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LINKAGE AND THE BASIC PART OF WITT RINGS

Abstract. The strength of two axioms placed on a quaternionic mapping is compared. The first is the linkage axiom (L), and the second is the structure of the basic part axiom (X). It is shown here that (L) is strictly stronger than (X).

Let $q\colon G\times G\to B$ be a quaternionic mapping in the terminology of [2]. Recall that this means G and B are groups of exponent two, G has a distinguished element -1 and q is a symmetric bilinear mapping satisfying $q(a,-a)=1$ for every $a\in G$. For $a\in G$, let $D\langle 1, a\rangle=\{b\in G\mid q(-a, b)=1\}$. $D\langle 1, a\rangle$ is a subgroup of G containing both 1 and a. If $D\langle 1, a\rangle=\{1, a\}$ then a is called *rigid* and if both a and $-a$ are rigid a is said to be *2-sided rigid.* The *basic part* of G is the set

$$B_G=\{\pm 1\}\cup\{a\in G\mid a \text{ is not 2-sided rigid}\}$$

and for any $a\in G$ we define the sets $X_i(a)$ as in [1]. We let $X_1(a)=D\langle 1,-a\rangle$ and inductively define

$$X_i(a)=\bigcup\{D\langle 1,-z\rangle\mid 1\neq z\in X_{i-1}(a)\}$$

for $i>1$.

If a quaternionic mapping q also satisfies:

(L) $q(a, b)=q(c, d)\Rightarrow\exists\ x\in G$ with $q(a, b)=q(a, x)$ and $q(c, d)=q(c, x)$

then q is said to be a *linked* quaternionic mapping. (L) is the most powerful of the axioms placed on q. Its full strength has yet to be determined.

In [1] it is shown that if q is a quaternionic mapping with $|G|<\infty$ then (L) implies

(X) $\qquad B_G=\pm X_1(a)X_3(a)\cup X_1(a)X_2(a)^2$ for every $a\in B_G\backslash\{1\}$.

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This result is quite strong and requires several clever usages of (L). Indeed it was thought that perhaps (X) was strong enough to imply (L) when $|G| < \infty$. The purpose of this note is to provide an example showing the contrary.

We begin with G any finite group of exponent two of dimension $n \geqslant 4$ over $\mathbf{F}_2$ and we fix a basis $A = \{a_1, a_2, \ldots, a_n\}$ for G. Let B be any other group of exponent two of dimension $n-2$ over $\mathbf{F}_2$ with fixed basis say $Q = \{q_2, q_3, \ldots, q_{n-1}\}$. Define q on $A \times A$ by

$$q(a_i, a_j) = \begin{cases} q_j, & \text{if } i = 1 \text{ and } 2 \leqslant j \leqslant n-1, \\ q_i, & \text{if } j = 1 \text{ and } 2 \leqslant i \leqslant n-1, \\ q_{n-j+1}, & \text{if } i = n \text{ and } 2 \leqslant j \leqslant n-1, \\ q_{n-i+1}, & \text{if } j = n \text{ and } 2 \leqslant i \leqslant n-1, \\ 1, & \text{otherwise,} \end{cases}$$

and extend q to $q\colon G \times G \to B$ by bilinearity. For future reference we display the matrix $[q(a_i, a_j)]$:

$$\begin{bmatrix} 1 & q_2 & q_3 & \ldots & q_{n-1} & 1 \\ q_2 & 1 & 1 & \ldots & 1 & q_{n-1} \\ q_3 & 1 & 1 & \ldots & 1 & q_{n-2} \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ q_{n-1} & 1 & 1 & \ldots & 1 & q_2 \\ 1 & q_{n-1} & q_{n-2} & \ldots & q_2 & 1 \end{bmatrix}$$

Notice that $q(a, a) = 1$ for every $a \in G$ hence q is a quaternionic mapping with $-1 = 1$. Also notice that $D\langle 1, a_2 \cdot a_3 \cdot \ldots \cdot a_{n-1} \rangle$ is spanned by $a_2, a_3, \ldots, a_{n-1}$, $a_1 a_n$ hence $D\langle 1, a_2 \cdot a_3 \cdot \ldots \cdot a_{n-1} \rangle$ has index 2 in G.

THEOREM. *q is a quaternionic mapping which satisfies* (X) *but not* (L).

