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THEOREM OF BERNSTEIN-DOETSCH IN BAIRE SPACES

Abstract. Theorem of Bernstein and Doetsch is one of the most important in the theory of convex (or convex in the sense of Jensen) functions. In this paper it is shown that the original proof of Bernstein and Doetsch of this theorem can be adapted in a more general situation. Also some parts of the proof may be of interest for themselves.

Introduction. The famous theorem of F. Bernstein and G. Doetsch [2] reads as follows. (Concerning the terminology used in this Introduction see §§ 1—2 below.)

THEOREM A. *Let $I \subset \mathbf{R}$ be an open interval, and let $f: I \to \mathbf{R}$ be a J-convex function. If f is bounded above on a non-empty open subinterval of I, then it is continuous in I.*

This theorem, dating from 1915, has been extended to more general spaces since, and shorter and simpler proofs have been found (cf. [10], [6], [5]). The original proof of Bernstein and Doetsch, however, certainly is interesting and ingenious and deserves to be paid a little attention.

In [5] the following extension of Theorem A has been proved.

THEOREM B. *Let X be a linear space endowed with a semilinear topology, let $D \subset X$ be an open and convex set, and let $f: D \to \mathbf{R}$ be a J-convex function. If f is bounded above on a non-empty open subset of D, then it is continuous in D.*

Our goal here is to investigate the question whether the original proof of Bernstein and Doetsch can be adapted to the more general situation as in Theorem B. It turns out that it is possible under the additional assumption that X is a Baire space. This condition, however, does not seem to be very restrictive, and we believe that the proof, although fairly long, may present an interest of its own. Also the particular parts of the proof may be of an independent interest.

1. Let X be a real linear space endowed with a topology $\mathscr{T}$. The topology $\mathscr{T}$ is called *linear* iff the function $\psi: \mathbf{R} \times X \times X \to X$,

(1) $$\psi(\lambda, x, y) = \lambda x + y,$$

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is continuous. If ψ is only separately continuous with respect to each variable, the topology $\mathscr{T}$ is said to be *semilinear* (cf. [5]).

REMARK 1. *If the topology $\mathscr{T}$ in X is semilinear, then*

a) *for every fixed $\lambda \neq 0$ and $y \in X$ function* (1) *is a homeomorphism from X onto X;*

b) *for every fixed $x, y \in X$ the function (from* $\mathbf{R}$ *into X) $\psi(\lambda, x-y, y) = \lambda x + (1-\lambda) y$ is continuous.*

We recall that a topological space X is called a *Baire space* whenever every non empty open subset of X is of the second category. If X is a linear space with a semilinear topology, then in view of Banach's first category theorem (cf. [8]) this is equivalent to the condition that X itself is of the second category. In the sequel a topology $\mathscr{T}$ in X such that the topological space $(X, \mathscr{T})$ is a Baire space will be referred to as a *Baire topology*.

Let X be a linear space, and let $A \subset X$ be an arbitrary set. A point $y \in A$ is said to be *algebraically interior* to A iff for every $x \in X$ there exists an $\varepsilon > 0$ such that

$$\lambda x + y \in A \quad \text{for} \quad \lambda \in (-\varepsilon, \varepsilon). \tag{2}$$

Put

$$\operatorname{core} A = \{y \in A : y \text{ is algebraically interior to } A\}. \tag{3}$$

It results directly from (2) and (3) that for arbitrary sets $A, B \subset X$ we have

$$A \subset B \text{ implies } \operatorname{core} A \subset \operatorname{core} B. \tag{4}$$

A set A is called *algebraically open* whenever $\operatorname{core} A = A$. The family of all algebraically open subsets of X is a topology in X and is called the *core topology* (cf. [1], [11], [10], [5], [4], [3]). The core topology is semilinear, and if there is another semilinear topology in X, then we have for every set $A \subset X$ (cf. [5])

$$\operatorname{int} A \subset \operatorname{core} A. \tag{5}$$

LEMMA 1. *Let X be a linear space endowed with a semilinear topology, and let $A \subset X$ be a convex set. If $\operatorname{int} A \neq \varnothing$, then*

$$\operatorname{int} A = \operatorname{core} A. \tag{6}$$

Proof. Take an arbitrary $y \in \operatorname{core} A$, and let $U \subset A$ be a non-empty open set. Let $u \in U$. By (3) and (2) (take $x = z - u$) there exist a $z \in A$ and a $\lambda \in (0,1)$ such that $y = \lambda u + (1-\lambda) z$. The set $V = \lambda U + (1-\lambda) z$ is open (cf. Remark 1a), and $V \subset A$ since A is convex. Moreover, $y \in V \subset \operatorname{int} A$. Due to the arbitrarness of $y \in \operatorname{core} A$ this means that $\operatorname{core} A \subset \operatorname{int} A$, which together with (5) yields (6).

