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ON THE SOLUTIONS OF CERTAIN FUNCTIONAL-DIFFERENTIAL EQUATIONS OF THE n-TH ORDER

Abstract. The classes of solutions in $[0, \infty)$ of the general functional-differential equation (1) are studied. The equation (1) includes various types of functional-differential equations with deviated argument. The solutions are functions with discontinuous derivative of the n-th order.

In the earlier paper [2] the classes of solutions in $[0, \infty]$ of an abstract functional-differential equation of the form

$$(1) \qquad \varphi^{(n)}(t) = \Theta F \varphi(t), \qquad \Theta^2 = 1,$$

were studied. A solution φ of (1) was understood to be a function of the class $C^{(n)}$ in an interval $[a, b) \subset \mathbf{R}^1$.

Let us introduce certain spaces of functions:

Φ^n, $n = 0, 1, \ldots$, denotes the space of functions $\varphi(t)$, $t \geqslant 0$, with continuous derivatives $\varphi^{(0)}$, φ', ..., $\varphi^{(n)}$.

We write $\varphi(t) \geqslant' \alpha$ $(\varphi(t) \leqslant' \alpha)$ if there exists a number $b \geqslant 0$ such that $\varphi(t) \geqslant \alpha$ $(\varphi(t) \leqslant \alpha)$ for $t \geqslant b$ and $\varphi(t) \not\equiv \alpha$ in any subinterval of $[0, \infty)$. Instead of $\varphi(t) \geqslant' 0$ $(\varphi(t) \leqslant' 0)$ we write $S[\varphi] = 1$ $(S[\varphi] = -1)$. As the limit $\lim \varphi(t)$ we always understand $\lim \varphi(t)$ as $t \to \infty$.

Ψ^n denotes the subspace of Φ^n consisting of functions φ such that $\varphi^{(k)}(t)$ have determined signs for sufficiently large t and $k = 0, 1, \ldots, n$.

Ψ^{nk} denotes the subspace of Ψ^n containing functions φ satisfying the following conditions:

1° $\varphi \in \Psi^n$,

2° $S[\varphi^{(i)}]\, S[\varphi] = 1$ for $i = 0, 1, \ldots, k$,

3° $S[\varphi^{(i)}]\, S[\varphi] = (-1)^{i-k}$ for $i = k+1, \ldots, n-1$, (when $k < n$),

4° $\lim \varphi^{(m)}(t) = 0$ for $m = k+1, \ldots, n-1$ (when $k < n-1$),

5° $\lim \varphi^{(k)}(t) = g \neq \pm\infty$ exists and $g\, S[\varphi] \geqslant 0$ (when $k \leqslant n-1$).

$\mathscr{B}^{nk}$ is the subspace of Ψ^{nk} consisting of functions φ for which $\lim \varphi^{(k)}(t) = 0$.

$\mathscr{A}^n$ is the subspace of Φ^n consisting of functions φ for which

$$\sup\{t : \varphi(t) = 0\} = \infty.$$

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The following theorem was demonstrated in [2].

If

1° *for any* $\varphi \in \Phi^n$, $S[\varphi] = 1$ *or* $S[\varphi] = -1$ *we have*

$$S[\varphi]\ S[F\varphi] = 1$$

and

2° *for any* $\varphi \in \Phi^n$, $S[\varphi] = 1$ *or* $S[\varphi] = -1$, $|\varphi(t)| \geqslant' c t^p$, $c = const > 0$, $p \in \mathrm{N}$, $0 \leqslant p < n-1$, *we have*

$$|\int^{\infty} s^{n-p-2} F\varphi(s)\ \mathrm{d}s| = \infty,$$

then all solutions of (1) *which exist in* $[0, \infty)$ *belong to the following classes:*

$\varphi^{(n)} = -F\varphi$	$\varphi^{(n)} = F\varphi$	
$\mathscr{A}^n$	$\mathscr{A}^n$, Ψ^{nn}, $\mathscr{B}^{n0}$	*n even*
$\mathscr{A}^n$, $\mathscr{B}^{n0}$	$\mathscr{A}^n$, Ψ^{nn}	*n odd*

The results formulated above will be extended to functional - differential equations of the form (1) but with a more general definition of a solution.

