3 we can observe that all linear spaces of the system $\{\mathscr{T}_D(M,\mathbf{K})\}_{D\in A}$ are of the same dimension.

DEFINITION 1.2. Common dimension of all linear spaces $\mathscr{T}_D(M,\mathbf{K})$, $D\in A$, will be called a *dimension of vector structure* (1.3).

Now, let $\mathscr{T}(M,\mathbf{K})$ be a linear space of translations of the set M over $\mathbf{K}$ and let for each $p\in M$ $\mathscr{A}_p(M)$ be a trivial group $\{\mathrm{id}_M\}$. Observe that the pair

$$(1.6)\qquad (\{\mathscr{T}(M,\mathbf{K})\},\ \{\mathscr{A}_p(M)\}_{p\in M}),\qquad \mathscr{A}_p(M):=\{\mathrm{id}_M\}\quad\text{for } p\in M,$$

satisfies axioms V1—V4 and is, therefore, a vector structure of the set M over the field $\mathbf{K}$. In [2] the pair (1.6) was called an *elementary vector structure.*

DEFINITION 1.3. The set M with determined n-dimensional vector structure (1.3) ever $\mathbf{K}$ will be called an n-dimensional space with vector structure over $\mathbf{K}$ and denoted by $M^n(\mathbf{K})$ or, shortly, M.

2. Examples of spaces with vector structure. It is well known that the affine space over the field $\mathbf{K}$ with free vector space V we call the set M with such a mapping $\omega: M\times M\to V$ (called an *atlas* of this space) that the following axioms are satisfied:

A1. For each $p\in M$ and $v\in V$ there exists exactly one point $q\in M$ such that $\omega(p,q)=v$.

A2. For each $p,q,r\in M$ the equality

$$\omega(p,q)+\omega(q,r)=\omega(p,r)$$

holds true.

In virtue of axiom A1, for arbitrarily fixed vector $v\in V$ we can define the mapping $\tau_v: M\to M$ as follows:

$$\tau_v(p)=q\ \Leftrightarrow\ \omega(p,q)=v.$$

In particular, for zero-vector Θ we have $\tau_{\Theta} = \mathrm{id}_M$. Moreover, by axiom A2, the equality $\tau_{v+w} = \tau_w \circ \tau_v$ holds. Since the additive group of linear space is Abelian, $\tau_v \circ \tau_w = \tau_w \circ \tau_v$. Therefore the mapping τ_v is a bijection and the set of all such mappings forms an Abelian group of transformations of the set M. This group will be denoted by $\mathscr{T}(M)$.

It is easily seen that group $\mathscr{T}(M)$ acts straightly transitively on the set M. Moreover, the group with multiplication by elements of $\mathbb{K}$ defined by the formula: $a \cdot \tau_v := \tau_{a \cdot v}$ forms a linear space $\mathscr{T}(M, \mathbb{K})$ over $\mathbb{K}$.

The transformation $\varphi: V \to \mathscr{T}(M, \mathbb{K})$, $\varphi(v) = \tau_v$ is a linear isomorphism. Hence, linear space V is isomorphic with defined above linear space of translations $\mathscr{T}(M, \mathbb{K})$ of M over $\mathbb{K}$. Let us note that the transformation $\bar{\omega}: M \times M \to \mathscr{T}(M, \mathbb{K})$ defined by the formula $\bar{\omega}(p, q) = \tau$, where τ is the unique translation satisfying condition $\tau(p) = q$, satisfies axioms A1 and A2. Therefore the set M with such defined transformation is an affine space (over $\mathbb{K}$) with the free vector space $\mathscr{T}(M, \mathbb{K})$, being equivalent with affine space with free vector space V and linear atlas ω.

Thus we can appropriate the following definition.

DEFINITION 2.1. Space with vector structure (1.6) will be called *affine space* over the field $\mathbb{K}$.

As the different example of the space with vector structure, consider an n-dimensional projective space over $\mathbb{K}$. In [2] the construction of vector structure of this space is discussed. By the obvious reasons, Euclidean, pseudo-Euclidean and elliptic spaces are also spaces with vector structures.