LEMMA 2. *Let X be a linear space endowed with a semilinear Baire topology. If $A \subset X$ is a closed convex set, then* (6) *holds.*

Proof. If $\operatorname{core} A = \varnothing$, then (6) is a consequence of (5), and if $\operatorname{int} A \neq \varnothing$, then (6) results from Lemma 1. It remains to rule out the case, where $\operatorname{core} A \neq \varnothing$, say $y \in \operatorname{core} A$, and $\operatorname{int} A = \varnothing$.

For every $x \in X$ there exists a positive integer n such that $\dfrac{1}{n} x + y \in A$ (compare (2)), or, $x \in n(A-y)$. Hence

(7) $$X = \bigcup_{n=1}^{\infty} n\ (A-y).$$

The set A is nowhere dense (as a closed set with empty interior), and in view of Remark 1a so are also all sets $n(A-y)$. Consequently (7) shows that X is of the first category, which is incompatible with the assumption that X is a Baire space.

REMARK 2. The above lemma remains valid also in the case, where the set A is assumed to be closed in D, where $D \subset X$ is an open set such that $A \subset D$.

2. Throughout this section X denotes a real linear space endowed with a semilinear topology $\mathscr{T}$, and $D \subset X$ is an open and convex set. Unless explicitely said so, we do not assume that $\mathscr{T}$ is a Baire topology.

A function $f: D \to [-\infty, \infty)$ is called *J-convex* iff the inequality

(8) $$f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2}$$

holds for every $x, y \in D$. f is called *convex* whenever it satisfies

(9) $$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda) f(y)$$

for all $x, y \in D$ and $\lambda \in [0, 1]$.

We agree that the constant function equal to $-\infty$ at every point of its domain is continuous and convex. And every function f is lower semicontinous at every point x at which $f(x) = -\infty$.

The following lemma, which essentially is due to N. Kuhn [7] (cf. also [4]), allows us to get rid of the value $-\infty$ from the range of f.

LEMMA 3. *If $f: D \to [-\infty, \infty)$ is a J-convex function and if $f(x_0) = -\infty$ for an $x_0 \in D$, then $f(x) = -\infty$ for every $x \in D$.*

COROLLARY 1. *If $f: D \to [-\infty, \infty)$ is a J-convex function, then either $f = -\infty$ in D, or $f: D \to \mathbf{R}$ is a finite function.*

COROLLARY 2. *If $f: D \to [-\infty, \infty)$ is a J-convex function, then* (9) *holds for every $x, y \in D$ and rational $\lambda \in [0, 1]$.*

Proof. If $f = -\infty$ this is trivial, and if f is finite this is well known (cf. [6] or [10]).

REMARK 3. In view of Lemma 3 the results in [5] remain valid also for J-convex functions $f: D \to [-\infty, \infty)$.

LEMMA 4. *If a J-convex function $f: D \to \mathbf{R}$ is bounded above in a neighbourhood of a point $x_0 \in D$, then it is bounded above in a neighbourhood of every point $x \in D$.*

This is proved in [5].

LEMMA 5. *If a J-convex function $f: D \to \mathbf{R}$ is bounded above in a neighbourhood of a point $x_0 \in D$, then it is also bounded below in a neighbourhood of x_0.*

Proof. If f is bounded above, say by α, on a set $V \subset D$, then it is readily seen from (8) that f is bounded below by $2f(x_0) - \alpha$ on $(2x_0 - V) \cap D$.

LEMMA 6. *If a J-convex function $f:D\to\mathbf{R}$ satisfies*

$$(10)\qquad f(v)\leqslant\alpha \quad \text{for } v\in V$$

with an $\alpha\in\mathbf{R}$ and an open set $V\subset D$, and if $f(x_0)=\alpha$ for an $x_0\in V$, then $f(x)=\alpha$ for every $x\in V$.