We define a solution φ of (1) as a real valued function which satisfies the following conditions:

1° φ is continuous for $t \geqslant 0$,

2° the derivatives $\varphi', \ldots, \varphi^{(n-1)}$ exist for $t \geqslant 0$,

3° $\varphi^{(n)}(t)$ exists at each point $t \in [0, \infty)$ with the possible exception of a sequence $\{t_1, \ldots, t_n, \ldots\} \subset [0, \infty)$ without any finite cluster point,

4° the right-hand derivatives $\varphi^{(n)}(t_j+)$ exist,

5° equation (1) is satisfied in every interval $[t_i, t_{i+1}) \subset [0, \infty)$.

REMARK 1. This type of generalization of a solution of (1) and $n = 1$ is necessary for a study of functional-differential equations which occur in mathematical models of certain biomedical phenomena ([6]).

REMARK 2. Some examples of equations of the form (1) for which the above generalization of a solution is useful are given below:

(a) $\varphi^{(n)}(t) = \Theta f(t, \varphi(t - E(\alpha(t))) + \Theta g(t, \varphi(t - \beta(t)))$,

(b) $\varphi^{(n)}(t) = p(t) f(\varphi(E(t)))$, $E(t) = \mathrm{Entier}(t)$,

(c) $\varphi^{(n)}(t) = f\big(t, \varphi(t), \ldots, \varphi^{(n-1)}(t), \varphi(t \pm \delta_0), \ldots, \varphi^{(n-1)}(t \pm \delta_{n-1}), \varphi(E(t)), \ldots,$

$$\varphi^{(n-1)}(E(t))) \cdot \int_{h(t)}^{k(t)} \varphi(t-s)\ \mathrm{d}_s r(t, s),$$

$S[f] = \mathrm{const}$, $h(t) \leqslant t \leqslant k(t)$, for fixed t the function r is non-decreasing in s.

Now we shall give some definitions and lemmas.

The expression $\varphi^{(n)}(t) \geqslant' \alpha$ ($\varphi^{(n)}(t) \leqslant' \alpha$) means here that $\varphi^{(n)}(t) \geqslant \alpha$ for b sufficiently large ($\varphi^{(n)}(t) \leqslant \alpha$ for $t \geqslant b$) except possibly a sequence $\{t_1, t_2, \ldots\} \subset [0, \infty]$ for which $\varphi^{(n)}(t_i+) \geqslant \alpha$ ($\varphi^{(n)}(t_i+) \leqslant \alpha$). When $\alpha = 0$, then we say that $\varphi(t)$ is of constant sign in $[0, \infty)$. $S[\varphi] = 1$ or $S[\varphi] = -1$.

Let us define the following spaces of functions:

$\hat{\Phi}^n$, $n = 1, 2, \ldots$, is the space of continuous functions $\varphi(t)$, $t \geqslant 0$, with continuous derivatives $\varphi', \ldots, \varphi^{(n-1)}$ for $t \geqslant 0$. The derivative $\varphi^{(n)}$ exists for $t \geqslant 0$ possibly except for a sequence of points $\{t_1, t_2, \ldots\} \subset [0, \infty)$ for which the right derivatives $\varphi^{(n)}(t_i+)$ exist.

$\hat{\Psi}^n$, $n = 0, 1, \ldots$, is the subspace of $\hat{\Phi}^n$ with functions φ such that $\varphi^{(i)}(t)$, $i = 0, 1, \ldots, n-1$ have determined signs for sufficiently large t and $\varphi^{(n)}(t) \geqslant' 0$ or $' \leqslant 0$.

$\hat{\Psi}^{nk}$ is defined by the conditions: $\varphi \in \hat{\Psi}^{nk}$ if and only if

1° $\varphi \in \hat{\Psi}^n$,

2° $S[\varphi^{(i)}]S[\varphi] = 1$ for $i = 0, 1, \ldots, k$,

3° $S[\varphi^{(i)}]S[\varphi] = (-1)^{i-k}$ for $i = k+1, \ldots, n-1$ (when $k < n$),

4° $\lim \varphi^{(m)}(t) = 0$ for $m = k+1, \ldots, n-1$ (when $k < n-1$),

5° the limit $\lim \varphi^{(k)}(t) = g \neq \pm\infty$ exists and $g S[\varphi] \geqslant 0$ (when $k \leqslant n-1$).

$\hat{\mathscr{B}}^{nk}$ is the subspace of $\hat{\Psi}^{nk}$ consisting of functions φ for which $\lim \varphi(t) = 0$.