3. Tangent bundle of the space with vector structure. Consider an n-dimensional space $M^n(\mathbb{K})$ (over $\mathbb{K}$) with vector structure and the set

$$\mathscr{T}M := \{(p, \tau): \;\; p \in M, \tau \in \mathscr{T}_D(M, \mathbb{K}), D \in \Lambda_p\}.$$

We will define a relation in this set, as follows:

DEFINITION 3.1. We say that (p, τ) and (q, τ') are in the relation $\sim$ iff $p = q$ and there exists an $\alpha \in \mathscr{A}_p(M)$ such that $\alpha \circ \tau \circ \alpha^{-1} = \tau'$.

It is easily seen that $\sim$ is an equivalence relation.

DEFINITION 3.2. Quotient set $TM := \mathscr{T}M/_{\sim}$ will be called a *tangent bundle* of the space with vector structure $M^n(\mathbb{K})$. Classes of abstraction $[(p, \tau)]$ will be called *tangent vectors* to the space $M^n(\mathbb{K})$ in the point p. The set of all tangent vectors to this space in an arbitrary fixed point p will be denoted by T_pM and called *tangent space* to the space $M^n(\mathbb{K})$ in the point p.

It follows from Corollary 1.2 and two above definitions that for each tangent vector $[(p, \tau)]$ and each $D' \in \Lambda_p$ there exists one and only one quasi-translation $\tau' \in \mathscr{T}_{D'}(M, \mathbb{K})$ such that $(p, \tau') \in [(p, \tau)]$. Moreover, if

$$v_p = [(p, \tilde{\tau})], \;\; w_p = [(p, \check{\tau})], \;\; \text{where } \tau, \tilde{\tau} \in \mathscr{T}_D(M, \mathbb{K}),$$

and

$$(p, \tau') \in v_p, \;\; (p, \tilde{\tau}') \in w_p, \;\; \text{where } \tau', \tilde{\tau}' \in \mathscr{T}_{D'}(M, \mathbb{K}),$$

then

$$(p, \tau' \circ \tilde{\tau}') \in [(p, \tau \circ \tilde{\tau})] \text{ and } (p, a \cdot \tau') \in [(p, a \cdot \tau)] \text{ for each } a \in \mathbb{K}.$$

Thus, in every space T_pM we can define the union of two vectors

(3.1) $$[(p,\tau)]+[(p,\tilde{\tau})]:=[(p,\tau\circ\tilde{\tau})], \text{ where } \tau,\tilde{\tau}\in\mathcal{T}_D(M,\mathbf{K})$$

and the product of a vector by the element a of the field $\mathbf{K}$

(3.2) $$a[(p,\tau)]:=[(p,a\cdot\tau)].$$

Note, that the following two corollaries are immediate consequences of the above.

COROLLARY 3.1. *In each point p of the space $M^n(\mathbf{K})$, the tangent space T_pM with operations* (3.1) *and* (3.2) *forms an n-dimensional linear space over* $\mathbf{K}$.

COROLLARY 3.2. *Vectors* $v_{i_p}=[(p,\tau_i)]$, *where* $\tau_i\in\mathcal{T}_D(M,\mathbf{K})$, $D\in\Lambda_p$, $i=1,2,\ldots,m$ *are linearly dependent iff quasi-translations* τ_i, $i=1,2,\ldots,m$ *are linearly dependent.*

In each point p the zero tangent vector O_p is one-element class of abstraction $[(p,\mathrm{id}_M)]$.

If $M^n(\mathbf{K})$ is an affine space, the equality $TM=M\times\mathcal{T}(M,\mathbf{K})$ holds true. Hence, for affine spaces we usually consider the free vector space instead of tangent bundle.

It is easy to note that for each $\tau\in\mathcal{T}_A(M,\mathbf{K})$ the mapping (cf. [2])

(3.3) $$L_\tau\colon TM\to TM,\quad L_\tau([(p,\tilde{\tau})]):=[(\tau(p),\tau\circ\tilde{\tau}\circ\tau^{-1})]$$

is well defined and is a bijection.

DEFINITION 3.3. Mapping (3.3) will be called a *parallel transfer* of tangent bundle TM. If $L_\tau(v_p)=w_{\tau(p)}$ we say that tangent vector $w_{\tau(p)}$ is a parallel transfer of tangent vector v_p from the point p to $\tau(p)$.

By axiom V2, each vector v_p can be parallelly transferred from arbitrary point p to arbitrary point q.