Proof. Let $f(x_0)=\alpha$, $x_0\in V$, and suppose that for an $x\in V$ we have $f(x)<\alpha$. By (5) $x_0\in\operatorname{core} V$, whence by (3) and (2) there exists an $\varepsilon>0$ such that

$$x_0+\tau(x_0-x)\in V \quad \text{for} \quad \tau\in(-\varepsilon,\ \varepsilon).$$

We fix a rational $\tau\in(0,\ \varepsilon)$ and write $y=x_0+\tau(x_0-x)\in V$. Then $x_0=\lambda x+(1-\lambda)y$, where $\lambda=\tau/(1+\tau)\in(0,\ 1)$ is a rational number. In view of Corollary 2 we get by (9) and (10)

$$\alpha=f(x_0)\leqslant\lambda f(x)+(1-\lambda)f(y)<\alpha,$$

a contradiction.

LEMMA 7. *If a J-convex function $f:D\to\mathbf{R}$ is lower semicontinuous in D, then it is convex.*

Proof. Let f be lower semicontinuous in D, and fix arbitrary $x,\ y\in D$ and $\lambda\in[0,\ 1]$. Write $z=\lambda x+(1-\lambda)y$. Given an $\varepsilon>0$, we can find a neighbourhood $W\subset D$ of z such that

$$(11)\qquad f(t)\geqslant f(z)-\varepsilon \quad \text{for} \quad t\in W.$$

In view of Remark 1b there is an $\eta>0$ such that

$$(12)\qquad \mu x+(1-\mu)y\in W \quad \text{for} \quad \mu\in(\lambda-\varepsilon,\ \lambda+\varepsilon).$$

Take a rational $\mu\in(\lambda,\ \lambda+\eta)$. We have by (11) and (12), according to Corollary 2,

$$f(z)-\varepsilon\leqslant f(\mu x+(1-\mu)y)\leqslant\mu f(x)+(1-\mu)f(y).$$

Letting μ tend to λ over rational values in $(\lambda,\ \lambda+\eta)$ and then letting $\varepsilon\to 0+$ we obtain (9).

PROPOSITION 1. *If X is a Baire space, and if $f:D\to\mathbf{R}$ is a lower semicontinuous J-convex function, then f is continuous in D.*

Proof. For every $\alpha\in\mathbf{R}$ we put

$$(13)\qquad A_\alpha=\{x\in D: f(x)<\alpha\},\qquad B_\alpha=\{x\in D: f(x)\leqslant\alpha\}.$$

In virtue of Lemma 7 f is convex, when it follows easily that sets (13) are convex. Moreover, B_α are closed in D, since f is lower semicontinuous, and A_α are algebraically open as has been proved in [4].

Fix an $\alpha\in\mathbf{R}$. We shall distinguish two cases.

I. $A_\alpha\neq\varnothing$. Then we have by (4) and Lemma 2 (cf. Remark 2)

$$A_\alpha=\operatorname{core} A_\alpha\subset\operatorname{core} B_\alpha=\operatorname{int} B_\alpha,$$

i.e.

$$(14)\qquad A_\alpha\subset\operatorname{int} B_\alpha.$$

On the other hand, if for an $x \in \operatorname{int} B_\alpha$ we had $f(x) = \alpha$, then by Lemma 6 we would have $f = \alpha$ in $\operatorname{int} B_\alpha$. This means that $\operatorname{int} B_\alpha \subset A_\alpha$, which together with (14) yields $A_\alpha = \operatorname{int} B_\alpha$. Consequently A_α is open.

II. $A_\alpha = \emptyset$. Then clearly A_α is open.

We have shown that for every $\alpha \in \mathbf{R}$ the set A_α is open. This means that f is upper semicontinuous in D, and hence, being lower semicontinuous, it is continuous in D.

REMARK 4. As far as we know, all the existing proofs (cf. [1], [9]) of the analogous result in the case, where the topology in X is linear, rely on a suitable version of the theorem of Bernstein and Doetsch. The above proof, apart from the facts explicitely mentioned, implicitely uses (when we refer to [4]) also the property that a convex function of a single real variable is continuous in the interior of its domain (cf. [10]). However, the algebraic opennes of the set A_α may also be derived directly from (9).