$\hat{\mathscr{A}}^n$ is the subspace of $\hat{\Phi}^n$ consisting of functions φ for which

$$\sup\{t\colon \varphi(t) = 0\} = \infty.$$

LEMMA 1. *If* $\varphi \in \hat{\Phi}^n$ *and* $\varphi^{(n)}(t) \geqslant' 0$ ($\leqslant' 0$) *in the set* $[0, \infty) \setminus \{t_1, t_2, \ldots\}$, *then* $\varphi \in \hat{\Psi}^n$.

Proof. Let us consider the case $\varphi^{(n)}(t) \geqslant' 0$. Then $\varphi^{(n)}(t) \geqslant 0$ for $t \geqslant b$, except possibly for the points $\{t_1, t_2, \ldots\}$. From the formula

$$\varphi^{(n-1)}(t) = \varphi^{(n-1)}(a) + \int_a^t \varphi^{(n)}(s)\,\mathrm{d}s$$

it follows that $\varphi^{(n-1)}(t)$, $t \geqslant a$, is monotonic and therefore of constant sign for sufficiently large t. The same is true for $\varphi^{(n-2)}(t), \ldots, \varphi(t)$.

LEMMA 2. *For every function* $\varphi \in \hat{\Psi}^n$ *there exists a number* $b \geqslant 0$ *and a natural number* k, $0 \leqslant k \leqslant n$, *such that* $\varphi \in \hat{\Psi}^{nk}$. *(In fact the class* $\hat{\Psi}^n \subset \hat{\Phi}^n$ *, so when* $\varphi \in \hat{\Psi}^n$ *the functions* $\varphi, \varphi', \ldots, \varphi^{(n)}$ *are of constatnt sign for* $t \geqslant b$, *when* b *is sufficiently large.)*

Proof. The lemma is true for $n = 1$. Let us suppose that it is for $n-1$. At first, when $S[\varphi^{(n-1)}]S[\varphi^{(n-2)}] = -1$ and $\varphi^{(n-2)}(t) \geqslant' 0$, ($\varphi^{(n-1)}(t) \leqslant' 0$), then $\varphi^{(n)}(t)$ is of constant sign and the limit $\lim \varphi^{(n-1)}(t) = g \leqslant 0$ exists. When $g < 0$, then $\varphi^{(n-1)}(t) \leqslant' c < 0$ and for b sufficiently large

$$0 < \varphi^{(n-2)}(t) < \varphi^{(n-2)}(b) - cb + ct,$$

which is impossible. So we have $\varphi^{(n-1)}(t) \leqslant' 0$, $\lim \varphi^{(n-1)}(t) = g = 0$, from which it follows that $\varphi^{(n)}(t) \geqslant' 0$. We see that in this case the signs of $\varphi^{(n-2)}$, $\varphi^{(n-1)}$, $\varphi^{(n)}$ alternate. The existence of the integer k follows from our assumption on $k-1$.

Now let us discuss the case $S[\varphi^{(n-1)}]S[\varphi^{(n-2)}] = 1$. We can take $k = n$, when $S[\varphi^{(n)}]S[\varphi^{(n-1)}] = 1$, and $k = n-1$ when $S[\varphi^{(n)}]S[\varphi^{(n-1)}] = -1$ and the lemma is again true.

LEMMA 3. *When* $\varphi(t) \in \hat{\Psi}^{nk}$, $0 \leqslant k \leqslant n$, $\varphi^{(n)}(t) \leqslant 0$ *and* $f(t)$ *is a continuous and non-negative function in the interval* $[b, \infty)$ *such that for* $b \leqslant t \leqslant v$

$$S[\varphi^{(n)}]\int_t^v \varphi^{(n)}(s)\,ds \geqslant \int_t^v f(s)\,ds,$$

then

$$S[\varphi^{(\varkappa)}]\,\varphi^{(\varkappa)}(t) \geqslant \frac{1}{(n-\varkappa-1)!}\int_t^\infty (s-t)^{n-\varkappa-1} f(s)\,ds, \tag{2}$$

$$t \geqslant b,\ \varkappa = n-1,\ n-2, \dots,\ k.$$

The last integral is convergent and

$$S[\varphi^{(\varkappa)}] = (-1)^{n-\varkappa} S[\varphi^{(n)}].$$

Proof. From the assumptions of the lemma it follows that $\varphi^{(n-1)} \geqslant' 0$ and

$$-\int_t^v \varphi^{(n)}(s)\,ds = -\varphi^{(n-1)}(v)+\varphi^{(n-1)}(t) \geqslant \int_t^v f(s)\,ds.$$