It is easy to note that for each $\tau\in\mathcal{T}_A(M,\mathbf{K})$, $p\in M$, $v_p,w_p\in T_pM$ and $a\in\mathbf{K}$ we have:

$$L_\tau(v_p+w_p)=L_\tau(v_p)+L_\tau(w_p),\qquad L_\tau(av_p)=aL_\tau(v_p).$$

Therefore the following corollary is true.

COROLLARY 3.3. *Each parallel transfer* (3.3) *is a bijection. Moreover, for each $p\in M$ its restriction $L_\tau|_{T_pM}$ is a linear isomorphism of tangent space T_pM onto tangent space $T_{\tau(p)}M$.*

4. Straight lines in spaces with vector structure. Using tangent bundle TM we can define m-dimensional hyperplane in n-dimensional space with vector structure $M^n(\mathbf{K})$ $(0<m<n)$. First, we will formulate and observe some basic properties of straight lines.

DEFINITION 4.1. Let v_p be a non-zero tangent vector to $M^n(\mathbf{K})$. The set

(4.1) $$H^1(v_p):=\{(c\cdot\tau)(p)\colon\ c\in\mathbf{K},\ (p,\tau)\in v_p\}$$

will be called a *straight line* or *one-dimensional hyperplane* of the space $M^n(\mathbf{K})$ determined by vector v_p. Tangent vector v_p will be called a *directional vector* of the straight line (4.1).

It can be shown that if $M^n(\mathbf{K})$ is an affine (projective) space, then the straight line as defined above is also affine (projective) straight line.

Since for each $(p,\tau)\in v_p=[(p,\tau_0)]$, we have

$$(c\cdot\tau)(p)=(c\cdot(\alpha\circ\tau_0\circ\alpha^{-1}))(p)=(\alpha\circ(c\cdot\tau_0)\circ\alpha^{-1})(p)$$
$$=(\alpha\circ c\cdot\tau_0)(p),\quad \text{where } \alpha\in\mathscr{A}_p(M),$$

then every straight line determined by vector v_p can be also defined by the formula

$$H^1([(p,\tau_0)])=\{(\alpha\circ c\cdot\tau_0)(p):\quad \alpha\in\mathscr{A}_p(M),\ c\in\mathbf{K}\}. \tag{4.2}$$

This equality does not depend of the choice of representative of the vector $v_p=[(p,\tau_0)]$.

Now, we will study some properties of straight lines.

LEMMA 4.1. *If there exists a common point of straight lines $H^1(v_p)$ and $H^1(w_p)$, different from p, then directional vectors v_p and w_p are linearly dependent.*

Proof. Let q be a common point of straight lines $H^1(v_p)$ and $H^1(w_p)$ different from p. Then, according to Definition 4.1, there exist non-zero c', $c''\in\mathbf{K}$ and $(p,\tau')\in v_p$, $(p,\tau'')\in w_p$ such that

$$q=(c'\cdot\tau')(p)=(c''\cdot\tau'')(p),$$

where $\tau'\in\mathscr{T}_{D'}(M,\mathbf{K})$, $\tau''\in\mathscr{T}_{D''}(M,\mathbf{K})$, $D', D''\in\Lambda_p$. Thus, by Corollary 1.2, there exists a transformation $\alpha\in\mathscr{A}_p(M)$ such that quasi-translations $\alpha\circ(c'\cdot\tau')\circ\alpha^{-1}$ and $c''\cdot\tau''$ are linearly dependent. Since $(p,\alpha\circ\tau'\circ\alpha^{-1})\in v_p$, then, in virtue of Corollary 3.2, tangent vectors v_p and w_p are linearly dependent.

Now, we will formulate the necessary and sufficient condition for the equality

$$H^1(v_p)=H^1(w_p),\quad \text{where } v_p, w_p\in T_pM. \tag{4.3}$$

THEOREM 4.1. *Straight lines $H^1(v_p)$ and $H^1(w_p)$ coincide iff vectors v_p and w_p are linearly dependent.*

Proof. If (4.3) holds, then the point $q=\tau(p)$, where $(p,\tau)\in v_p$ is a common point of these straight lines. Then, by Lemma 4.1, vectors v_p and w_p are linearly dependent.