3. Now X, endowed with a topology $\mathscr{T}$, may be an arbitrary topological space, and $D \subset X$ is an open set. For every $x \in X$ the symbol $\mathscr{T}_x$ denotes the family of all open subsets of X containing x.

For any function $f: D \to [-\infty, \infty)$ *the lower hull* m_f of f is defined by the formula (cf. [4], and also [2], [6])

$$m_f(x) = \sup_{U \in \mathscr{T}_x,\, U \subset D} \inf_U f, \qquad x \in D. \tag{15}$$

Thus m_f is a function $m_f: D \to [-\infty, \infty)$.

LEMMA 8. *For every function $f: D \to [-\infty, \infty)$ the function m_f given by* (15) *is lower semicontinuous in D.*

This has been proved in [4].

When the function f is J-convex the basic properties of m_f are expressed by the following (cf. also [4])

PROPOSITION 2. *Let X be a real linear space with a semilinear topology $\mathscr{T}$, let $D \subset X$ be an open and convex set, and let $f: D \to [-\infty, \infty)$ be a J-convex function. The lower hull of f is convex and lower semicontinuous in D. If, moreover, X is a Baire space, then m_f is continuous in D.*

Proof. Take arbitrary $x, y \in D$ and arbitrary $\alpha, \beta \in \mathbf{R}$ such that

$$\alpha > m_f(x), \quad \beta > m_f(y). \tag{16}$$

Write $z = \frac{1}{2}(x+y)$, and let $W \subset D$ be an arbitrary neighbourhood of z contained in D.

The function $\varphi_1: X \to X$,

$$\varphi_1(t) = \frac{1}{2}t + \frac{1}{2}y, \qquad t \in X,$$

is continuous and $\varphi_1(x) = z$. Thus there exists a neighbourhood $U \subset D$ of x such that $\varphi_1(U) \subset W$. Accordnig to (15) and (16) we can find in U a point u such that

$$(17) \qquad f(u) < \alpha.$$

Since $u \in U$ the point $s = \varphi_1(u)$ belongs to W.

The function $\varphi_2 : X \to X$,

$$\varphi_2(t) = \frac{1}{2}u + \frac{1}{2}t, \qquad t \in X,$$

is continuous and $\varphi_2(y) = \varphi_1(u) = s \in W$. Thus there exists a neighbourhood $V \subset D$ of y such that $\varphi_2(V) \subset W$. According to (15) and (16) we can find in V a point v such that

$$(18) \qquad f(v) < \beta.$$

Since $v \in V$ the point $w = \varphi_2(v)$ belongs to W.

By (17) and (18) we have, since f is J-convex,

$$f(w) = f\left(\frac{u+y}{2}\right) \leqslant \frac{f(u)+f(y)}{2} < \frac{\alpha+\beta}{2},$$

whence

$$(19) \qquad \inf_W f < \frac{\alpha+\beta}{2}.$$

Letting in (19) $\alpha \to m_f(x)+$, $\beta \to m_f(y)+$ (cf. (16)), and then taking the supremum over all $W \in \mathscr{T}_z$, $W \subset D$, we obtain according to (15), since $z = \frac{1}{2}(x+y)$,

$$m_f\left(\frac{x+y}{2}\right) \leqslant \frac{m_f(x)+m_f(y)}{2}.$$

Consequently m_f is J-convex.

By Corollary 1 either $m_f = -\infty$ in D, and then clearly it is continuous and convex, or $m_f : D \to \mathbf{R}$ is finite. In the latter case m_f is a lower semicontinuous convex function in virtue of Lemmas 8 and 7. If, moreover, X is a Baire space, then the continuity of m_f results from Proposition 1.

4. We will need one more lemma.

LEMMA 9. *Let X, D, f be as in Proposition 2, and let m_f be given by* (15). *If for a $\xi \in D$ we have $f(\xi) \neq m_f(\xi)$, then f is not bounded above in any neighbourhood of ξ.*