So we have

$$\varphi^{(n-1)}(t) \geqslant \int_t^v f(s)\,ds.$$

The last formula is a particular case of (2) with $\varkappa = n-1$. Let us suppose that (2) is true for $\varkappa > k$. First we discuss the case $S[\varphi^{(\varkappa)}] = 1$ and, for the induction proof, assume that the formula

$$\varphi^{(\varkappa)}(t) \geqslant \frac{1}{(n-\varkappa-1)!}\int_t^\infty (s-t)^{n-\varkappa-1} f(s)\,ds$$

is true. By integration in the interval $[t, v]$ we get

$$\varphi^{(\varkappa-1)}(v)-\varphi^{(\varkappa-1)}(t) = \frac{1}{(n-\varkappa-1)!}\int_t^v \varphi^{(\varkappa)}(s)\,ds$$

$$\geqslant \frac{1}{(n-\varkappa-1)!}\int_t^v\left(\int_s^\infty (u-s)^{n-\varkappa-1} f(u)\,du\right)ds.$$

But $S[\varphi^{(\varkappa-1)}] = -1$ for $\varphi \in \hat{\Psi}^{nk}$ and

$$\varphi^{(\varkappa-1)}(t) \geqslant \frac{1}{(n-\varkappa-1)!}\int_t^v\left(\int_s^\infty (u-s)^{n-\varkappa-1} f(u)\,du\right)ds \geqslant 0.$$

So we have

$$\varphi^{(\varkappa-1)}(t) \geqslant \frac{1}{(n-\varkappa-1)!}\int_t^\infty\left(\int_s^\infty (u-s)^{n-\varkappa-1} f(u)\,du\right)ds \geqslant 0.$$

Let us estimate the integral

$$\lim_{v\to\infty}\int_t^v\left(\int_s^\infty (u-s)^{n-\varkappa-1} f(u)\,du\right)ds = \lim_{v\to\infty}\int_s^\infty\left(\int_t^v (u-s)^{n-\varkappa-1} f(u)\,ds\right)du$$

$$= \lim \int_u^\infty f(u)\left(\int_t^\infty (u-s)^{n-\varkappa-1}\,ds\right)du = \lim_{v\to\infty}\int_u^\infty f(u)\frac{(u-t)^{n-\varkappa}-(u-v)^{n-\varkappa}}{n-\varkappa}\,du$$

$$\geqslant \int_u^\infty \frac{(u-t)^{n-\varkappa}}{n-\varkappa} f(u)\,du.$$

This finishes the induction . The case $S[\varphi^{(\varkappa)}] = -1$ is similar.

Now we impose some hypotheses for the operation F in (1).

The operation F is in the space $\hat{\Phi}^n$ with values in the same space, $n \geqslant 1$.

Hypothesis H_1. For $\varphi \in \hat{\Phi}^n$, $S[\varphi] S[F\varphi] = 1$.

Hypothesis H_2. When $\varphi \in \hat{\Phi}^n$, $S[\varphi] = 1$ or $S[\varphi] = -1$, $|\varphi(t)| \geqslant' c t^p$, $c = \text{const} > 0$, $p \in \mathrm{N}$, $0 \leqslant p < n-1$, then

$$\int^{\infty} |s^{n-p-2} F\varphi(s) \, ds| = \infty.$$

THEOREM 1. *When the hypothesis* H_1 *is true, then every solution* φ *of equation* (1) *which exists in the interval* $[b, \infty)$ *belongs to one of the classes* $\hat{\mathscr{A}}^n$ *or* $\hat{\Psi}^{nk}$ *i.e.*

$$\text{if } \varphi^{(n)} = -F\varphi, \text{ then } \varphi \in \hat{\mathscr{A}}^n \text{ or } \varphi \in \hat{\Psi}^{nk}, \; 0 \leqslant k < n,$$

$$\text{if } \varphi^{(n)} = F\varphi, \text{ then } \varphi \in \hat{\mathscr{A}}^n \text{ or } \varphi \in \hat{\Psi}^{nk}, \; 0 \leqslant k \leqslant n.$$

Proof. Suppose that $\varphi \notin \hat{\mathscr{A}}^n$. Then $S[\varphi] = 1$ or $S[\varphi] = -1$ and from Lemmas 1 and 2 it follows that $\varphi \in \hat{\Psi}^{nk}$, $0 \leqslant k \leqslant n$.