Conversely, if $v_p=[(p,\tau)]$ and $w_p=[(p,\tilde{\tau})]$, where $\tau,\tilde{\tau}\in\mathscr{T}_D(M,\mathbf{K})$, $D\in\Lambda_p$, are linearly dependent, then, by Corollary 3.2, quasi-translations τ and $\tilde{\tau}$ are linearly dependent as well. Hence, by Definition 4.1, we get (4.3).

THEOREM 4.2. *For any quasi-translation $\tau\in\mathscr{T}_\Lambda(M,\mathbf{K})$, the image $\tau(H^1(v_p))$ of the straight line $H^1(v_p)$ is a straight line determined by vector $L_\tau(v_p)$, being the parallel transform* (3.3) *of v_p from point p to point $\tau(p)$, i.e. the equality*

$$(4.4) \qquad \tau(H^1(v_p)) = H^1(L_\tau(v_p))$$

holds true.

Proof. If $v_p = [(p, \tilde{\tau})]$, $q = \tau(p)$, $w_q = L_\tau(v_p)$, then, according to Definitions 4.1 and 3.3 and axiom V1 we get

$$\begin{aligned} H^1(w_q) &= \{(c\cdot\tau')(q): \quad c\in\mathbf{K}, (q,\tau')\in w_q\} \\ &= \{(c\cdot(\tau\circ\tilde{\tau}\circ\tau^{-1}))(\tau(p)): \quad c\in\mathbf{K}, \ (p,\tilde{\tau})\in v_p\} \\ &= \{\tau((c\cdot\tilde{\tau})(p)): \quad c\in\mathbf{K}, \ (p,\tilde{\tau})\in v_p\} = \tau(H^1(v_p)), \end{aligned}$$

which proves equality (4.4).

LEMMA 4.2. *If* $(p,\tau)\in v_p$, *then*

$$(4.5) \qquad H^1(v_p) \subset \tau(H^1(v_p)).$$

Proof. Let $(p,\tau)\in v_p$, where $\tau\in\mathcal{T}_D(M,\mathbf{K})$, $D\in\Lambda_p$ and let q be an arbitrary point of the straight line $H^1(v_p)$. Then there exist such $a\in\mathbf{K}$, $\tau'\in\mathcal{T}_{D'}(M,\mathbf{K})$, $D'\in\Lambda_p$ that $(p,\tau')\in v_p$ and

$$(4.6) \qquad q = (a\cdot\tau')(p).$$

Consider two cases:

a) $q\in D$. Then there exists such $\tilde{\tau}\in\mathcal{T}_D(M,\mathbf{K})$ that $\tilde{\tau}(p) = q$. Then, by (4.6), $(a\cdot\tau')(p) = \tilde{\tau}(p)$. In virtue of Corollary 1.2, there exists a transformation $\alpha: D'\to D$, $\alpha\in\mathcal{A}_p(M)$ such that quasi-translations $\alpha\circ(a\cdot\tau')\circ\alpha^{-1}$ and $\tilde{\tau}$ are linearly dependent. It is easily seen that $\alpha\circ\tau'\circ\alpha^{-1} = \tau$. Thus, for some $b\in\mathbf{K}$, $\tilde{\tau} = b\cdot\tau$. Therefore

$$q = \tilde{\tau}(p) = (b\cdot\tau)(p) = \tau(((b-1)\cdot\tau)(p)),$$

which means that $q\in\tau(H^1(v_p))$.

b) $q\notin D$. Then $\tau(q) = q$, and also $q\in\tau(H^1(v_p))$. Thus, we have proved the implication

$$q\in H^1(v_p) \Rightarrow q\in\tau(H^1(v_p)),$$

proving inclusion (4.5).

LEMMA 4.3. *If* $(p,\tau)\in v_p$, *then*

$$(4.7) \qquad H^1(v_p) = \tau(H^1(v_p)).$$

Proof. Let $v_p = [(p,\tau)]$, $q = \tau(p)$, $w_q = L_\tau(v_p)$. Hence, obviously, $-w_q = [(q,\tau^{-1})]$. By Lemma 4.2,

$$H^1(-w_q) \subset \tau^{-1}(H^1(-w_q)).$$

Simultaneously, in virtue of Theorems 4.1 and 4.2 we know that $H^1(-w_q) = H^1(w_q) = \tau(H^1(v_p))$. Hence $\tau(H^1(v_p)) \subset \tau^{-1}(\tau(H^1(v_p))) = H^1(v_p)$. Thus, by Lemma 4.2, we get equality (4.7).