Proof. Supposing the contrary, let $f(\xi) \neq m_f(\xi)$ and let $V \subset D$ be a neighbourhood of ξ such that (10) holds with an $\alpha \in \mathbf{R}$. In view of Corollary 1 f is finite, since by (15) $f(\xi) > m_f(\xi) \geqslant -\infty$, and by Lemma 5 f is bounded below in a neighbourhood of ξ so that, in fact, $m_f(\xi)$ is finite and we have

$$(20) \qquad 2\varepsilon = f(\xi) - m_f(\xi) > 0.$$

We fix a positive integer n such that

$$(21) \qquad (n+1)\varepsilon + m_f(\xi) > \alpha.$$

Put

$$U=\left(\frac{n}{n-1}\xi-\frac{1}{n-1}V\right)\cap D.$$

U is an open set (Remark 1a); moreover, $\xi\in U$ so that $U\in\mathcal{T}_\xi$, $U\subset D$. According to (15) we can find a $u\in U$ with the property

$$f(u)<m_f(\xi)+\varepsilon. \tag{22}$$

Write $v=n\xi-(n-1)u\in n\xi-(n-1)U\subset n\xi-(n\xi-V)=V$ so that

$$v\in V. \tag{23}$$

In view of Corollary 2

$$f(\xi)=f\left(\frac{n-1}{n}u+\frac{1}{n}v\right)\leqslant\frac{n-1}{n}f(u)+\frac{1}{n}f(v),$$

whence by (20), (22) and (21)

$$f(v)\geqslant nf(\xi)-(n-1)f(u)>(n+1)\varepsilon+m_f(\xi)>\alpha,$$

which is incompatible with (23) and (10).

Now we are in position to prove our main result.

THEOREM. *Let X be a real linear space endowed with a semilinear Baire topology, let $D\subset X$ be an open and convex set, and let $f{:}D\to[-\infty,\infty)$ be a J-convex function. If f is bounded above on a non-empty open subset of D, then it is continuous in D.*

Proof. In view of Corollary 1 we may restrict ourselves to the case, where $f{:}D\to\mathbf{R}$ is a finite function. By Lemmas 4 and 5 f is locally bounded at every point of D, whence it follows that also its lower hull m_f is a finite function. By Lemma 9 $f=m_f$ in D, and this function is continuous in virtue of Proposition 2.

It is the step from lower semicontinuity to continuity (Proposition 1) that requires the assumption that X is a Baire space. All the proofs of the aforesaid implication we know about (cf. [1], [9]) use this assumption, so it is seems reasonable to conjecture that this assumption about X is essential for the validity of Proposition 1. On the other hand, we know of no example showing that without this condition on X Proposition 1 is invalid.

REFERENCES

[1] V. BARBU and TH. PRECUPANU, *Convexity and optimization in Banach spaces,* Editura Academiei Bucureşti and Sijthoff-Noordhoff International Publishers, Bucureşti, 1978.

[2] F. BERNSTEIN und G. DOETSCH, *Zur Theorie der konvexen Funktionen,* Math. Ann. 76 (1915), 514—526.

[3] Z. KOMINEK, *On additive and convex functionals,* Radovi Mat. 3 (1987), 267—279.

[4] Z KOMINEK and M. KUCZMA, *On the lower hull of convex function,* Aequationes Math. 38 (1989), 192—210.

[5] Z. KOMINEK and M. KUCZMA, *Theorems of Bernstein-Doetsch, Piccard and Mehdi, and semilinear topology,* Arch. Math. (Basel) 52 (1989), 595—602.

[6] M. KUCZMA, *An Introduction to the Theory of Functional Equations and Inequalities. Canchy's equation and Jensen's inequality*, PWN (Polish Scientific Publishers) i Uniwersytet Śląski, Warszawa—Kraków—Katowice, 1985.
[7] N. KUHN, *A note on t-convex functions*, International Ser. Numer. Math. 71. Birkhauser Verlag, Basel-Stuttgart, 1984. General Inequalities 4, 269—276. [Proceedings of the Fourth International Conference on General Inequalities held in the Mathematical Institute at Obervolfach, Black Forest, May 8—14, 1983]. Edited by W. Walter.
[8] K. KURATOWSKI, *Topologie*, vol. I. Monografie Mat. 20. PWN (Polish Scientific Publishers), Warszawa, 1958.
[9] M. R. MEHDI, *On convex functions*, J. London Math. Soc. 39 (1964), 321—326.
[10] A. W. ROBERTS and D. E. VARBERG, *Convex functions*, Academic Press, New York and London, 1973.
[11] F. A. VALENTINE, *Convex sets*, McGraw-Hill Book Company, New York-San Francisco-Toronto-London, 1964.