THEOREM 2. *When the hypotheses* H_1 *and* H_2 *are satisfied, then every solution* φ *of equation* (1) *which exists in the interval* $[b, \infty)$ *belongs to one of the classes* $\hat{\mathscr{A}}^n$, $\hat{\Psi}^{nn}$, $\hat{\mathscr{B}}^{n0}$, *i.e.*

$\varphi^{(n)} = -F\varphi$	$\varphi^{(n)} = F\varphi$	
$\hat{\mathscr{A}}^n$	$\hat{\mathscr{A}}^n$, $\hat{\Psi}^{nn}$, $\hat{\mathscr{B}}^{n0}$	*n even*
$\hat{\mathscr{A}}^n$, $\hat{\mathscr{B}}^{n0}$	$\hat{\mathscr{A}}^n$, $\hat{\Psi}^{nn}$	*n odd*

Proof. Let us consider a solution φ of equation (1) in the interval $[b, \infty)$, such that $\varphi \notin \hat{\mathscr{A}}^n$. From Theorem 1 it follows that $\varphi \in \hat{\Psi}^{nk}$, $0 \leqslant k \leqslant n$. It is sufficient to demonstrate that the index k is equal to zero. When $\varphi \in \hat{\Psi}^{nk}$ then $S[\varphi^{(k)}] = 1$ or $S[\varphi^{(k)}] = -1$. Suppose that $S[\varphi^{(k)}] = 1$, $\varphi^{(k)} \geqslant' 0$ and $\varphi^{(k-1)}(t)$ is positive and non-decreasing. There exists an $\alpha > 0$ such that $\varphi^{(k-1)} \geqslant' 2\alpha$. The last inequality gives $\varphi(t) \geqslant t^{k-1}$ for $t \geqslant b$ and b sufficiently large. Let us take $f(t) = F\varphi(t)$. From the form of equation (1) and the hypothesis H_2, for $p = k-1$ we get

$$\int^{\infty} s^{n-k-1} f(s) \, ds = \infty.$$

Integrating "per partes" the integral

$$\int_{\tau}^{t} s^{n-k-1} \varphi^{(n)}(s) \, ds$$

we get

$$\int_{\tau}^{t} s^{n-k-1} \varphi^{(n)}(s) \, ds = t^{n-k-1} \varphi^{(n-1)}(t) - (n-k-1)(n-k-2) t^{n-k-2}$$

$$\cdot \varphi^{(n-2)}(t) + \ldots \pm (n-k-1)! \, \varphi^{(k)}(t) + \int_{\tau}^{t} \varphi^{(k)}(s) \, ds + C$$

$$> (n-k-1)! \, \varphi^{(k)}(t) + C = C + \int_{\tau}^{t} (s-\tau)^{n-k-1} \varphi^{(n)}(s) \, ds.$$

For $t \to \infty$ these inequalities lead to a contradiction $\infty < \infty$ and hence our assumptions about $k: k \geqslant 1$ is false. A similar contradiction is obtained when $S[\varphi^{(k)}] = -1$.

Suppose now that $\varphi(t) \geqslant' \alpha > 0$. From the hypothesis H_2, with $p = 0$, it follows that

$$(3) \qquad \infty = \int^{\infty} s^{n-2} F\varphi(s)\,ds < \int^{\infty} s^{n-1} F\varphi(s)\,ds.$$

But from Lemma 3 and $f(t) = \alpha > 0$, $\varkappa = 0$ it follows that

$$\varphi(t) \geqslant \frac{1}{(n-1)!} \int_t^{\infty} (s-t)^{n-1} \alpha\,ds \geqslant 0.$$

From the last inequality and Lemma 3 we get

$$(4) \qquad \infty > \frac{1}{(n-1)!} \int^{\infty} s^{n-1} F\varphi(s)\,ds.$$

Conditions (3) and (4) leads to

$$\infty = \int^{\infty} s^{n-2} F\varphi(s)\,ds < \int^{\infty} s^{n-1} F\varphi(s)\,ds < \infty.$$

This contradiction finishes the demonstration.

REMARK. Equation (1) is a generalization of a great number of functional--differential equations with or without deviation of the argument. The formulation of Theorem 2 is very general. The theorem includes not only the classical results of W. B. Fite [4], J. G. Mikusiński [5], A. Bielecki and T. Dłotko [2], T. Dłotko [3], but also the newest results reported by B. G. Zhang and N. Parhi [6], A. R. Aftabizadeh and J. Wiener [1], B. G. Zhang [7].

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