Using above lemmas and theorems, we will prove the following:

LEMMA 4.4. *If $p \neq q \in H^1(v_p)$ and τ is a quasi-translation satisfying condition $\tau(p) = q$, then*

$$(4.8) \qquad H^1(v_p) = H^1(L_\tau(v_p)).$$

Proof. Since q is a common point of straight lines $H^1(v_p)$ and $H^1([(p, \tau)])$, different from p, then, by Lemma 4.1, vectors v_p and $[(p, \tau)]$ are linearly dependent. Thus, by Theorem 4.1, we get the equality $H^1(v_p) = H^1(([(p, \tau)])$. In virtue of Lemma 4.3, $H^1([(p, \tau)]) = \tau(H^1([(p, \tau)])$ and, therefore, $H^1(v_p) = \tau(H^1(v_p))$. Hence, by Theorem 4.2, we get equality (4.8).

Now, we will show the necessary and sufficient condition for the equality

$$(4.9) \qquad H^1(v_p) = H^1(w_q) \quad \text{with } p \neq q$$

to hold true.

THEOREM 4.3. *Two straight lines $H^1(v_p)$ and $H^1(w_q)$ determined by tangent vectors in different points, coincide iff $q \in H^1(v_p)$ and vectors $L_\tau(v_p)$ and w_q are linearly dependent, where τ is an arbitrary quasi-translation satisfying condition $\tau(p) = q$.*

Proof. If the given straight lines coincide, then $q \in H^1(v_p)$. Thus, by Lemma 4.4, we get $H^1(v_p) = H^1(L_\tau(v_p)) = H^1(w_q)$. Hence, in virtue of Theorem 4.1, vectors $L_\tau(v_p)$ and w_q are linearly dependent.

Conversely, if $q \in H^1(v_p)$ and vectors $L_\tau(v_p)$ and w_q are linearly dependent, then, by Theorem 4.1, $H^1(L_\tau(v_p)) = H^1(w_q)$, whereas in virtue of Lemma 4.4 equality (4.8) holds. That proves (4.9).

According to axiom V2, through every two points p_1 and p_2 runs a straight line, namely $H_1([(p_1, \tau)])$, where τ is a quasi-translation satisfying $\tau(p_1) = p_2$. It is easy to note that for each straight line running through points p_1 and p_2 the equality $H^1(v_p) = H^1([(p_1, \tau)])$ holds true. It follows that we can consider straight lines in a space with vector structure, without considering their directional vectors. Moreover, we have the following:

THEOREM 4.4. *Through every two distinct points of space with vector structure one and only one straight line runs.*

It can be shown that similar properties are possessed by m-dimensional hyperplanes defined (cf. (4.2)) as follows:

DEFINITION 4.2. Let $v_{ip} = [(p, \tau_i)]$, where $\tau_i \in \mathcal{T}_D(M, \mathbf{K})$, $D \in A_p$ for $i = 1, 2, \ldots, m$, $0 < m < n$, be a system of linearly independent tangent vectors in the point p to the space $M^n(\mathbf{K})$. The set

$$H^m(v_{1p}, v_{2p}, \ldots, v_{mp})$$
$$:= \{(\alpha \circ (c_1 \cdot \tau_1 \circ c_2 \cdot \tau_2 \circ \ldots \circ c_m \cdot \tau_m))(p) \colon \ \alpha \in \mathscr{A}_p(M), \ \ c_i \in \mathbf{K} \text{ for } i = 1, \ldots, m\}$$

will be called an *m-dimensional hyperplane* in the space $M^n(\mathbf{K})$, determined by vectors v_{ip}, $i = 1, 2, \ldots, m$.

REFERENCES

[1] B. SZOCIŃSKI, *Elementary Klein spaces*, Demonstratio Math. (in print).
[2] B. SZOCIŃSKI, *On some generalization of elementary Klein space*, Zeszyty Nauk. Politech. Śląsk. Mat.-Fiz. (in print).
[3] B. SZOCIŃSKI, *Some properties of the projective space*, Demonstratio Math. (in print).