Proof. For $a \in G$ let $G_i(a) = \{b \in G \mid q(a, b) = q_i\}$, $i = 2, 3, \ldots, n-1$. Hence $G_2(a_1) = \{a_2, a_1 a_2, a_2 a_n, a_1 a_2 a_n\}$ and $G_2(a_n) = \{a_{n-1}, a_1 a_{n-1}, a_{n-1} a_n, a_1 a_{n-1} a_n\}$. Notice that $q(a_1, a_2) = q(a_n, a_{n-1}) = q_2$ but there is no $x \in G$ with $q(a_1, a_2) = q(a_1, x) = q(a_n, x) = q(a_n, a_{n-1})$ since $G_2(a_1) \cap G_2(a_n) = \varnothing$ for $n \geqslant 4$. Consequently (L) is not satisfied. We now show that (X) does hold.

Step 1. There are no rigid elements in G thus $B_G = G$.

Let $x \in G$ and write $x = a_{i_1} a_{i_2} \ldots a_{i_s}$, $i_1 < i_2 < \ldots < i_s$. If $i_1 = 1$ and $i_s = n$ then $a_2 \cdot a_3 \cdot \ldots \cdot a_{n-1} \in D\langle 1, x \rangle \backslash \{1, x\}$. If exactly one of $i_1 = 1$ or $i_s = n$ holds then $a_{n-i_1+1} \cdot a_{n-i_2+1} \cdot \ldots \cdot a_{n-i_s+1} \in D\langle 1, x \rangle \backslash \{1, x\}$. If $i_1 \neq 1$ and $i_s \neq n$ then $a_2 \cdot a_3 \cdot \ldots \cdot a_{n-1} \in D\langle 1, x \rangle \backslash \{1, x\}$ unless $x = a_2 \cdot a_3 \cdot \ldots \cdot a_{n-1}$ in which case $D\langle 1, x \rangle$ has index 2 in G. Consequently, x is not rigid.

Step 2. $D\langle 1, x \rangle \cap D\langle 1, a_2 \cdot a_3 \cdot \ldots \cdot a_{n-1} \rangle \neq \{1\}$ for every $x \in G$.

If $x = 1$ or $x \in D\langle 1, a_2 \cdot a_3 \cdot \ldots \cdot a_{n-1} \rangle$ the result is clear so assume otherwise. By Step 1 there is $y \in D\langle 1, x \rangle \backslash \{1, x\}$. Since $D\langle 1, a_2 \cdot a_3 \cdot \ldots \cdot a_{n-1} \rangle$ has index 2 in

G either $y \in D\langle 1, a_2 \cdot a_3 \cdot \ldots \cdot a_{n-1}\rangle$ or $xy \in D\langle 1, a_2 \cdot a_3 \cdot \ldots \cdot a_{n-1}\rangle$. Hence one of y or xy is in the intersection.

Step 3. $G = X_1(a)X_3(a)$ for every $a \in G$.

First suppose $a \in D\langle 1, a_2 \cdot a_3 \cdot \ldots \cdot a_{n-1}\rangle$. Then $a_2 \cdot a_3 \cdot \ldots \cdot a_{n-1} \in D\langle 1, a\rangle$ and thus $D\langle 1, a_2 \cdot a_3 \cdot \ldots \cdot a_{n-1}\rangle \subseteq X_2(a)$. Let $x \in G$. By Step 2 there exists $y \in D\langle 1, x\rangle \cap D\langle 1, a_2 \cdot a_3 \cdot \ldots \cdot a_{n-1}\rangle \setminus \{1\}$ hence $x \in D\langle 1, y\rangle$ and $y \in D\langle 1, a_2 \cdot a_3 \cdot \ldots \cdot a_{n-1}\rangle \subseteq X_2(a)$. This implies $x \in X_3(a)$ and $G = X_3(a)$. Suppose now that $a \notin D\langle 1, a_2 \cdot a_3 \cdot \ldots \cdot a_{n-1}\rangle$. By Step 2 there exists $y \in D\langle 1, a\rangle \cap D\langle 1, a_2 \cdot a_3 \cdot \ldots \cdot a_{n-1}\rangle \setminus \{1\}$. Consequently, $a_2 \cdot a_3 \cdot \ldots \cdot a_{n-1} \in D\langle 1, y\rangle$ and $y \in D\langle 1, a\rangle$ so $a_2 \cdot a_3 \cdot \ldots \cdot a_{n-1} \in X_2(a)$ and $D\langle 1, a_2 \cdot a_3 \cdot \ldots \cdot a_{n-1}\rangle \subseteq X_3(a)$. Since $D\langle 1, a_2 \cdot a_3 \cdot \ldots \cdot a_{n-1}\rangle$ has index 2 in G, $G = \{1, a\} D\langle 1, a_2 \cdot a_3 \cdot \ldots \cdot a_{n-1}\rangle \subseteq X_1(a)X_3(a)$.

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